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CONTIGUITIES AND PROXIMITIES ON MAPPINGS. T_1-COMPACTIFICATIONS OF MAPPINGS

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Abstract. Some notions and assertions concerning topological spaces are extended to continuous mappings. In particular, we show that there is a natural one-to-one correspondence between the family of all Ivanov proximities on a mapping f and the set of all compactifications of f.

1. Let X and Y be topological spaces, $\mathcal{T}$ the topology on Y and $f:X \to Y$ a continuous mapping such that $f \in T_0$ [1] and $f \in R_0$ (i.e. for any closed subset F of X and any $x \in X \setminus F$ there exists a set $O \in \mathcal{T}$ such that $f(x) \in O$ and $[\{x\}]_X \cap F \cap f^{-1}(O) = \emptyset$).

REMARK 1. If f is a regular (completely regular, normal T_0-) mapping [1], then, obviously, $f \in R_0$.

REMARK 2. For any closed (in particular, perfect) T_1-mapping [1] f one has $f \in R_0$.

THEOREM 1. *Let $f \in T_0$. Then the following conditions are equivalent:*

1. *There exists a T_1-compactification [1] of f ;*
2. *$f \in R_0$.*

2. Let $f:X \to Y$ be a mapping such that $f \in T_0$ and $f \in R_0$. For any $O \in \mathcal{T}$ let $Fin_2^\infty Or(f,O)$ denote the family of all ordered systems γ, $|\gamma|>1$, of closed subsets of $f^{-1}(O)$.

DEFINITION 1. *For any $O \in \mathcal{T}$ let $\sigma_O: Fin_2^\infty Or(f,O) \to \{0,1\}$ be a mapping satisfying axioms (C1)-(C5) from [2] ($\sigma_O(\gamma) = 0$ iff γ is a contiguity system [2]) and the following axioms (CM1)-(CM3):*

CM1. *If $\gamma = \{F_1,\dots,F_n\} \in Fin_2^\infty Or(f,O)$, $\sigma_O(\gamma) = 1$, $O' \in \mathcal{T}$, $O' \subset O$, then $\sigma(F_1 \cap f^{-1}(O'),\dots,F_n \cap f^{-1}(O')) = 1$;*

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CM2. *If* $\gamma = \{F_1,\dots,F_n\} \in Fin_2^{\infty}Or(f,O)$ *and for any* $y \in O$ *there is a neighbourhood* O' *of* y, $O' \subset O$, *such that* $\sigma_{O'}(F_1 \cap f^{-1}(O'),\dots,F_n \cap f^{-1}(O'))$ $= 1$, *then* $\sigma_O(\gamma) = 1$;

CM3. *If* $F \subset X$ *is closed, then* $x \in F$ *iff for any* $O \in \mathcal{T}$ *with* $f(x) \in O$ *it holds* $\sigma_O([\{x\}]_X \cap f^{-1}(O), F \cap f^{-1}(O)) = 0$.

The system $\sigma = \{\sigma_O : O \in \mathcal{T}\}$ *is said to be a* **contiguity on f**.

EXAMPLE 1. Let for any $O \in \mathcal{T}$ and any $\gamma = \{F_1,\dots,F_n\} \in Fin_2^{\infty}Or(f,O)$ the following formula holds: $\sigma_O^i(\gamma) = 1$ iff $\cap\{F_i : 1 \le i \le n\} = \varnothing$. Then the system $\sigma^i = \{\sigma_O^i : O \in \mathcal{T}\}$ is a contiguity on mapping f - the contiguity "on the intersection" (compare with [2]).

Let $cl(f,O)$ be the family of all closed subsets of $f^{-1}(O)$, $O \in \mathcal{T}$, and $cl(f,y) = \cup\{cl(f,O) : O \in \mathcal{T},\ y \in O\}$, $cl(f) = \cup\{cl(f,y) : y \in Y\}$. Let σ be a contiguity on f.

DEFINITION 2. *Let* $y \in Y$, $\mu \subset cl(f,y)$. *The pair* (μ,y) *is said to be a* **contiguity system** *(or* **σ-system***)* **of f** *(over y) if the following conditions are satisfied:*

a) For any neighbourhood O *of* y , $f^{-1}(O) \subset \mu$;

b) If $\gamma = \{F_1,\dots,F_n\} \subset \mu$, *then there exists a neighbourhood* O *of* y *such that for any neighbourhood* O' *of* y *with* $O' \subset O$ *the following holds:* $\sigma_{O'}(F_1 \cap f^{-1}(O'),\dots,F_n \cap f^{-1}(O')) = 0$.

PROPOSITION 1. A. *If for any* σ*-system* (μ,y) *of* f, $(\cap\mu) \cap f^{-1}(y) \ne \varnothing$ *holds, then f is a perfect mapping.*

B. *The following conditions are equivalent:*

b1) f is a perfect mapping ;

b2) $(\cap\mu) \cap f^{-1}(y) \ne \varnothing$ *for* $\sigma = \sigma^i$ *and any* σ*-system* (μ,y) *of* f.

