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POINT GROUPS OF SIMPLE AND MULTIPLE ANTIHOMOLOGY $H_{30}{}^{l}$

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By the use of the antisymmetric characteristic method (*AC*-method), point groups of simple and multiple antihomology $H_{30}{}^{l}$, are derived.

1. INTRODUCTION

Crystallographic point groups of simple and multiple antisymmetry $G_{30}{}^{l}$ are discussed in the works [1,2,3,4]. By the use of the *AC*-method, the complete derivation of simple and multiple antisymmetry point groups $G_{30}{}^{l}$, without crystallographic restriction, is given [5].

Crystallographic homology groups introduced by V.I.Miheev [6] are discussed, in detail, by P.A.Zabolotnyj and his colleagues and generalized to the simple and multiple antihomology groups [7,8,9,10,11]. In this work, by the use of the *AC*-method, all the point groups of simple and multiple antihomology $H_{30}{}^{l}$, without crystallographic restriction, are derived.

2. HOMOLOGY POINT GROUPS H_{30}

Every point group of homology $H \in H_{30}$ can be derived from the

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corresponding point group of symmetry $G \in G_{30}$ by some affine transformation, and $H \cong G$. All homology groups derived from the same symmetry group G forme the homology class of G. Since all the point groups of symmetry G_{30} can be divided into 10 symmetry classes [1,5,12], the same principle will be used for classifying the point groups of homology H_{30}. Every point group of homology is uniquely defined and denoted by a system of its elements [6,7,8,9,10,11], where a reflection is denoted by R, a slanting reflection by r, an inversion by C, a (circular) rotation of order n by S_n, a (circular) slanting rotation by s_n, an elliptic rotation by E_n, an elliptic slanting rotation by e_n, and the corresponding rotatory reflections respectively by $S_{\bar{n}}$, $s_{\bar{n}}$, $E_{\bar{n}}$, $e_{\bar{n}}$.

3. *AC*-METHOD

After some adjustment, the *AC*-method [5,13] can be used for a derivation of simple and multiple antihomology groups:

Definition 1 Let all products of the generators of a homology group H be formed, and then separate the subsets of transformations that are equivalent with respect to homology. After all powers of generators are reduced by (mod 2), the resulting system is called the antihomology characteristic $AC(H)$ of the group H.

A formation of $AC(H)$ and reduced $AC(H)$ [5,13] will be illustrated by example of two already discussed crystallographic point groups of homology [7,8,9,10,11]. Let us consider first the homology group $H_1=(E_4, R_1, r_2, R_3, r_4)$. This group can be defined by a minimal generator set $\{E_4, R_1\}$ as

$H_1=(E_4, R_1, r_2, R_3, r_4)=(E_4, R_1, E_4R_1, E_4{}^2R_1, E_4{}^3R_1)$.

In line with Definition 1,

$AC(H_1)=\{E_4\}\{R_1, R_3\}\{r_2, r_4\}=\{E_4\}\{R_1, E_4{}^2 R_1\}\{E_4 R_1, E_4{}^3 R_1\}=$

$=\{E_4\}\{R_1, R_1\}\{E_4 R_1, E_4 R_1\}=\{E_4\}\{R_1\}\{E_4 R_1\}=\{E_4\}\{R_1\}.$

For the group

$H_2=(E_6, R_1, r_2, r_3, R_4, r_5, r_6)=(E_6, R_1, E_6 R_1, E_6{}^2 R_1, E_6{}^3 R_1, E_6{}^4 R_1, E_6{}^5 R_1),$

$AC(H_2)=\{E_6\}\{R_1, R_4\}\{r_2, r_3, r_5, r_6\}=$

$=\{E_6\}\{R_1, E_6{}^3 R_1\}\{E_6 R_1, E_6{}^2 R_1, E_6{}^4 R_1, E_6{}^5 R_1\}=$

$=\{E_6\}\{R_1, E_6 R_1\}\{E_6 R_1, R_1, R_1, E_6 R_1\}=\{E_6\}\{R_1, E_6 R_1\}\{R_1, E_6 R_1\}=$

$=\{E_6\}\{R_1, E_6 R_1\}=\{R_1, E_6 R_1\}.$

Theorem 1 Groups that posses isomorphic AC generate the same number of simple and multiple antihomology groups of the M^m-type, with the same structure.

Hence we can simply conclude that

$AC(H_1)=\{E_4\}\{R_1\}\cong\{A\}\{B\}$, 2.1, $N_1(H_1)=3$, $N_2(H_1)=6$;

$AC(H_2)=\{R_1, E_6 R_1\}\cong\{A,B\}$, 2.2, $N_1(H_2)=2$, $N_2(H_2)=3$ [13].

