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TWO THEOREMS ON RELATIVE CARDINAL INVARIANTS IN C_p-THEORY

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Abstract. Two well-known theorems of C_p-theory on the relationship between topological properties of X and $C_p(X)$ are extended to the case of relative topological invariants.

All spaces under consideration are assumed to be Tychonoff topological spaces. By $C_p(X)$ we denote the space of all continuous real-valued functions on a space X in the topology of pointwise convergence. N^+ is the set of all positive integers. The notations and terminology are the same as in [1].

Let $f:X \to Y$ be a continuous mapping from a space X into a space Y. The **tightness t(f)** of f is defined to be the smallest infinite cardinal number τ such that whenever $A \subset X$ and $x \in \bar{A}$ there is $B \subset A$ satisfying conditions: $|B| \le \tau$ and $f(x) \in \overline{f(B)}$.

PROPOSITION 1. *For every continuous mapping $f:X \to Y$ we have: $t(f) \le t(X)$ and $t(f) \le t(Y)$.*

The **(relative) Lindelöf number** $\ell(Z,X)$ of a subspace $Z \subset X$ is the smallest infinite cardinal number τ such that for every open covering γ of X one can find a subfamily $\mu \subset \gamma$ with the properties: $Z \subset \cup\mu$ and $|\mu| \le \tau$.

If $f:X \to Y$ is a continuous mapping, then the dual mapping $f^{\#}$ from $C_p(Y)$ to $C_p(X)$ is defined by the rule: if $g \in C_p(Y)$, then $(f^{\#}(g))(x) = g(f(x))$ for all $x \in X$.

THEOREM 1. *For each continuous mapping $f:X \to Y$ the following inequality holds: $t(f) \le \ell(f^{\#}(C_p(Y), C_p(X))$.*

Proof. Let $\tau = \ell(C_p(Y), C_p(X))$, $A \subset X$ and $x \in A$. The set

$$\Phi = \{g \in C_p(X) : g(x_0) = 1\}$$

AMS Subject Classification (1980): 54C35, 54A25 is closed in $C_p(X)$. For every $x \in X$ we put:

$$V_x = \{h \in C_p(X): h(x) > 0\}.$$

The set V_x is open in $C_p(X)$.

Take any $g \in \Phi$. From $x_0 \in A$ and the continuity of g it follows that there exists a point $x \in A$ such that $g(x) > 0$. Then $g \in V_x$ and this implies $\Phi \subset \cup\{V_x : x \in A\}$.

Thus the family $\{C_p(X) \setminus \Phi\} \cup \{V_x : x \in A\}$ is an open covering of the space $C_p(X)$. By our assumption we can find a subset $B \subset A$ such that

$$|B| \le \tau \text{ and } f^{\#}(C_p(Y)) \cap \Phi \subset \cup\{V_x : x \in B\}.$$

Let us show that $f(x_0) \in \overline{f(B)}$. Assume to the contrary $f(x_0) \notin \overline{f(B)}$. Then there exists a neighbourhood V of $f(x_0)$ in Y with $f^{-1}(V) \cap B = \emptyset$. We fix a function $\psi \in C_p(Y)$ satisfying the conditions: $\psi(f(x_0)) = 1$ and $\psi(Y \setminus V) \subset \{0\}$. Let us put $\varphi = f^{\#}(\psi) \in f^{\#}(C_p(Y))$. From $\varphi(x_0) = \psi(f(x_0)) = 1$ it follows that $\varphi \in \Phi \cap f^{\#}(C_p(Y)) \subset \cup\{V_x : x \in B\}$. Hence $\varphi(x) > 0$ for some $x \in B$. On the other hand, we have:

$$B \subset X \setminus f^{-1}(V) \text{ and } \varphi(X \setminus f^{-1}(V)) = \psi(f(X \setminus f^{-1}(V)) \subset \psi(Y \setminus V) \subset \{0\}$$

which is a contradiction.

COROLLARY 1. (D. Pavlovskij, see [1]) *For every space X we have*

$$t(X) \le \ell(C_p(X)).$$

In what follows Y is a subspace of a space X and $\pi: C_p(X) \to C_p(Y)$ is the restriction mapping [1], i.e. $\pi(f) = f|Y$ for every $f \in C_p(X)$.