DEFINITION 3. *A* σ*-system* (μ,y) *of f is said to be a* **σ-end of f** *(or maximal contiguity system of f) (over* $y \in Y$*) if for any* σ*-system* (ν,y) *of f with* $\nu \supset \mu$ *we have* $\nu = \mu$.

Let $\sigma_f(X)$ be the family of all σ-ends of f and let $\sigma f : \sigma_f(X) \to Y$ be a mapping such that $\sigma f(\mu,y) = y$ for any $(\mu,y) \in \sigma_f(X)$. The mapping σf is said to be **put up by means of the contiguity** σ on f.

For any $F \subset cl(f)$ let $\Phi(F) = \{(\mu,y) \in \sigma_f(X) : F \in \mu\}$ (see also [2]).

LEMMA 1. *The family* $B = \{(\sigma_f(X) \setminus \Phi(F)) \cap (\sigma f)^{-1}(O) : O \in \mathcal{T},\ F = \bar{F} \subset X\}$ *is a base of some topology* $\mathcal{T}(\sigma)$ *on* $\sigma_f(X)$.

DEFINITION 4. *An extension* $g : Z \to Y$ *of a mapping* $f : X \to Y$ *is said to be a* **correct extension** *of f if* $B(g) = \{Z \setminus [A]_X : A \subset X\}$ *is a base [1] of g.*

PROPOSITION 2. *Any T_2-compactification* [1] *g of a mapping f is a correct extension of f.*

THEOREM 2. *The mapping $\sigma f:(\sigma_f(X),\mathcal{T}(\sigma)) \to Y$ is a correct T_1-compactification of $f:X \to Y$.*

THEOREM 3. *There is a natural one-to-one correspondence between the family of all contiguities on a continuous(T_0- and R_0-) mapping f and the family of all correct T_1-compactifications of f. To every correct T_1-compactification $g:Z \to Y$ we assign the contiguity $\sigma = \{\sigma_O:O \in \mathcal{T}\}$ on f for which the following condition holds: if $O\in\mathcal{T}$, $\{F_1,\dots,F_n\}\in Fin_2^{\infty}Or(f,O)$ then $\sigma_O(F_1,\dots,F_n) = 1$ iff $(\cap\{[F_i]_Z:1\le i\le n\}) \cap g^{-1}(O) = \emptyset$.*

If $|Y| = 1$, $f \in T_1$, then Theorem 3 coincides with Theorem 6 in [2].

3. Let $Fin_2Or(f,O)$ be the family of all systems $\gamma \in Fin_2^{\infty}Or(f,O)$ such that $|\gamma| = 2$.

DEFINITION 5. *For any $O \in \mathcal{T}$ let $\delta_O:Fin_2Or(f,O) \to \{0,1\}$ be a mapping such that for any $\gamma \in Fin_2Or(f,O)$ one and only one of equalities $\delta_O(\gamma)=0$ or $\delta_O(\gamma)=1$ holds and the following axioms are satisfied:*

IP1. $\delta_O(F,H) = \delta_O(H,F)$;

IP2. $\delta_O(F,H\cup\Phi) = 1$ iff $\delta_O(F,H) = \delta_O(F,\Phi) = 1$;

IP3. $\delta_O(f^{-1}(O),\emptyset) = 1$;

IP4. If $\delta_O(F,H) = 1$, $O'\in\mathcal{T}$, $O'\subset O$, then $\delta_{O'}(F\cap f^{-1}(O'),H\cap f^{-1}(O')) = 1$;

IP5. If $\{F,H\} \in Fin_2Or(f,O)$ and for any $y \in O$ there is a neighbourhood O' of y, $O' \subset O$, such that $\delta_{O'}(F\cap f^{-1}(O'),H\cap f^{-1}(O')) = 1$, then $\delta_O(F,H) = 1$;

IP6. If F is closed in X, then $x \in F$ iff for any $O \in \mathcal{T}$ containing $f(x)$ the following holds: $\delta_O([\{x\}]_X\cap f^{-1}(O),F\cap f^{-1}(O)) = 0$.

The system $\delta = \{\delta_O:O \in \mathcal{T}\}$ is said to be an **Ivanov proximity on f.**

REMARK 3. Let $\sigma = \{\sigma_O:O \in \mathcal{T}\}$ be a contiguity on f and let for any $O \in \mathcal{T}$ and any $\{F,H\} \in Fin_2Or(f,O)$ the condition

$$\delta_O(F,H) = \sigma_O(F,H) \tag{1}$$

holds. Then the system $\delta = \{\delta_O:O \in \mathcal{T}\}$ is an Ivanov proximity on f.

Let δ be an Ivanov proximity on f. Let $P(\delta)$ be the family of all contiguities on f for which condition (1) holds.

