Zbornik radova Filozofskog fakulteta u Nišu
Serija Matematika 4(1990), 55-64

Nevena Pušić

ON AN INVARIANT TENSOR OF A CONFORMAL TRANSFORMATION OF A HYPERBOLIC KAEHLERIAN MANIFOLD

(Received 24.03.1990.)

Abstract. We construct components of a conformal connection in a hyperbolic Kaehlerian space and an invariant curvature-type tensor of such a transformation. We also investigate its algebraic properties.

1. On conformal transformations of spaces with structure tensor

It is not easy to transfere the theory of conformal transformations and their invariants from Riemannian spaces to spaces with a structure. Only in locally product spaces a naturally conformal transformation raises.

A locally product space is a differentiable manifold M_n with positive definite Riemannian metric

(1.1) $$ds^2 = g_{ij}dx^i dx^j$$

and with structure tensor $(F_i^{\ j})$, where the following is satisfied:

(1.2) $$F_j^{\ t}F_i^{\ s}g_{ts} = g_{ji} \ ,$$

(1.3) $$F_j^{\ t}F_t^{\ i} = \delta_j^{\ i} \ ,$$

(1.4) $$\nabla_k F_j^{\ i} = 0 \text{ for locally decomposable spaces.}$$

From (1.2) and (1.3) it follows that the covariant structure tensor (F_{ij}) is symmetric.

Tachibana [6] defined a PC (product-conformal) transformation as a transformation of the metric

(1.5) $$\bar{g}_{ij} = \rho g_{ij} + \sigma F_{ij} \ ,$$

where ρ and σ are scalar functions satisfying

$$\partial_i\rho = \partial_m\sigma F_i^{\ m} \text{ and } \rho^2-\sigma^2 \neq 0.$$

AMS Subject Classification (1980): 53A30

There is also a tensor invariant under a PC-transformation called a PC-curvature tensor. For further informations the reader can see [4], [6].

There is a difference when we deal with Kaehlerian spaces. A Kaehlerian space is a differentiable manifold M_n of an even dimension and with structure tensor $(F_i{}^j)$ which satisfies

$$F_j{}^t F_i{}^s g_{ts} = -g_{ji}\ , \tag{1.6}$$

$$F_j{}^t F_t{}^i = \delta_j{}^i\ , \tag{1.7}$$

$$\nabla_k F_i{}^j = 0. \tag{1.8}$$

From (1.6) and (1.7) it follows that the covariant structure tensor is skew-symmetric (and parallel because the metric tensor is parallel). This is a real obstruction to define a complex conformal transformation. If we introduce a conformal transformation as it was done in Riemannian spaces, then two conformally correspondent Kaehlerian spaces are homothetic.

Yano [7] introduced a transformation satisfying

$$\bar{g}_{ij} = e^{2p} g_{ij}\ ,\ \bar{F}_j{}^i = F_j{}^i. \tag{1.9}$$

He also constructed a connection which satisfies

$$D_k \bar{g}_{ij} = 0,\ D_k F_j{}^i = 0,\ \frac{1}{2}(\Gamma_j{}^i{}_k - \Gamma_k{}^i{}_j) = -F_{jk} q^i. \tag{1.10}$$

Such a connection is called a complex conformal connection.

Yano also proved [7] that, if a oomplex conformal connection has a vanishing curvature tensor, then the Bochner curvature tensor

$$\begin{aligned} B^i{}_{jkl} = K^i{}_{jkl} - (n+4)^{-1}\{ & \delta_l{}^i K_{kj} - \delta_k{}^i K_{lj} + K_l{}^i g_{kj} - K_k{}^i g_{lj} + F_l{}^i H_{kj} - \\ & - F_k{}^i H_{lj} - H_l{}^i F_{kj} - H_k{}^i F_{lj} - 2H_{lk} F_j{}^i - 2H_j{}^i F_{lk} - \\ & (n+2)^{-1} K(\delta_l{}^i g_{kj} - \delta_k{}^i g_{lj} + F_l{}^i F_{kj} - F_k{}^i F_{lj} + 2F_{lk} F_j{}^i))\} \end{aligned} \tag{1.11}$$

(where K_{kj} is the Ricci tensor and $H_{kj} = -K_{ka} F_j{}^a$) vanishes.