4. DERIVATION OF SIMPLE AND MULTIPLE ANTIHOMOLOGY POINT GROUPS $H_{30}{}^1$

Crystallographic point groups of homology H_{30} (n=1,2,3,4,6) are well known [7,8,9,10,11]. For the derivation of all point groups of homology (without crystallographic restriction), distributed into 7 infinite classes, two of them, **n** and **nm**, i.e. homology groups of the category H_{20}, are sufficient. All the remaining classes can be obtained by the use of antihomology which makes possible the dimensional transition $H_{20}{}^1 \longrightarrow H_{320}$ [4]. The category H_{30} consists of homology groups H_{320}, treated as H_{30}, and polyhedral homology groups derived from the symmetry groups $[3,q]$ and $[3,q]^+$ (q=3,4,5) [5,12].

For the every class is given a list of the corresponding homology groups H, comprising $AC(H)$ and its AC-isomorphism class number [13]. The existential conditions for simple and multiple antihomology groups of the M^m-type are the same as for their generating symmetry groups [5].

1) n

$n=1$	(S_1);	
$n=2$	(S_2)	AC: $\{S_2\}$, 1.1;
	(e_2)	AC: $\{e_2\}$, 1.1;
$n\geq3$	(S_n)	AC: $\{S_n\}$, 1.1, $n=2k$;
	(E_n)	AC: $\{E_n\}$, 1.1, $n=2k$;
	(s_n)	AC: $\{s_n\}$, 1.1, $n=2k$;
	(e_n)	AC: $\{e_n\}$, 1.1, $n=2k$.

By the use of antihomology, by interpreting the antiidentity transformation as a reflection or slanting reflection in the invariant plane of the homology rosette n, homology groups of the classes $\tilde{n}$ ($\underline{n}\longrightarrow\tilde{n}$) and n:m (nx$\underline{1}\longrightarrow$n:m), are obtained:

2) $\tilde{n}$

$n=2$	(C)	AC: $\{C\}$, 1.1;
$n=2k$ $(k\geq2)$	$(S_{\tilde{n}})$	AC: $\{S_{\tilde{n}}\}$, 1.1;
	$(E_{\tilde{n}})$	AC: $\{E_{\tilde{n}}\}$, 1.1;
	$(s_{\tilde{n}})$	AC: $\{s_{\tilde{n}}\}$, 1.1;
	$(e_{\tilde{n}})$	AC: $\{e_{\tilde{n}}\}$, 1.1.

3) n:m

$n=2$	$(S_2)\times(R)$	AC: $\{S_2\}\{R\}$, 2.1;
	$(e_2)\times(R)$	AC: $\{e_2\}\{R\}$, 2.1;
$n\geq3$	$(S_n)\times(R)$	AC: $\{R\}$, 1.1, $n=2k+1$, $\{S_n\}\{R\}$, 2.1, $n=2k$;
	$(E_n)\times(R)$	AC: $\{R\}$, 1.1, $n=2k+1$, $\{E_n\}\{R\}$, 2.1, $n=2k$;
	$(s_n)\times(R)$	AC: $\{R\}$, 1.1, $n=2k+1$, $\{s_n\}\{R\}$, 2.1, $n=2k$;
	$(e_n)\times(R)$	AC: $\{R\}$, 1.1, $n=2k+1$, $\{e_n\}\{R\}$, 2.1, $n=2k$.

If H is a homology group derived from G, and if $AC(G)$ is trivial [5,13], then, certainly, $AC(H)\cong AC(G)$ (e.g. homology groups of the classes 1,2,3).

4) nm

$n=1$	(R)	AC: $\{R\}$, 1.1;
	(r)	AC: $\{r\}$, 1.1;
$n=2$	(S_2, R_1, R_2)	AC: $\{R_1, S_2 R_1\}$, 2.2;
	(S_2, r_1, r_2)	AC: $\{r_1, S_2 r_1\}$, 2.2;
	(e_2, r_1, r_2)	AC: $\{r_1, e_2 r_1\}$, 2.2.

At every $n \geq 3$ there are 6 "standard" groups:

$(S_n, R_1, \dots, R_n)$	AC: $\{R_1\}$, 1.1, $n=2k+1$, $\{R_1, S_n R_1\}$, 2.2, $n=2k$;
$(s_n, r_1, \dots, r_n)$	AC: $\{r_1\}$, 1.1, $n=2k+1$, $\{r_1, s_n r_1\}$, 2.2, $n=2k$;
$(s_n, R_1, r_2, \dots, r_n)$	AC: $\{R_1\}$, 1.1, $n=2k+1$, $\{R_1\}\{s_n\}$, 2.1, $n=2k$;
$(E_n, r_1, \dots, r_n)$	AC: $\{r_1\}$, 1.1, $n=2k+1$, $\{r_1, E_n r_1\}$, 2.2, $n=2k$;
$(e_n, r_1, \dots, r_n)$	AC: $\{r_1\}$, 1.1, $n=2k+1$, $\{r_1, e_n r_1\}$, 2.2, $n=2k$;
$(e_n, R_1, r_2, \dots, r_n)$	AC: $\{R_1\}$, 1.1, $n=2k+1$, $\{R_1\}\{e_n\}$, 2.1, $n=2k$.

