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A NATURALLY INDUCED STRUCTURE ON A SUBSPACE OF A SPACE WITH AN F (2K+1,-1) - STRUCTURE

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Abstract: In section 1 the results of the author's previous paper [4] are given, where the structure $f(2k+1,-1)$ was defined on the manifold M^n. In section 2, 3 and 4 there are new results which refer to structure $-f^k=\phi$ of a rank $f=n-1$. The hypersurface N^{n-1} of the manifold M^n is observed and it is shown that ϕ induces on N^{n-1} a natural structure $F=B^{-1}\phi B$, which is a $F(3,-1)$-structure, or an almost product structure, which depends on the choise of the hypersurface.

Examined also are the integrability conditions of the naturally induced $F(3,-1)$-structure.

1. Preliminaries. Let us first observe the structure f satisfying the condition $f^{2k+1}-f = 0$.

DEFINITION 1. Let M^n be a differentiable manifold of class C^∞, and let there be a tensor field $f \neq 0$ of the type (1,1) and of class C^∞ such that

(1.1) $\quad f^{2k+1}-f = 0, \quad f^{2i+1}-f \neq 0 \quad \text{for} \quad 1 \leq i < k,$

where k is a fixed positive integer, greater then 1. Let rank $f = r$ be constant. We call such a structure an $f(k+1,-1)$-structure or an f-structure of rank r and of degree $2k+1$. For a tensor field f, $f \neq 0$ satisfying (1.1), the operators

(1.2) $\quad m = 1-f^{2k}, \quad l = f^{2k}$

are complementary projection operators where 1 denotes the identity operator applied to the tangent space at a point of the manifold.

It is easily seen that $l+m=1$, $l^2=l$, $m^2=m$, $ml=lm=0$ by virtue of (1.1), which proves the Theorem.

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Let L and M be the complementary distributions corresponding to the operators l and m, respectively. If rank $f = r$ is constant, then $\dim L = r$ and $\dim M = n - r$.

THEOREM 1.1. *For f satisfying (1.1) and l, m, defined by (1.2), we have*

$$lf = fl = f, \qquad mf = fm = 0, \qquad f^2m = 0$$

For f satisfying (1.1) and m, defined by (1.2), we have

$$(1.3) \qquad (m + f^k)^2 = 1, \qquad fm = mf = 0.$$

THEOREM 1.2 *Suppose that there is a projection operator m on M^n and that there exists a tensor feild f such that (1.3) is satisfied; then f satisfies (1.1).*

PROPOSITION 1.1. *Let an f-structure of rank r and of degree $2k+1$ be given on M^n, then $f^{2k}l = l$ and $f^{2k}m = 0$. Then f^k acts on L as an almost product structure operator and on M as a null operator.*

In [4] an adapted frame $u_1, \dots, u_n$ for an $f(2k+1,-1)$-structure is chosen and matrices of tensors g_{ij} and f_i^j are given with respect to this adapted frame:

$$(1.4) \qquad f = \left[\begin{array}{ccccccccccc} 0 & E_{2p-\frac{2p}{2^q}} & & & & & & & & & \\ -E_{\frac{2p}{2^p}} & 0 & & & & & & & & & \\ & & \ddots & & & & & & & & \\ & & & 0 & E_{2p-\frac{p}{2}} & & & & & & \\ & & & -E_{\frac{p}{2}} & 0 & & & & & & \\ & & & & & 0 & E_p & & & & \\ & & & & & -E_p & 0 & & & & \\ & & & & & & & E_p & 0 & & \\ & & & & & & & 0 & -E_p & & \\ & & & & & & & & & 0_{n-r} \end{array}\right] \Bigg\} q+1$$

$$g = \left[\begin{array}{cccc} E_{2p} & & & \\ & \ddots & & \\ & & E_{2p} & \\ & & & E_{n-r} \end{array}\right] \Bigg\} q+1 .$$

Let $\bar{u}_1, \ldots, \bar{u}_n$ be another adapted frame with respect to wich the metric tensor g and the tensor f have the same components as (1.4). We put $\bar{u}_i = \gamma_i^j u_j$, then we find that γ has a form

$$\gamma = \begin{bmatrix} S_{\left(\frac{2p}{2^q}\right)} & & & & & & & \\ & S_{\left(\frac{2p}{2^{q-1}}\right)} & & & & & & \\ & & \ddots & & & & & \\ & & & S_{\left(\frac{p}{2}\right)} & & & & \\ & & & & A_p & B_p & & \\ & & & & -B_p & A_p & & \\ & & & & & & 0_{2p} & \\ & & & & & & & 0_{n-r} \end{bmatrix}$$

where $S_{\left(\frac{2p}{i}\right)}$, $i = 2, 4, \ldots, 2^q$ is a matrix of the form

$$S_{\left(\frac{2p}{i}\right)} = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 & \cdots & A_i \\ -A_i & A_1 & A_2 & A_3 & \cdots & A_{i-1} \\ -A_{i-1} & -A_i & A_1 & A_2 & \cdots & A_{i-2} \\ \vdots & & & & & \\ -A_3 & -A_4 & -A_5 & & \cdots & A_2 \\ -A_2 & -A_3 & -A_4 & & \cdots & A_1 \end{bmatrix}$$

where each matrix A_t, $t = 1, \ldots, i$ has a format $\left(\frac{2p}{i}\right) \times \left(\frac{2p}{i}\right)$, i.e. $s \times s$.