A hyperbolic Kaehlerian space is a space of even dimension (n=2m) with indefinite metric and with structure which satisfies

$$F_i{}^t F_t{}^j = \delta_i{}^j\ , \tag{1.12}$$

$$\nabla_k F_j{}^i = 0\ , \tag{1.13}$$

$$F_i{}^a g_{aj} = -F_j{}^a g_{ai}\ . \tag{1.14}$$

In this paper, following the idea and the method of Yano [7], we will construct a connection which will be called a **conformal connection of a hyperbolic Kaehlerian space**. We will also find, under certain conditions, an invariant of all these connections.

2. About a hyperbolic Kaehlerian space

In the previous paragraph, we defined a hyperbolic Kaehlerian space as a space with indefinite metric and with structure. The structure is non-degenerate and possesses n (dimension of the space is n=2m) linearly independent eigen vectors in the tangent space. We can prove the following properties of the structure.

PROPOSITION 1. *(A) Every vector in the tangent space of a hyperbolic Kaehlerian space is transformed by a structure into an orthogonal vector.*

(B) The scalar square of a vector-original is opposite to the scalar square of the vector-image.

Proof. (A) $a_j F_i{}^j = b_j$.

$$a_j b^j = a_j a_s F_t{}^s g^{tj} = a_j a_s F^{js} = -a_j a_s F^{sj} = 0.$$

(B) $b_s b^s = b_s b_t g^{ts} = b_s a_j F_t{}^j g^{ts} = b_s a_j F^{sj} = -a_j b_s F^{js} = -a_j a^j$.

In accordance to Proposition 1 eigen vectors of the structure are isotropic (null-vectors). As the structure possesses n linearly independent eigen-vectors, there exists a basis of the tangent space of a hyperbolic Kaehlerian space where these isotropic vectors serve as basic vectors. In such a basis metric tensor is hybrid and the structure tensor is pure. Covariant structure tensor is also hybrid. Using this coordinate system, we can show that a hyperbolic Kaehlerian space admits isotropic vector fields which are not eigen for the structure. Such a basis is called a **separated basis.** Also, according to Proposition 1 (B), there exist vectors of positive scalar square (space-like vectors) and vectors of negative scalar square (time-like vectors). Space-like vectors may serve as a domain for the involution $(F_j{}^i)$ and its co-domain will be the set of time-like vectors. We may choose such a basis; then the metric tensor will be a pure tensor of signature (n,n) and $(F_j{}^i)$ will be a hybrid tensor. Such a basis is called an **adapted basis.**

For all the considerations in this paper we will use an arbitrarely chosen basis - it will be neither separated nor adapted. However, all our results can be transferred into these special bases and some of them even may look simpler.

3. On a metric connection with a given torsion

Let a connection with coefficients $\{\Gamma_{j\ k}^{\ i}\}$ be non-symmetric and let

(3.1) $$\Gamma_{i\,k}^{\ \ s} - \Gamma_{k\,i}^{\ \ s} = 2S_{ik}^{\ \ s},$$

where $S_{ik}^{\ \ s}$ are components of the torsion tensor.

Our goal is to construct connection coefficients, if the connection is metric and the torsion tensor is chosen in advance. As the connection is metric, we have

(A) $$\frac{\partial g_{ij}}{\partial x^k} - \Gamma_{i\,k}^{\ \ s} g_{sj} - \Gamma_{j\,k}^{\ \ s} g_{is} = 0.$$

If we permute in (A) i, j, k cyclicly, then we obtain

(B) $$\frac{\partial g_{jk}}{\partial x^i} - \Gamma_{j\,i}^{\ \ s} g_{sk} - \Gamma_{k\,i}^{\ \ s} g_{js} = 0$$

and

(C) $$\frac{\partial g_{ki}}{\partial x^j} - \Gamma_{k\,j}^{\ \ s} g_{si} - \Gamma_{i\,j}^{\ \ s} g_{ks} = 0.$$