THEOREM 4. *There are a maximal contiguity* [8] *$\sigma^{max} \in P(\delta)$ and a minimal* [8] *contiguity $\sigma^{min} \in P(\delta)$ on f.*

Recall that two compactifications $g:Z \to Y$ and $g':Z' \to Y$ of a mapping $f:X \to Y$ are said to be **identical** (in another way $g = g'$) if there exists a homeomorphism $h:Z \to Z'$ such that $h(X) \equiv X$ and $g = g'h$.

DEFINITION 6. *A compactification $g:Z \to Y$ of a mapping $f:X \to Y$ is said to be* **put up by means of a contiguity** *σ on f if $g = \sigma f$.*

DEFINITION 7. *A (correct) T_1-compactification $g:Z \to Y$ of a mapping $f:X \to Y$ is said to be a* **major compactification** *(see [2]) if it is put up by means of a contiguity $\sigma^{min} \in P(\delta)$ (for some Ivanov proximity δ) on f.*

PROPOSITION 3. *Any T_2-compactification (in particular, any completely regular compactification) g of a mapping f is a major compactification of f.*

PROPOSITION 4. (1) *Let ωf be the Wallman compactification of f (see [5], [9]). Then $\omega f = \sigma^i f$.*

(2) *Let $\delta^i = \{\delta^i_O : O \in \mathcal{T}\}$ be an Ivanov proximity on f such that for any $O \in \mathcal{T}$ and any $\{F,H\} \in Fin_2 Or(f,O)$ one has $\delta^i_O(F,H) = \sigma^i_O(F,H)$. Then we have: $\omega f = \sigma^i f = \sigma^{max}(\delta^i)$.*

THEOREM 5. *There is a natural one-to-one correspondence between the family of all Ivanov proximities on a (T_0- and R_0-) mapping f and the family of all major (correct T_1-) compactifications of f.*

REMARK. Theorem 5 is an obvious corollary of Theorems 3 and 4.

LEMMA 2. *Let $g:Z \to Y$ be a major (T_1-) compactification of a mapping $f:X \to Y$ and let δ be an Ivanov proximity on f such that $g = \sigma^{min} f$, where $\sigma^{min} \in P(\delta)$. The following conditions are equivalent:*

1) $g \in T_2$;

2) *There exists a NP-proximity [3] $\hat{\delta} = \{\hat{\delta}_O : O \in \mathcal{T}\}$ on f such that $\hat{\delta}_O(F,H) = \delta_O(F,H)$ for any $O \in \mathcal{T}$ and any $\{F,H\} \in Fin_2 Or(f,O)$.*

Now the theorem of Norin-Pasynkov in [3] follows from Theorem 5, Proposition 3 and Lemma 2.

Recall that for $|Y| = 1$ the theorem of Norin-Pasynkov [3] coincides with a theorem of Smirnov (see [4;T.8.4.13]).

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REFERENCES

[1] B.A. PASYNKOV, *Extensions of certain notions and assertions concerning spaces to mappings*, In: Mappings and functors, MGU, Moscow, 1984, 72-102 (In Russian).

[2] V.M. IVANOVA & A.A. IVANOV, *Contiguity spaces and bicompact extensions*, Izvestya AN SSSR, Ser. Matem. 23(1959), 613-634 (In Russian).

[3] V.P. NORIN & B.A. PASYNKOV, *Hausdorff compactifications of mappings and proximities*, Abstracts Nauchno-Methodich. Confer. Kirov 1990.

[4] R. ENGELKING, *General Topology*, M., Moscow, 1986 (In Russian).

[5] I.M. JAMES, *Fibrewise Topology*, Cambrige Tracts in Math. 91, Cambr. Univ. Press, 1989.

[6] V.P. NORIN, *On proximities for mappings*, Vestnik Mosc. Univ., Ser. Mat. 4(1982), 33-36 (In Russian).

[7] B.A. PASYNKOV, *On proximities on mappings*, Comp. Rend. Acad. Bulg. Sci. 42(1989), 5-6 (In Russian).

[8] Y.P. PERSHIN, *On contiguities and proximities for mappings*, In: Sbor. Nauch. Trud. Mosc. Priborostr. Inst. (1990) (In Russian)(to appear).

[9] N.I. ILYINA, *A construction of the extension ωf of a continuous mapping f by means of ultrafilters*, In: Sbornik Nauch. Trud. Mosc. St. Pedag. Inst. (1989), 115-124.

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SUSEDNOSTI I BLISKOSTI NA PRESLIKAVANJIMA.

T_1-KOMPAKTIFIKACIJE PRESLIKAVANJA

Neki pojmovi i tvrdjenja iz teorije topoloških prostora prenose se na neprekidna preslikavanja medju topološkim prostorima. Specijalno, pokazano je da postoji prirodna uzajamno jednoznačna korespondencija izmedju skupa svih bliskosti Ivanova na preslikavanju f i jednog skupa kompaktifikacija tog f.

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