Let $\bar{S}_{\left(\frac{2p}{i}\right)}$ be the tangent group defined by $S_{\left(\frac{2p}{i}\right)}$. then we can say that the group of the tangent bundle of the manifold can be reduced to

$$\bar{S}_{\left(\frac{2p}{2^q}\right)} \times \bar{S}_{\left(\frac{2p}{2^{q-1}}\right)} \times \ldots \times \bar{S}_{\left(\frac{2p}{4}\right)} \times U_p \times O_{2p} \times O_{n-r}.$$

THEOREM 1.3. *A necessary and sufficient condition for an n-dimensional manifold M^n to admit a tensor field $f \neq 0$ of the type (1,1) and of rank r, such that $f^{2\cdot 2^q+1} - f = 0$, is that*

(i) $r = (q+1)2p$

(ii) $p = s \cdot 2^q = s \cdot k$ *and*

(iii) the group of the tangent bundle of the manifold be reduced to the group

$$\bar{S}_{\binom{2p}{2^q}} \times \bar{S}_{\frac{2p}{2^{q-1}}} \times \ldots \times \bar{S}_{\binom{2p}{4}} \times U_p \times O_{2p} \times O_{n-r}.$$

2. Almost paracontact strusture - f^k. In what follows let $k = 2^q$, $q \in \mathbb{N}$.

THEOREM 2.1. *Denote $-f^k = \phi$. The structure ϕ satisfies the condition $\phi^3 - \phi = 0$, i.e. ϕ is an $\phi(3,-1)$-structure.*

Proof. We can see that

$$f^k = \left[\begin{array}{cccccc} -E_p & & & & & 0 \\ & -E_p & & & & \\ & & E_{2p} & & & \\ & & & \ddots & & \\ & & & & E_{2p} & \\ 0 & & & & & 0 \end{array}\right] \Big\} q+1 \qquad f^{2k} = \left[\begin{array}{ccccc} E_p & & & & \\ & E_p & & & \\ & & E_{2p} & & \\ & & & E_{2p} & \\ 0 & & & & 0 \end{array}\right] \Big\} q+1$$

THEOREM 2.2. *The structure ϕ is an almost paracontact Riemannian structure if rank* f = n-1.

Proof. Let

$$m = 1 - f^{2k} = \left[\begin{array}{ccccccc} 0 & 0 & & & & & \\ 0 & 0 & & & & & \\ & & \ddots & & & & \\ & & & 0 & 0 & & \\ & & & 0 & 0 & & \\ & & & & & & 1 \end{array}\right], \qquad \xi = \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{array}\right], \qquad \mu = (0,\ 0,\ \ldots,\ 0,\ 1)$$

M is an 1-dimensional distribution.

Multiplying the coresponding matrices, it is clear that $m = \mu \otimes \xi$, $\phi^2 = 1-m = 1 - \mu \otimes \xi$, $\phi\xi - 0$, $\mu\phi = 0$, $\xi(\mu) = 1$, $\mu(x) = g(\xi, x)$ and $g(\phi x, \phi y) = g(x,y) - \mu(x) \cdot \mu(y)$ which prove the Theorem.

3. The structure on the hypersurface

THEOREM 3.1. *Let M^n be a manifold with $\phi(3,-1)$-structure of rank r and let N^m be a hypersurface in M^n, $m = n-1$. If the dimension of $T(N^m)_p \cap f(T(N^m))_p$ is constant, say s, for all $p \in N^m$, then N^m possesses a natural $F(3,-1)$-structure of rank s.*

Proof. Let C ba a transversal defined on N^m, i.e. $C \in T(M^n)_p$, but $C \notin T(N^m)_p$ for all $p \in N^m$. Let B be a differential of tha imbedding of N^m in M^n. Then B is a map of $T(N^m)$ into $T_R(M^n)$, where $T_R(M^n)$ denotes the rastriction of $T(M^n)$, the tangent bundle of M^n to N^m. Then we can find a locally 1-form C^* defined on N^m such that:

$$B^{-1}B = I \qquad BB^{-1} = I - C^* \otimes C,$$

$$C^*B = B^{-1}C = 0, \qquad C^*(C) = 1.$$

Let F be defined locally on $T(N^{n-1})$ by $F = B^{-1}\phi B$. Then:

$$F^2X = B^{-1}\phi BB^{-1}\Phi BX = B^{-1}\phi\ (I - C^* \otimes C)\ \phi\ (BX) =$$

$$= B^{-1}\phi^2\ (BX) - C^*\phi\ (BX)\ B^{-1}\phi C.$$

If C is in distribution M, then $\phi C = 0$; so we have that

$$(F^3 - F)X = B^{-1}\phi BB^{-1}\phi^2 BX - B^{-1}\phi BX =$$

$$= B^{-1}\phi\ (I - C^* \otimes C)\ \phi^2 BX - B^{-1}\phi BX = B^{-1}((\phi^3 - \phi)BX) = 0$$

for all X. On the other hand, supose that C is in distribution L. Then:

$$(F^3 - F)X = (B^{-1}\phi B)B^{-1}\phi^2(BX) - (B^{-1}\phi B)\ C^*\ (\phi BX)\ B^{-1}\phi C - B^{-1}\phi BX =$$

$$B^{-1}(\phi^3 - \phi)BX - C^*(\phi^2 BX)B^{-1}\phi C - C^*(\phi BX)B^{-1}\phi^2 C + C^*(\phi BX)C^*(\phi C)B^{-1}\phi C = 0$$

since $\phi^2 C = C$ on L and $C^*B = B^{-1}C = 0$, and since we can choose C^* so that $C^*(\phi C) = 0$. Also $C^*(\phi^2 BX) = C^*\ (BX + (\phi^2 - 1)BX) = 0$

THEOREM 3.2. *If (ϕ, ξ, μ) is an almost paracontact structure on M^n, hten N^{n-1} possesses a natural $F(3,-1)$-structure if ξ is tangent to N^{n-1}. The hypersurface N^{n-1} possesses a natural almost product structure if ξ is not tangent to N^{n-1}.*

Proof. When ξ is not tangent to N^{n-1}, ξ can be chosen for a pseudonormal. Then we have from Theorem 4.1. that $T(N^{n-1}) \cap f(T(N^{n-1})) = T(N^{n-1})$, and rank F = dim $N^{n-1} = n - 1$. The almost paracontact structure F has a maximal rank, i.e. F is an almost product structure.

4. Integrability conditions

The structure ϕ is integrable if $[\phi, \phi] = 0$; $[\phi, \phi]$ denotes the Nijenhuis tensor of ϕ, i.e.

$$[\phi, \phi]\ (X, Y) = [\phi X, \phi Y] - \phi\ [\phi X, Y] - \phi\ [X, \phi Y] + \phi^2[X, Y]$$

for all vector fields X and Y on M^n.

In this section we shall assume that ϕ is an $\phi(3,-1)$-structure on M^n and that $F = B^{-1}\phi B$ is the naturally induced $F(3,-1)$-structure.

THEOREM 4.1. *Let N^{n-1} be a hypersurface in M^n, and suppose ϕ is integrable. If, locally, the transversal to N^{n-1} can be found to lie in the distribution M, then the induced $F(3,-1)$-structure on N^{n-1} is integrable.*

Proof. We see that

$$[F,F]\,(X,Y) = [FX,FY] - F\,[FX,Y] - F\,[X,FY] + F^2[X,Y] =$$
$$= B^{-1}[BB^{-1}\phi BX,\ BB^{-1}\phi BY] - B^{-1}\phi\,[BB^{-1}\phi BX,\ BY] -$$
$$- B^{-1}\phi\,[BX,\ BB^{-1}\phi BY] + B^{-1}\phi BB^{-1}\phi\,[BX,\ BY] =$$
$$= B^{-1}\,\{[\phi,\ \phi]\,(BX,BY) - [C^*(\phi BX)C,\ \phi BY] - [\phi BX,\ C^*(\phi BY)C] +$$
$$+ [C^*(\phi BX)C,\ C^*(\phi BY)C] + \phi\,[C^*(\phi BX)C,\ BY] + \phi\,[BX,\ C^*(\phi BY)C] -$$
$$- C^*(\phi\,[BX,\ BY])\,\phi C\},$$

where we have used the fact that $B[X,\ Y] = [BX,\ BY]$, for vector fields X, Y on N^{n-1}, and that $BB^{-1} = I - C^*\otimes C$. If the transversal C lies in the distribution M, then the form C^* can be chosen so that $C^*\phi = 0$. We see that: $[F,\ F]\,(X,\ Y) = B^{-1}([\phi,\ \phi]\,(BX,BY)) = 0$.

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PRIRODNO INDUKOVANA STRUKTURA NA POTPROSTORU

PROSTORA SA f(2k+1,-1)-STRUKTUTOM

U jednom ranijem radu autora definisana je struktura f(2k+1,-1) na mnogostrukosti M^n. Ovde se daju novi razultati koji se odnose na strukturu $-f^k = \phi$ ranga f=n-1. Utvrdjeni su takodje uslovi integragilnosti prirodno indukovane F(3,-1)-strukture.

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