From (A), (B) and (C) we obtain

$$\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ki}}{\partial x^j} = (\Gamma_{i\,k}^{\ \ s} + \Gamma_{k\,i}^{\ \ s}) g_{js} + 2S_{jk}^{\ \ s} g_{is} + 2S_{ji}^{\ \ s} g_{ks}$$

or, in accordance to (3.1),

$$\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ki}}{\partial x^j} = 2(\Gamma_{k\,i}^{\ \ s} + S_{ik}^{\ \ s}) g_{js} + 2S_{jk}^{\ \ s} g_{is} + 2S_{ji}^{\ \ s} g_{ks}.$$

Then, trancvecting this result by $\frac{1}{2}g^{aj}$, we obtain

$$\left\{ {a \atop i\,k} \right\} = \Gamma_{k\,i}^{\ \ a} + S_{ik}^{\ \ a} + S_{jk}^{\ \ s} g_{is} g^{aj} + S_{ji}^{\ \ s} g_{ks} g^{aj},$$

or, taking into account that the torsion tensor is skew-symmetric in lower indices,

(3.2) $$\Gamma_{k\,i}^{\ \ a} = \left\{ {a \atop k\,i} \right\} + S_{ki}^{\ \ a} + S_{k\ i}^{\ a} + S_{i\ k}^{\ a},$$

where

(3.3) $$S_{k\ i}^{\ a} = S_{kl}^{\ \ t} g^{la} g_{ti}.$$

We have proved

PROPOSITION 2. *The metric connection with torsion tensor components* $S_{ki}^{\ \ j}$ *has coefficients of the form (3.2).*

(This result is due to Hayden [2]. Here it is adapted to a different covariant differentiation convention, used by this author.)

4. Conformal connection of a hyperbolic Kaehlerian space

We introduce a transformation of a hyperbolic Kaehlerian space

(4.1) $$\bar{g}_{ji} = e^{2p} g_{ji}, \quad \bar{F}_j^{\ i} = F_j^{\ i}, \quad \bar{F}_{ji} = e^{2p} F_{ji},$$

where p is a C^r, $r \geq 3$, scalar function of n variables.

We are looking for an affine connection for which the new metric tensor $\bar{g}_{ji}$ will be parallel and whose torsion tensor will be $S_{ji}{}^{a} = F_{ji}q^{a}$ where q^{a} is a vector field.

According to Proposition 2 such a connection will have coefficients of this form:

$$\Gamma_{j\ k}^{\ i} = \overline{\left\{ {}_{j\ k}^{\ i} \right\}} + F_{j}{}^{i}q_{k} + F_{k}{}^{i}q_{j} + F_{jk}q^{i} ,$$

where $\overline{\left\{ {}_{j\ k}^{\ i} \right\}}$ are Christoffel symbols formed of $\bar{g}_{ji}$, i.e.

(4.2) $$\Gamma_{j\ k}^{\ i} = \left\{ {}_{j\ k}^{\ i} \right\} + \delta_{j}^{i}p_{k} + \delta_{k}^{i}p_{j} - g_{jk}p^{i} + F_{j}{}^{i}q_{k} + F_{k}{}^{i}q_{j} + F_{jk}q^{i},$$

where

(4.3) $$p_{k} = \frac{\partial p}{\partial x^{k}} , \ p^{i} = g^{ia}p_{a} , \ q_{j} = g_{ja}q^{a}.$$

We also want the connection with coefficients (4.2) to be an F-connection, which means that the covariant structure tensor (as well as the mixed one) needs to be parallel. If we differentiate F_{ij} covariantly, then

$$D_{k}\bar{F}_{ij} = D_{k}e^{2p}F_{ij} = e^{2p}[2p_{k}F_{ij} + D_{k}F_{ij}] =$$

$$= e^{2p}[g_{ik}(q_{j}-F_{js}p^{s})-g_{kj}(q_{i}-p^{s}F_{is})-F_{ik}(p_{j}-q^{s}F_{js})+F_{jk}(p_{i}-q^{s}F_{is})] = 0$$

what means: the connection with the coefficients of the form (4.2) is an F-connection if and only if

