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# ON THE NEIGHBORHOOD STRUCTURE OF FUZZY TOPOLOGICAL SPACES

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**Abstract.** In [5] a fuzzy topology on a set X was defined as a fuzzy subset $\tau$ of the family $I^X$ of fuzzy subsets of X ($\tau: I^X \to I$) satisfying some axioms. In this paper the notions of a neighborhood structure and of a q-neighborhood structure are introduced and studied. These structures are applied to obtain local description of fuzzy topological spaces. As a special case the obtained theory contains in itself the local theory of Chang fuzzy spaces developed by Pu and Liu [3].

Following [5]-[7] by a **fuzzy topological space** we understand here a pair $(X,\tau)$ where X is a set and $\tau: I^X \to I$ is a **fuzzy topology**, i.e. a mapping satisfying the next three axioms:

(1) $\tau(0)=\tau(1)=1$;

(2) $\tau(U \wedge V) \geq \tau(U) \wedge \tau(V)$ for any $U, V \in I^X$;

(3) $\tau(\bigvee_i U_i) \geq \bigwedge_i \tau(U_i)$ for any family $\{U_i : i \in I\} \subset I^X$.

A fuzzy topology satisfying the additional axiom

(C) $\tau(I^X) \subset 2 := \{0,1\} \subset I$

can be considered as a fuzzy topology in Chang's sense [1]; in the sequel a fuzzy topology $\tau$ satisfying (C) is called a **Chang fuzzy topology** and the corresponding pair (X, ) is called a **Chang fuzzy topological space.**

Local characterization of Chang fuzzy spaces by means of neighborhood systems was first obtained by Pu and Liu in the fundamental work [3],[4]. The aim of this note is to introduce the notions of

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a neighborhood structure and of a q-neighborhood structure and to use them for the study of arbitrary fuzzy topological spaces. However it seems more convenient and natural to consider this problem in a more general propounding: namely, for fuzzy subtopological spaces as they are introduced below.

**DEFINITION 1.** *A mapping $\tau: I^X \to I$ such that*

(1) $\tau(0) = \tau(1) = 1$ *and*

(2) $\tau(U \wedge V) \geq \tau(U) \wedge \tau(V)$ *for any* $U, V \in I^X$

*is called a* **fuzzy subtopology** *on a set X and the corresponding pair* $(X,\tau)$ *is called a* **fuzzy subtopological space.**

**REMARK 1.** A "Chang" fuzzy subtopology , i.e. a mapping $\tau: I^X \to 2$ satisfying (1) and (2), in a natural way generates a Chang fuzzy topology $\tau: I^X \to 2$ for which $\tau$ is a base. Therefore whe considering Chang fuzzy topological spaces (and moreover, when considering ordinary topological spaces) the notion of a (fuzzy) subtopology seems to be superfluous: one can always manage with (fuzzy) topologies. However the situation with general fuzzy topologies as they are defined at the begining of the paper is essentially diferent and the relations between fuzzy topologies and fuzzy subtopologies are far less clear. In particular we need the notion of a subtopology to make our exposition more conventien and natural one.

Let $(X,\tau)$ be a fuzzy subtopological space and $\mathcal{X}$ be the set of all its fuzzy points. (Following [3], [4] by a fuzzy point of a set X we mean a mapping $p := x_0^t : X \to I$, where $x_0 \in X$ and $t \in (0,1]$ such that $p(x)=0$ if $x=x_0$ and $p(x_0)=t$; $x_0$ is called the support and t is called the value of the fuzzy point $x_0^t$. A fuzzy point $p := x_0^t$ belongs to a fuzzy set $U \in I^X$ (in symbols $p \tilde{\in} U$) if $U(x_0) \leq t$; a fuzzy point $p := x_0^t$ is quasicioincident, or q-coincident with a fuzzy set $U \in I^X$ (in symbols $pqU$) if $U(x_0)+t > 1$ [3].)

**DEFINITION 2.** *A mapping $Q: \mathcal{X} \times I^X \to I$ defined by the equality $N(x_0^t, U) = \sup \{\tau(V) : V \leq U,\ V(x_0) \geq 1-t\}$ is called the* **neighborhood structure** *of the fuzzy subtopological space $(X,\tau)$.*

**DEFINITION 2$^q$.** *A mapping $Q : \mathcal{X} \times I^X \to I$ defined by the equality $Q(x_0^t, U) = \sup \{\tau(V) : V \leq U,\ V(x_0) > 1-t$ is called the* **q-neighbordhood structure** *of the fuzzy topological space $(X,\tau)$.*

In the sequel we write usually $N_p(U)$ and $Q_p(U)$ instead of $N(p,U)$ and $Q(p,U)$ respectively.