At $n=2k+1$ there is one more homology group:

$(E_n, R_1, r_2, \dots, r_n)$ AC: $\{R_1\}$, 1.1.

At n- an even natural number ($n \geq 4$) there are certain additional homology groups deserving to be considered apart. If n is an even number given by decomposition:

$n=2\exp a p_1 \exp a_1 \dots p_k \exp a_k$,

where $p_1, \dots, p_k$ are distinct prime numbers ($p_i \geq 3$, $i=1, \dots, k$), then there are also h homology groups. By h is denoted the number of nontrivial factors d of n, such that n/d is an even number. For every such d there is the homology group:

$H=(E_n, R_1, r_2, \dots, r_d, R_{d+1}, r_{d+2}, \dots, r_{2d}, \dots, R_{n-d}, r_{n-d+1}, \dots, r_n)$.

<u>Theorem 2</u> $AC(H)=\{R_1, E_n R_1\}$, 2.2, at d- an odd natural number, and

$AC(H)=\{R_1\}\{E_n\}$, 2.1, at d- an even natural number.

Proof:

$H=(E_n, R_1, r_2, \dots, r_d, R_{d+1}, r_{d+2}, \dots, r_{2d}, \dots, R_{n-d}, r_{n-d+1}, \dots, r_n)=$

$=(E_n, R_1, E_n R_1, \dots, E_n^{d-1} R_1, E_n^{d} R_1, E_n^{d+1} R_1, \dots, E_n^{2d-1} R_1, \dots,$

$E_n^{n-d-1} R_1, E_n^{n-d} R_1, \ldots, E_n^{n-1} R_1)$

$AC(H)=\{E_n\}\{R_1, R_{d+1}, \ldots, R_{n-d}\}\{r_2, \ldots, r_d, r_{d+2}, \ldots, r_{2d}, \ldots,$

$r_{n-d-1}, r_{n-d+1}, \ldots, r_n\}=$

$=\{E_n\}\{R_1, E_n^d R_1, \ldots, E_n^{n-d-1} R_1\}\{E_n R_1, \ldots, E_n^{d-1} R_1, E_n^{d+1} R_1,$

$\ldots, E_n^{2d-1} R_1, \ldots, E_n^{n-d-2} R_1, E_n^{n-d}, \ldots, E_n^{n-1} R_1\}.$

At d– an odd natural number,

$AC(H)=\{E_n\}\{R_1, E_n R_1\}\{R_1, E_n R_1\}=\{E_n\}\{R_1, E_n R_1\}=\{R_1, E_n R_1\}$, 2.2.

At d– an even natural number,

$AC(H)=\{E_n\}\{R_1\}\{E_n R_1\}=\{E_n\}\{R_1\}$, 2.1■

Hence, at d– an odd number, $N_1(H)=2$, $N_2(H)=3$, and at d– an even number, $N_1(H)=3$, $N_2(H)=6$. Therefore, at $n=2\exp a p_1 \exp a_1 \ldots p_k \exp a_k$ there are $h=a(a_1+1)\ldots(a_n+1)+5$ homology groups of the class **nm**, consisting of $(a_1+1)\ldots(a_n+1)+3$ groups with d– an odd number, and $(a-1)(a_1+1)\ldots(a_n+1)+2$ groups with d– an even number. These h homology groups generate $(3a-1)(a_1+1)\ldots(a_n+1)+12$ antihomology groups of the M^1-type, and $3(2a-1)(a_1+1)\ldots(a_n+1)+21$ multiple antihomology groups of the M^2-type.

The remaining 3 infinite classes of homology point groups 5) $\tilde{\mathbf{n}}\mathbf{m}$, 6) **mn:m**, 7) **n:2**, can be derived from the class 4) **nm** by the use of antihomology ($\underline{\mathbf{n}}\mathbf{m}$, $\mathbf{n}\underline{\mathbf{m}} \longrightarrow \tilde{\mathbf{n}}\mathbf{m}$; $\mathbf{nm}\times\underline{1} \longrightarrow \mathbf{mn:m}$; $\mathbf{n}\underline{\mathbf{m}} \longrightarrow \mathbf{n:2}$).