(4.4) $$g_{ik}(q_{j}-F_{js}p^{s})-g_{kj}(q_{i}-p^{s}F_{is})-F_{ik}(p_{j}-q^{s}F_{js})+F_{jk}(p_{i}-q^{s}F_{is}) = 0.$$

Transvecting (4.4) with g^{ik} we get

$$(n-1)(q_{j}-F_{js}p^{s}) + (F_{j}{}^{i}p_{i}-g_{js}q^{s}) = 0$$

which means $$(n-2)(q_{j}-F_{js}p^{s}) = 0.$$

As $n > 2$ it follows

(4.5) $$q_{j} = F_{js}p^{s} = F_{j}{}^{s}p_{s} , \ p_{j} = F_{js}q^{s} = F_{j}{}^{s}q_{s} .$$

We have proved

THEOREM 1. *In a hyperbolic Kaehlerian space with metric tensor (g_{ij}) and structure tensor $(F_{j}{}^{i})$ an affine connection which satisfies*

$$D_{k}e^{2p}g_{ij} = 0, \ D_{k}e^{2p}F_{ij} = 0 \ \text{ and } \ \Gamma_{j\ k}^{\ i} - \Gamma_{k\ j}^{\ i} = 2F_{jk}q^{i},$$

where p is a C^{r} scalar function of n variables, is given by (4.2) and the conditions (4.3) and (4.5) are satisfied.

Such a connection we call a conformal connection on a hyperbolic Kaehlerian space.

5. Curvature tensor of a conformal connection on a hyperbolic Kaehlerian space

Suppose we have given a conformal connection on a hyperbolic Kaehlerian space (4.2) and that the conditions (4.3) and (4.5) are satisfied.

We calculate the curvature tensor of such a connection according to formula

$$(5.1)\qquad R^i{}_{jkl} = \frac{\partial \Gamma^i{}_{jk}}{\partial x^l} - \frac{\partial \Gamma^i{}_{jl}}{\partial x^k} + \Gamma^s{}_{jk}\Gamma^i{}_{sl} - \Gamma^s{}_{jl}\Gamma^i{}_{sk} .$$

By a straightforward calculation we obtain:

$$\begin{aligned} R^i{}_{jkl} = K^i{}_{jkl} &- \delta_l{}^i(\nabla_k p_j - p_k p_j - q_k q_j + \tfrac{1}{2}g_{kj}p_s p^s - \tfrac{1}{2}F_{jk}q_s p^s) + \\ &+ \delta_k{}^i(\nabla_l p_j - p_l p_j - q_l q_j + \tfrac{1}{2}g_{lj}p_s p^s - \tfrac{1}{2}F_{jl}q_s p^s) - \\ &- g_{jk}(\nabla_l p^i - p_l p^i - q_l q^i + \tfrac{1}{2}\delta_l{}^i p_s p^s - \tfrac{1}{2}F_l{}^i p_s q^s) + \\ &+ g_{jl}(\nabla_k p^i - p_k p^i - q_k q^i + \tfrac{1}{2}\delta_k{}^i p_s p^s - \tfrac{1}{2}F_k{}^i p_s q^s) - \\ &- F_l{}^i(\nabla_k q_j - p_k q_j - q_k p_j + \tfrac{1}{2}F_{kj}q_s q^s - \tfrac{1}{2}g_{kj}p_s q^s) + \\ &+ F_k{}^i(\nabla_l q_j - p_l q_j - q_l p_j + \tfrac{1}{2}F_{lj}q_s q^s - \tfrac{1}{2}g_{lj}p_s q^s) - \\ &- F_{kj}(\nabla_l q^i - p_l q^i - q_l p^i + \tfrac{1}{2}F_l{}^i q_s q^s - \tfrac{1}{2}\delta_l{}^i p_s q^s) + \\ &+ F_{lj}(\nabla_k q^i - p_k q^i - q_k p^i + \tfrac{1}{2}F_k{}^i q_s q^s - \tfrac{1}{2}\delta_k{}^i p_s q^s) + \\ &+ 2F_{kl}(p_j q^i - q_j p^i) + F_j{}^i(\nabla_l q_k - \nabla_k q_l) , \end{aligned}$$

where $K^i{}_{jkl}$ are coefficients of curvature tensor of the original hyperbolic Kaehlerian space.