**THEOREM 1.** *The neighbordhood structure $N$ of a fuzzy subtopological space $(X,\tau)$ satisfies the following conditions ($p:=x_0^t$ is an arbitrary fuzzy point in $X$):*

*(1N) if $N_p(U)>0$, then $p\tilde{\in}U$;*

*(2N) $\sup_{U\in I^X} N_p(U) = 1$;*

*(3N) $N_p(U_1\wedge U_2)\geq N_p(U_2)\wedge N_p(U_2)$ for any $U_1$, $U_2\in I^X$;*

*(4N) if $U\leq U'\in I^X$, then $N_p(U')\geq N_p(U)$;*

*(5N) $N_p(U)\leq \sup_{\substack{V\leq U\\ V\in I^X}} N_p(V)\wedge(\bigwedge_{\substack{r\tilde{\in}V\\ r\in\mathcal{X}}} N_p(V))$ for any $U\in I^X$.*

*(in fact the equality holds in (5N)).*

**Proof.** The properties (1N), (2N) and (4N) are obvious. The third property follows from the next series of inequalities:
$N_p(U_1\wedge U_2) = \sup\{\tau(V):\ V\leq U_1\wedge U_2,\ V(x_0)\geq t\} = \sup\{\tau(V_2\wedge V_2):\ V_i\leq U_i,\ V_i(x_0)\geq t,\ i=1,2\} \geq \sup\{\tau(V_1)\wedge\tau(V_2):\ V_i\leq U_i,\ V_i(x_0)\geq t,\ i=1,2\} = \sup\{\tau(V_1):\ V_1\leq U_1,\ V_1(x_0)\geq t\} \wedge \sup\{\tau(V_2):\ V_2\leq U_2,\ V_2(x_0)\geq t\} = N_p(U_1)\wedge N_p(U_1)\wedge N_p(U_2)$.

To show (5N) assume that $N_p(U) = \alpha$ and fix $\varepsilon>0$. Then by the definition of $N_p(U)$ there exists $V\in I^X$ such that $V\leq U$, $V(x_0)\geq t$, and $\tau(V)\geq\alpha - \varepsilon$. However this means that for each $r\tilde{\in}V$ it holds $N_r(V)\geq\tau(V)\geq\alpha-\varepsilon$ and, in particular, $N_p(V)\geq\alpha-\varepsilon$. To finish the proof notice that since $\varepsilon>0$ is arbitrary, it follows that $\sup_{V\leq U} N_p(V)\wedge(\bigwedge_{r\tilde{\in}V} N_r(V)) \geq \alpha$

Quite similarly one can check the next statement:

**THEOREM 1^q.** *The q-neighborhood structure $Q$ of a fuzzy subtopological space $(X,\tau)$ satisfies the next conditions ($p:=x_0^t$ is an arbitrary fuzzy point in $X$):*

*(1Q) if $Q_p(U)>0$, then $pqU$;*

*(2Q) $\sup_{U\in I^X} Q_p(U) = 1$;*

*(3Q) $Q_p(U_1\wedge U_2)\geq Q_p(U_1)\wedge Q_p(U_2)$ for any $U_1$, $U_2\in I^X$;*

*(4Q) if $U\leq U'\in I^X$, then $Q_p(U')\geq Q_p(U)$;*

(5Q) $Q_p(U) \le \sup_{\substack{V\le U \\ V\in I^X}} Q_p(V) \wedge (\bigwedge_{\substack{r q V \\ r\in \mathcal{X}}} Q_r(V))$ *for any* $U\in I^X$.

*(in fact the equality holds in (5Q)).*

**THEOREM 2.** *Let X be a set,* $\mathcal{X}$ *be the set of all its fuzzy points, and let* $N : \mathcal{X}\times I^X \to I$ *be a mapping satisfying the conditions (1N)-(5N). Then the mapping* $\tau: I^X: \to I$ *defined by* $\tau(U) = \inf_{p\tilde{\in} U} N_p(U)$ *is a fuzzy topology on X. Besides N is exactly the neighborhood structure of the fuzzy subtopological space* $(X,\tau)$.