The following result was obtained by A.V. Arhangel'skii and E.G. Pytkeev (see [1]): for every space X, $\sup\{\ell(X^n): n \in N^+\} = t(C_p(X))$. The next theorem extends this assertion to the case of relative invariants which, incidentally, testifies that the definitions of relative cardinal invariants involved in the theorem were chosen correctly.

THEOREM 2. *For any subspace Y of a space X we have:*

$$\sup\{\ell(Y^n, X^n): n \in N^+\} = t(\pi).$$

Proof. The proof of the inequality $t(\pi) \le \sup\{\ell(Y^n, X^n): n \in N^+\}$ requires only trivial changes in the proof of the corresponding inequality in the Arhangel'skii-Pytkeev theorem given in [1]. So we pase to the proof of the remaining inequality.

Let $t(\pi) = \tau$ and $n \in N^+$. We have to show that $\ell(Y^n, X^n) \le \tau$. Let γ be any open covering of the space X^n. We are going to choose a subfamily of γ of cardinality $\le\tau$ covering Y^n.

A finite system μ of open sets in X is called **γ-small** if for every $V_1, \dots, V_n \in \mu$ there exists $G \in \gamma$ such that $V_1 \times \dots \times V_n \subset G$. Let us denote by $\mathcal{E}$ the family of all finite γ-small systems of open sets in X. For $\mu \in \mathcal{E}$ we put

$$A_\mu = \{f \in C_p(X): f(X \setminus \cup\mu) = \{0\}\}.$$

Let $A = \cup\{A_\mu : \mu \in \mathcal{E}\}$. Then $\bar{A} = C_p(X)$ - this is proved by an argument in the proof of Arhangel'skii-Pytkeev theorem in [1]. Consider the function $f_1 \in C_p(X)$ defined by $f_1(X) = \{1\}$. As $f_1 \in \bar{A}$ and $t(\pi) = \tau$, there exists $B \subset A$ such that $|B| \le \tau$ and $\pi(f_1) \in \overline{\pi(B)}$. We can also fix a family $\mathcal{E}_0 \subset \mathcal{E}$ satisfying the conditions: $|\mathcal{E}_0| \le \tau$ and $B \subset \cup\{A_\mu : \mu \in \mathcal{E}_0\}$.

Let $\mu \in \mathcal{E}_0$. For every $\xi = (V_1, \dots, V_n) \in \mu^n$ fix $G_\xi \in \gamma$ such that $V_1 \times \dots \times V_n \subset G_\xi$ and put $\gamma_\mu = \{G_\xi : \xi \in \mu^n\}$. We also consider the family $\gamma^* = \cup\{\gamma_\mu : \mu \in \mathcal{E}_0\}$. Every family γ_μ is finite, hence $|\gamma^*| \le \tau$.

Let us show that $Y^n \subset \cup\gamma^*$ which will complete the proof. Take any $(y_1, \dots, y_n) \in Y^n$. The set $U = \{f \in C_p(Y): f(y_i) > 0 \text{ for all } i \le n\}$ is open in $C_p(Y)$ and $\pi(f_1)|Y \equiv 1 \in U$. From $\pi(f_1) \in \overline{\pi(B)}$ it follows that $\pi(B) \cap U \neq \emptyset$. Fix any $g \in \pi(B) \cap U$ and any $f^* \in \pi^{-1}(g) \cap B$. As $g \in U$ and $f^*|Y = g$ we have $f^*(y_i) = g(y_i) > 0$ for all $i \le n$. From $f^* \in B \subset A$ it follows that $f^* \in A_{\mu_0} \in \mathcal{E}_0$. Then $f^*(X \setminus \cup\mu_0) = \{0\}$ which implies $y_i \in \cup\mu_0$ for every $i \le n$. For each $i = 1, \dots, n$ fix $V_i \in \mu_0$ such that $y_i \in V_i$. Then $(y_1, \dots, y_n) \in V_1 \times \dots \times V_n \subset G_\xi \in \gamma_{\mu_0} \subset \gamma^*$. The proof is complete.

REFERENCES

[1] A.V. ARHANGEL'SKII, *Topological spaces of functions,* MGU, Moscow, 1989 (In Russian).

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DVE TEOREME O RELATIVNIM KARDINALNIM INVARIJANTAMA U C_p-TEORIJI

Dve poznate teoreme o vezi izmedju topoloških osobina prostora X i $C_p(X)$ prenose se na slučaj relativnih topoloških invarijanti. Specijalno pokazan je analogon teoreme Arhangel'skii-Pytkeeva o tesnomi prostora $C_p(X)$.

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