According to Proposition 1, $p_s p^s = -q_s q^s$ and $p_s q^s = 0$. If we introduce new notation

$$(5.2)\qquad \begin{aligned} p_{kj} &= \nabla_k p_j - p_k p_j - q_k q_j + \tfrac{1}{2}p_s p^s g_{kj} , \\ q_{kj} &= \nabla_k q_j - p_k q_j - q_k p_j - \tfrac{1}{2}p_s p^s F_{kj} , \\ \beta_{ji} &= 2(p_j q_i - q_j p_i) , \\ \alpha_{kl} &= -(\nabla_k q_l - \nabla_l q_k) \end{aligned}$$

and take into account that the metric and structure are parallel, then

$$(5.3)\qquad \begin{aligned} R^i{}_{jkl} = K^i{}_{jkl} &- \delta_l{}^i p_{kj} + \delta_k{}^i p_{lj} - g_{kj}p_l{}^i + g_{lj}p_k{}^i - F_l{}^i q_{kj} + \\ &+ F_k{}^i q_{lj} - F_{kj}q_l{}^i + F_{lj}q_k{}^i + F_{kl}\beta_j{}^i + F_j{}^i\alpha_{kl} . \end{aligned}$$

p_{kj} and q_{kj} are related as follows:

$$(5.4)\qquad p_{ka}F_j{}^a = q_{kj} \ ; \ q_{ka}F_j{}^a = p_{kj} .$$

For α_{kj} and β_{kj} we have

$$(5.5)\qquad \begin{aligned} \alpha &= F^{kj}\alpha_{kj} = -2\nabla_t p^t , \quad \alpha_{kj} = -\alpha_{jk} , \\ \beta &= F^{kj}\beta_{kj} = 4p_s p^s , \quad \beta_{kj} = -\beta_{jk} . \end{aligned}$$

After lowering the upper index i in (5.3) we obtain

$$\begin{aligned} R_{ijkl} = K_{ijkl} &- g_{li}p_{kj} + g_{ki}p_{lj} - g_{kj}p_{li} + g_{lj}p_{ki} - F_{li}q_{kj} + \\ &+ F_{ki}q_{lj} - F_{kj}q_{li} + F_{lj}q_{ki} + F_{kl}\beta_{ji} + F_{ji}\alpha_{kl} , \end{aligned}$$

where K_{ijkl} is the Riemann-Christoffel tensor of a hyperbolic Kaehlerian space. We can notice that the tensor R_{ijkl} is skew-symmetric in indices i,j. Suppose now that it is invariant under change of the first and second pair of indices.

$$R_{ijkl} = R_{klij},$$
$$R_{klij} = K_{klij} - g_{jk}p_{il} + g_{ik}p_{jl} - g_{il}p_{jk} + g_{jl}p_{ki} - F_{jk}q_{il} + F_{ik}q_{jl} - F_{il}q_{jk} + F_{jl}q_{ik} + F_{ij}\beta_{lk} + F_{lk}\alpha_{ij}.$$

As p_{kj} and q_{kj} are symmetric and F_{kj}, α_{kj} and β_{kj} skew-symmetric, we obtain

$$(5.7)\quad F_{li}(q_{kj} + q_{jk}) - F_{ki}(q_{lj} + q_{jl}) + F_{kj}(q_{il} + q_{li}) - F_{lj}(q_{ki} + q_{ik}) - F_{ji}(\alpha_{kl} - \beta_{kl}) + F_{lk}(\alpha_{ij} - \beta_{ij}) = 0.$$

Transvecting the relation (5.7) with F^{il} we obtain

$$(n-2)(q_{kj} + q_{jk}) = 0$$

and consequently

$$(5.8)\quad q_{kj} + q_{jk} = 0,$$

i.e. q_{kj} is skew-symmetric.