**Proof.** Obviosly $\tau(0) = \inf 0 = 1$. From (2N) and (4N) it easyly follows that $N_p(1) = 1$ for each $p\in\mathcal{X}$ and hence $\tau(1) = \inf_{p\tilde{}} N_p(1) = 1$. Let $U_1, U_2\in I^X$. Applying (3N) and noticing that $p\tilde{\in} U_1\wedge U_2$ iff $p\tilde{\in} U_1$ and $p\tilde{\in} U_2$ (see e.g. [3]) we obtain the next series of inequalities:

$\tau(U_1\wedge U_2) = \inf\{N_p(U_1\wedge U_2): p\tilde{\in} U_1\wedge U_2\} \ge \inf\{N_p(U_1)\wedge N_p^{\sim}(U_2): p\tilde{\in} U_1,\ p\tilde{\in} U_2\}$

$\ge \inf\{N_p(U_1): p\tilde{\in} U_1\} \wedge \inf\{N_p(U_2): p\tilde{\in} U_2\} = \tau(U_1) \wedge \tau(U_2)$.

Thus $\tau$ is a fuzzy subtopology.

Let now $N_\tau$ denote the fuzzy neighbordhood structure of the fuzzy subtopological space $(X,\tau)$. Fix a fuzzy point $p:=x_0^t\in\mathcal{X}$ and a fuzzy set $U\in I^X$. Then applying (5N) we have

$N_p(p,U) = \sup\{\tau(V): V\le U,\ V(x_0)\ge t\} = \sup\{\inf_{r\tilde{\in} V} N_r(V): V\le U,\ V(x_0)\ge t\} \le$

$N_p(U) \le \sup_{V\le U} N_p(V) \wedge (\bigwedge_{r\tilde{\in} V} N_r(V)) = \sup\{\inf_{r\tilde{\in} V} N_r(V):\ V\le U,\ p\tilde{\in} V\} =$

Hence $N = N_\tau$.

Notice that, as it is shown by an easy example below, $\tau$ need not be a fuzzy topology even in Chang's situation (cf also 2).

**EXAMPLE.** Let X be a set and fix some $\alpha\in(0,1)$. Take a fuzzy point $p:=x^t$ and define $N_p: I^X \to I$ as follows. For $t\in(0,\alpha)$ let $N_p(U)= 1$ iff $U\ge t$ and $N_p(U) = 0$ otherwise; for $t \in [\alpha,1]$ let $N_p(U) = 1$ iff $U = 1$ and $N_p(U) = 0$ otherwise.

Obviosly a mapping $N:\mathcal{X}\times I^X \to I$ thus defined is a neigborhood structure. Let $\tau$ be the corresponding fuzzy subtopology. Then, by letting $U_c(x) = c$ for each $x\in X$, where $c\in(0,\alpha)$ we obtain a family of fuzzy sets such that $\tau(\bigvee_{c<\alpha} U_c) = 0$ while $\tau(U_c) = 1$ for each $c\in[0,\alpha)$ and hence $\tau$ is not a fuzzy topology.

**THEOREM $2^q$.** *Let $X$ be a set, $\mathcal{X}$ be the set of all its fuzzy points and let a mapping $Q : \mathcal{X} \times I^X \to I$ satisfy conditions (1Q) - (5Q). Then the mapping $\tau$: $I^X \to I$ defined by $\tau(U) = \inf_{pqU} Q_p(U)$ is a fuzzy topology. besides $Q$ is exactly the q-neighbordhood structure of the fuzzy topological space $(X,\tau)$.*

**Proof.** Obviosly $\tau(0) = \inf 0 = 1$. From (2Q) and (4Q) it follows that $Q_p(1) = 1$ for each $p \in \mathcal{X}$ and hence $\tau(1) = \inf_{pqU} Q_p(1) = 1$.

Let $U_1, U_2 \in I^X$. Applying (3Q) and noticing that $pq(U \wedge U)$ iff $pqU_1$ and $pqU_2$ (see e.g. [3]) we get

$\tau(U_1 \wedge U_2) = \inf\{Q_p(U_1 \wedge U_2): pq(U_1 \wedge U_2)\} \geq \inf\{Q_p(U_1) \wedge Q_p(U_2): pqU_1, pqU_2\} \geq \inf\{Q_p(U_1): pqU_1\} \wedge \inf\{Q_p(U_2): pqU_2\} = \tau(U_1) \wedge \tau(U_2)$.