Besides, from the relation (5.7) there holds

$$F_{lk}(\alpha_{ij} - \beta_{ij}) = F_{ji}(\alpha_{kl} - \beta_{kl})$$

and, after transvecting with F^{kl}, we obtain

$$(5.9)\quad \alpha_{ji} = \beta_{ji} + \tfrac{2}{n}(\nabla_t p^t + 2p_t p^t)F_{ji}.$$

Also, as q_{ji} is skew-symmetric, using (5.2) we can obtain

$$(5.10)\quad \begin{cases} \alpha_{ji} = -2q_{ji} - p_s p^s F_{ji} \\ \beta_{ji} = -2q_{ji} - \tfrac{2}{n}(\nabla_s p^s + 2p_s p^s)F_{ji}. \end{cases}$$

Suppose now that R_{ijkl} satisfies the first Bianchi identity. Then we obtain (by straightforward calculation):

$$R_{ijkl} + R_{iklj} + R_{iljk} = K_{ijkl} + K_{iklj}\ K_{iljk} - F_{li}(2q_{kj} + \alpha_{kj}) + F_{ki}(2q_{lj} + \alpha_{lj}) - F_{ji}(2q_{lk} + \alpha_{lk}) - F_{lk}(2q_{ji} + \beta_{ji}) + F_{lj}(2q_{ki} + \beta_{ki}) - F_{kj}(2q_{li} + \beta_{li}) = 0$$

what gives us

$$(5.11)\quad F_{li}(2q_{kj} + \alpha_{kj}) - F_{ki}(2q_{lj} + \alpha_{lj}) + F_{ji}(2q_{lk} + \alpha_{lk}) + F_{lk}(2q_{ji} + \beta_{ji}) - F_{lj}(2q_{ki} + \beta_{ki}) + F_{kj}(2q_{li} + \beta_{li}) = 0.$$

According to (5.10) one has

$$2q_{kj} + \alpha_{kj} = -p_s p^s F_{kj} \text{ and } 2q_{ji} + \beta_{ji} = -\tfrac{2}{n}(p_s{}^s + 2p_s p^s)F_{ji}.$$

Substituting this into (5.11) we obtain

$$-\tfrac{2}{n}(F_{li}F_{kj} - F_{ki}F_{lj} + F_{ji}F_{lk})(p_s{}^s + \tfrac{n+4}{2}p_s p^s) = 0.$$

If $F_{li}F_{kj} - F_{ki}F_{lj} + F_{ji}F_{lk} = 0$, then $(n-2)F_{kj} = 0$, what is senseless. Then we can conclude that

$$(5.12)\quad p_t = -\tfrac{n+4}{2}p_t p^t$$

$$(5.13)\quad \beta_{ji} = -2q_{ji} + p_t p^t F_{ji}.$$

We can calculate the Ricci tensor of the conformal connection on hyperbolic Kaehlerian space. If we transvect (5.6) with g^{il}, then we have

$$R_{jk} = K_{jk} - (n+4)p_{kj} - g_{kj}p_s^{\ s}. \tag{5.14}$$

Transvecting this expression again with g^{jk} we obtain

$$R = K - 2(n+2)p_s^{\ s}. \tag{5.15}$$

From (5.12), (5.14) and (5.2) we obtain

$$\text{(5.16)}\quad \begin{cases} \text{(i)} & p_{kj} = \frac{1}{n+4}(K_{jk} - R_{jk}) - \frac{1}{2(n+2)(n+4)}(K - R)g_{jk}\,, \\ \text{(ii)} & q_{kj} = \frac{1}{n+4}(S_{jk} - \bar{S}_{jk}) + \frac{1}{2(n+2)(n+4)}(K - R)F_{kj}\,, \\ \text{(iii)} & \alpha_{kj} = \frac{2}{n+4}(S_{kj} - \bar{S}_{kj})\,. \\ \text{(iv)} & \beta_{kj} = \frac{2}{n+4}(S_{kj} - \bar{S}_{kj}) - \frac{2}{(n+2)(n+4)}(K - R)F_{kj}\,. \end{cases}$$

S_{kj} and $\bar{S}_{kj}$ are tensors:

$$S_{kj} = K_{ka}F_j^{\ a}\,,\quad \bar{S}_{kj} = R_{ka}F_j^{\ a}. \tag{5.17}$$

Their difference, according to (5.16), is skew-symmetric.