Let now $\{U_i: i \in I\} \subset I^X$. Applying (4Q) and noticing that $pq(\vee U_i)$ iff $pqU_i$ for some $i_0 \in I$ (see e.g. [3]) we get

$$\tau(\vee_i U_i) = \inf\{Q_{p_i}(\vee_i U_i): pq(\vee_i U_i)\} = \inf\{Q_{p_i}(\vee_i U_i): \exists i_0 \in I, pqU_i\} \geq$$

$$\geq \inf\{Q_p(U_{i_0}): pqU_{i_0}\} = \tau(U_{i_0}) \text{ and hence } \tau(\vee_i U_i) \geq \wedge_{i \in I} \tau(U_i).$$

Thus $\tau$ is a fuzzy topology.

Let $Q_\tau$ denote the q-neighborhood structure of the fuzzy topological space $(X,\tau)$. Fix a fuzzy point $p := x_0^t \in \mathcal{X}$ and a fuzzy set $U \in I^X$. Then applying (5Q) we get

$Q_\tau(p,U) = \sup\{\tau(V): V \leq U, V(x_0) + t > 1\} = \sup\{\inf_{rqV} Q_r(V): V \leq U, V(x_0) + t > 1\} \leq Q_p(U) \leq \sup_{V \leq U} Q_p(V) \wedge (\wedge_{rqV} Q_r(V)) =$

$= \sup\{\inf_{rqV} Q_r(V): V \leq U, pqV\} = Q_\tau(p,U)$.

Hence $Q_\tau = Q$.

For a fuzzy neighbordhood structure $N : \mathcal{X} \times I^X \to I$ let $\tau_N : I^X \to I$ denote the corresponding fuzzy subtopology defined in the proof of Theorem 2. For a fuzzy subtopology $\tau : I^X \to I$ let $N_\tau$: $\mathcal{X} \times I^X \to I$ denote the corresponding fuzzy neighborhood structure constructed in Definition 2. Analogous meaning is attached to the notation $\tau_Q$ and $Q_\tau$. Theorems 2 and $2^q$ state in fact that for any neighbordhood structure $N$ and for any q-neighbordhood structure $Q$ the equalities $N_{\tau_N} = N$ and $Q_{\tau_Q} = Q$ hold. The next Theorems 3 and $3^q$ supplement to these results.

**THEOREM 3.** *For each fuzzy subtopology $\tau: I^X \to I$ the inequality $\tau_{N_\tau} \geq \tau$ holds. If, besides $\tau$ is a fuzzy topology, then $\tau = \tau_{N_\tau}$.*

**Proof.** From the definitions it follows that

$$\tau_{N_\tau}(U) = \inf_{p \tilde{\in} U} N_\tau(p,U) = \inf_{p \tilde{\in} U} \sup\{\tau(V)\colon V \leq U,\ V(x_0) \geq t\} \geq \tau(U),$$

for each $u \in I^X$ and hence $\tau_{N_\tau} \geq \tau$.

Let now $\tau$ be a fuzzy topology and assume that there exists $U \in I^X$ such that $\tau(U) < \tau_{N_\tau}(U) = \inf_{p \tilde{\in} U} \sup\{\tau(V)\colon V \leq U,\ V(x_0) \geq t\}$.

Take some $a \in (\tau(U),\ \tau_{N_\tau}(U))$ and for each $p \tilde{\in} U$ find $V_p \leq U$ such that $p \tilde{\in} V_p$ and $\tau(V_p) \geq a$. Hence $U = \bigvee_{p \tilde{\in} U} V_p$ and since $\tau$ is a fuzzy topology, we have $\tau(U) \geq \bigwedge_{p \tilde{\in} U} \tau(V_p) \geq a$. The obtained contraindiction implies that $\tau = \tau_{N_\tau}$.

**THEOREM $3^q$.** *For each fuzzy subtopology $\tau: I^X \to I$ the the inequality $\tau_{Q_\tau} \geq \tau$ holds. If besides $\tau$ is a fuzzy topology, then $\tau_{Q_\tau} = \tau$.*

**Proof.** The first atatement can be proved in the same way as in Theorem 3.

Let $\tau$ be a fuzzy topology and assume that there exist $U \in I^X$ and $a \in I$ such that $\tau(U) < a < \tau_{N_\tau} = \inf_{pqU} \sup\{\tau(U)\colon V \leq U,\ V(x_0) + t > 1\}$.

Then for each $pqU$ there exists $V$ such that $V_p \leq U$, $pqV$ and $\tau(V\ ) \geq a$. It is easy to notice That $U = \bigvee_{pqU} V_p$ (otherwise $U(x_0) > \bigvee_{pqU} V_p(x_0)$ for some $x_0 \in X$ and hence there would exist a fuzzy point $p_0 := x_0^t$ for which $p_0 qU$ but $p_0 q \bigvee V$ ). Since $\tau$ is a fuzzy topology it follows that $\tau(U) \geq \bigwedge_{pqU} \tau(V_p) \geq a$. The obtained contradiction completes the proof.

**REMARK 2.** For a fuzzy subtopology $\tau$ the equalities $\tau_{N_\tau} = \tau$ and $\tau_{Q_\tau} = \tau$ do not generally hold. In the second case one can be make sure himself about this indirectly in the following way.

Take a fuzzy subtopology $\tau$ which is not a fuzzy topology. Then nevertheless by Theorem $1^q$ the q-neighborhood structure $Q_\tau$ satisfies the conditions (1Q) - (5Q) and hence by Theorem $2^q$ $\tau_{Q_\tau}$ is a fuzzy topology, i.e. $\tau_{Q_\tau} \neq \tau$.

**REMARK 3.** Notice that if one starts with a Chang fuzzy (sub)topological space $(X,\tau)$ then corresponding fuzzy neighbordhood and q-neighbordhood structures $N$ and $Q$ constructed in Definitions 2 and $2^q$ are also two-valued and hence can be considered as systems of fuzzy subsets of X:

$$N = \cup_{p\in\mathcal{X}} N_p = \cup_{p\in\mathcal{X}} \{U: \exists V\in\tau,\ V\leq U,\ p\tilde{\in}V\};$$

$$Q = \cup_{p\in\mathcal{X}} Q_p = \cup_{p\in\mathcal{X}} \{U: \exists V\in\tau,\ V\leq U,\ pqU\}$$

The properties (1N) - (5N) can be written in this case in the following way:

(1NC) if $U\in N_p$, then $p\tilde{\in}U$;

(2NC) $N_p \neq \emptyset$ for each $p\in\mathcal{X}$;

(3NC) if $U_1, U_2 \in N_p$, then $U_1\wedge U_2\in N_p$;

(4NC) if $U\in N_p$ and $U\leq U'$, then $U' \in N_p$;

(5NC) for each $U \in N_p$ there exists $V \in N_p$ such that $V\leq U$ and $V\in N_r$ for each $r \tilde{\in} V$.

The properties (1Q) - (5Q) of a q-neighbordhood structure can be written in this case in the following way:

(1QC) if $U \in Q_p$, then pqU;

(2QC) $Q_p \neq \emptyset$ for each $p \in \mathcal{X}$;

(3QC) if $U_1, U_2 \in Q_p$, then $U_1\wedge U_2 \in Q_p$;

(4QC) if $U \in Q_p$ and $U\leq U'$, then $U' \in Q_p$;

(5QC) for each $U \in Q_p$ there exists $V \in Q_p$ such that $V\leq U$ and $V \in N_r$ for each rqV.

Comparing these conditions with the conditionals in Theorem 2.2 of [3] one can notice that in case of Chang fuzzy topological spaces our results reduce on the whole to the results of Pu and Liu (see [3], Theorem 2.2). However, it is necessary to take note that there is an error in the formulation of Theorem 2.2 [3]. In fact $\tau$ constructed there by means of neighbordhoods is not a Chang fuzzy topology, but only its base, i.e. a Chang fuzzy subtopology.

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**O OKOLINSKOJ STRUKTURI FAZI TOPOLOŠKOH PROSTORA**


Uvedeni su i izučavani pojmovi okolinske strukture i q-okolinske strukture fazi topoloških prostora. Ove strukture su primenjene za dobijanje lokalnih opisa fazi topoloških prostora.



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