Substituting (5.16) into (5.6) we obtain

$$\begin{aligned} &R_{ijkl} - \frac{1}{n+4}[g_{li}R_{kj} - g_{ki}R_{lj} + g_{kj}R_{li} - g_{lj}R_{ki} + F_{li}\bar{S}_{kj} - F_{ki}\bar{S}_{lj} + F_{kj}\bar{S}_{li} - \\ &F_{lj}\bar{S}_{ki} + 2\bar{S}_{ji}F_{kl} + 2\bar{S}_{kl}F_{ji} - \frac{R}{n+2}(g_{li}g_{kj} - g_{ki}g_{lj} + F_{lj}F_{ki} - F_{li}F_{kj} - 2F_{ji}F_{kl})] \\ &= K_{ijkl} - \frac{1}{n+4}[g_{li}K_{kj} - g_{ki}K_{lj} + g_{kj}K_{li} - g_{lj}K_{ki} + F_{li}S_{kj} - F_{ki}S_{lj} + F_{kj}S_{li} - \\ &F_{lj}S_{ki} + 2S_{ji}F_{kl} + 2S_{kl}F_{ji} - \frac{K}{n+2}(g_{li}g_{kj} - g_{ki}g_{lj} + F_{lj}F_{ki} - F_{li}F_{kj} - 2F_{ji}F_{kl})] \end{aligned} \tag{5.18}$$

The tensor on the right-hand side of (5.18) we will call the **Bochner curvature tensor of a hyperbolic Kaehlerian space**, for the sake of analogy with a result of Yano [7]. We will denote it by HB_{ijkl}. We have just proved

THEOREM 2. *If the curvature tensor of a conformal connection on a hyperbolic Kaehlerian space is invariant under change of the first and second pair of indices and if it satisfies the first Bianchi identity, then the tensor* HB_{ijkl} *is an invariant of all these connections.*

6. Algebraic properties of the tensor HB_{ijkl}

In order to investigate algebraic properties of the tensor HB_{ijkl} we have to do some investigation of the tensor S_{ij}.

As the structure tensor is parallel there holds

$$K^i_{\ tkl}F_j^{\ t} - K^t_{\ jkl}F_t^{\ i} = 0\,, \tag{6.1}$$

$$K^i_{\ jkl} - K^t_{\ akl}F_t^{\ i}F_j^{\ a} = 0. \tag{6.2}$$

After lowering the index i and taking into account properties of the curvature tensor in a (pseudo)Riemannian space, we have

(6.3) $$K_{ijkl} - K_{atkl}F_j^{\ a}F_i^{\ t} = 0$$

and for the Ricci tensor

(6.4) $$K_{jk} - K_{atkl}F_j^{\ a}F^{lt} = 0\ .$$

Transvecting (6.1) with g^{jk} we get

$$K_l^{\ t}F_t^{\ i} = K^i_{\ tkl}F^{kt} = \frac{1}{2}(K^i_{\ tkl} - K^i_{\ ktl})F^{kt}$$

or

$$K_l^{\ t}F_t^{\ i} = -\frac{1}{2}K^i_{\ ltk}F^{tk} \quad \text{and} \quad K_l^{\ t}F_{ti} = -\frac{1}{2}K_{iltk}F^{tk}.$$

Then one has

$$K_l^{\ t}F_{ti} + K_i^{\ t}F_{tl} = 0\ .$$

As the tensor F_{ti} is skew-symmetric, then

(6.5) $$K_{lt}F_i^{\ t} + K_{it}F_l^{\ t} = 0 \quad \text{or} \quad S_{li} + S_{il} = 0 \ ([7],\ [8]).$$

We have proved

THEOREM 3. *In a hyperbolic Kaehlerian space, the tensor* $S_{li} = K_{la}F_j^{\ a}$ *is skew-symmetric.*

From (5.16) it follows

LEMMA 1. *The tensor* $\bar{S}_{li} = R_{la}F_j^{\ a}$ *is skew-symmetric.*

Now it will be easy to prove

THEOREM 4. *The Bochner tensor of a hyperbolic Kaehlerian space has the following algebraic properties:*

(a) $HB_{ijkl} = -HB_{ijlk}$,

(b) $HB_{ijkl} = -HB_{jikl}$,

(c) $HB_{ijkl} = HB_{klij}$,

(d) $HB_{ijkl} + HB_{iklj} + HB_{iljk} = 0$,

(e) $HB^t_{\ tkl} = 0$,

(f) $HB^i_{\ tkl}F_j^{\ t} - HB^t_{\ jkl}F_t^{\ i} = 0.$

Proof. We are going to prove just this last equality; the all rest is very easy to prove. We have

$$HB^i_{\ tkl}F_j^{\ t} = K^i_{\ tkl}F_j^{\ t} - \frac{1}{n+4}[\delta_l^{\ i}S_{kj} - \delta_k^{\ i}S_{lj} + F_{jk}K_l^{\ i} - F_{jl}K_k^{\ i} + F_l^{\ i}K_{kj} - F_k^{\ i}K_{lj} - g_{kj}S_l^{\ i} + g_{lj}S_k^{\ i} - 2K^i_{\ j}F_{kl} - 2S_{kl}\delta^i_{\ j}\frac{K}{n+2}(\delta_l^{\ i}F_{jk} - \delta_k^{\ i}F_{jl} - g_{lj}F_k^{\ i} + F_l^{\ i}g_{kj} - 2\delta_j^{\ i}F_{kl})],$$

$$HB^t_{\ jkl}F_t = K^t_{\ jkl}F_t - \frac{1}{n+4}[F_l^{\ i}K_{kj} - F_k^{\ i}K_{lj} - g_{kj}S_l^{\ i} + g_{lj}S_k^{\ i} + \delta_l^{\ i}S_{kj} - \delta_k^{\ i}S_{lj} - F_{kj}K_l^{\ i} + F_{lj}K_k^{\ i} - 2K_j^{\ i}F_{kl} - 2S_{kl}\delta_j^{\ i}\frac{K}{n+2}(F_l^{\ i}g_{kj} - F_k^{\ i}g_{lj} + F_{lj}\delta_k^{\ i} - F_{kj}\delta_l^{\ i} - 2\delta_j^{\ i}F_{kl})].$$

As the tensors S_{kj} and F_{kj} are skew-symmetric and the tensor $K^i_{\ jkl}$ satisfies the equality analogous to (f), then (f) holds.

REFERENCES

[1] G. GANCHEV & A. BORISOV, *Isotropic sections and curvature properties of hyperbolic Kaehlerian manifolds,* Publ. Inst. Math. (Belgrade) 38 (52)(1985), 183-192.

[2] H.A. HAYDEN, *Subspaces of a space with torsion,* Proc. London Math. Soc. 34(1934), 27-50.

[3] P.K. RASHEVSKII, *Skaljarnoe pole v rassloennom prostranstve,* Trudy seminara po vektornomu i tenzornomu analizu 6(1948), 225-248.

[4] V. PETROVIĆ, *Konformne transformacije prostora sa kompleksnom i produkt strukturom,* Magistarska teza, PMF, University of Belgrade, 1980.

[5] M. PRVANOVIĆ, *Holomorphically projective transformations in a locally product space,* Math. Balkanica 1(1971), 195-213.

[6] S. TACHIBANA, *Some theorems on locally product Riemannian spaces,* Ochanomizu University, 1959.

[7] K. YANO, *On complex conformal connections,* Kodai Math. Sem. Rep. 26 (1975), 137-151.

[8] K. YANO, *Differential geometry of complex and almost complex spases,* Pergamon Press, New York, 1965.

Nevena Pušić

O INVARIJANTNOM TENZORU KONFORMNE TRANSFORMACIJE

HIPERBOLIČNE KELEROVE MNOGOSTRUKOSTI

U radu je konstruisana konformna koneksija za hiperbolični Kelerov prostor. Pod odredjenim uslovima pronadjena je invarijanta svih takvih koneksija. Za invarijantni tenzor su utvrdjene algebarske osobine.

Prirodno-matematički fakultet
Univerzitet u Novom Sadu
21000 Novi Sad, Yugoslavia