At $n=2$ there is only one homology group. At $n=2\exp a p_1 \exp a_1 \ldots p_k \exp a_k$ there are $(2a-1)(a_1+1)\ldots(a_n+1)+7$ homology groups. Since AC of every symmetry group of the class $\tilde{\mathbf{n}}\mathbf{m}$ is trivial and isomorphic to $\{A\}\{B\}$, 2.1, $N_1(H)=3$, $N_2(H)=6$. Therefore, at $n=2\exp a p_1 \exp a_1 \ldots p_k \exp a_k$, there are $3(2a-1)(a_1+1)\ldots(a_n+1)+21$ antihomology groups of the M^1-type and $6(2a-1)(a_1+1)\ldots(a_n+1)+42$ multiple antihomology groups of the M^2-type.

At n=2, in the class **mn:m** there are 3 homology groups:

$H_1=(S_2, R_1, R_2)\times(R)$ AC: $\{R, R_1, S_2R_1\}$, 3.7;

$H_2=(S_2, r_1, r_2)\times(R)$ AC: $\{R\}\{r_1, S_2r_1\}$, 3.2;

$H_3=(e_2, r_1, r_2)\times(r)$ AC: $\{r, r_1, e_2r_1\}$, 3.7.

$N_1(H_1)=N_1(H_3)=3$, $N_2(H_1)=N_2(H_3)=10$, $N_3(H_1)=N_3(H_3)=28$;

$N_1(H_2)=5$, $N_2(H_2)=24$, $N_3(H_2)=84$ [13].

In line with **nmx**$\underline{1}$$\longrightarrow$**mn:m**, at $n\geq3$ every group H_1 of the class **mn:m** is a direct product $H_1=H\times(R)$ or $H_1=H\times(r)$. Moreover, $AC(H_1)=AC(H)\{R\}$ or $AC(H_1)=AC(H)\{r\}$. At $n=2k+1$ ($n\geq3$) in the class **nm** there are 7 homology groups with trivial $AC(H)\doteq\{A\}$, 1.1, so that in the class **mn:m** there will be also 7 homology groups H_1 with trivial $AC(H_1)\doteq\{A\}\{B\}$, 2.1. At n- an even number of the form $n=2\exp a p_1 \exp a_1 \ldots p_k \exp a_k$, having in mind that $AC(H_1)=AC(H)\{R\}$ or $AC(H_1)=AC(H)\{r\}$, in line with Theorem 3 [13] we have: $N_1(H_1)=2N_1(H)+1$, $N_2(H_1)=4N_2(H)+6N_1(H)$, $N_3(H_1)=28N_1(H)$. Since there are $(a_1+1)\ldots(a_n+1)+3$ homology groups of the class **nm** with $N_1(H)=2$, $N_2(H)=3$, and for all remaining $(a-1)(a_1+1)\ldots(a_n+1)+2$ homology groups of the same class $N_1(H)=3$, $N_2(H)=6$, at every such n there are h homology groups of the class **mn:m**, which generate $(7a-1)(a_1+1)\ldots(a_n+1)+29$ antihomology groups of the M^1-type, $6(7a-3)(a_1+1)\ldots(a_n+1)+156$ multiple antihomology groups of the M^2-type and $84(2a-1)(a_1+1)\ldots(a_n+1)+588$ multiple antihomology groups of the M^3-type.

At n=2 in the class **n:2** there are 3 homology groups:

(S_2, S_2', S_2'') AC: $\{S_2, S_2', S_2S_2'\}$, 2.3;

(S_2, e_2', e_2'') AC: $\{e_2', S_2e_2'\}$, 2.2;

(e_2, e_2', e_2'') AC: $\{e_2, e_2', e_2e_2'\}$, 2.3.

In line with the relationship **n**$\underline{\mathbf{m}}$$\longrightarrow$**n:2**, the complete discussion about homology groups of the class **nm** can be immediately transferred to the class n:2. Hence, at n- an odd number ($n \geq 3$) there are 7 homology groups which generate 7 antihomology groups of the M^1-type. At n- an even number of the form $n=2\exp\alpha p_1\exp\alpha_1 \ldots p_k\exp\alpha_k$ there are h homology groups of the class n:2 which generate $(3\alpha-1)(\alpha_1+1)\ldots(\alpha_n+1)+12$ antihomology groups of the M^1-type and $3(2\alpha-1)(\alpha_1+1)\ldots(\alpha_n+1)+21$ multiple antihomology groups of the M^2-type.

As the final result, we can conclude that in 7 infinite classes of homology point groups H_{30}, at $n=1$ there are 3 homology groups which generate 2 antihomology groups of the M^1-type, at $n=2$ there are 15 homology groups which generate 33 antihomology groups of the M^1-type, 76 multiple antihomology groups of the M^2-type and 140 multiple antihomology groups of the M^3-type. At n- an odd number ($n \geq 3$) there are 29 homology groups which generate 39 antihomology groups of the M^1-type and 42 multiple antihomology groups of the M^2-type. At n- an even number of the form $n=2\exp\alpha p_1\exp\alpha_1 \ldots p_k\exp\alpha_k$ there are $(5\alpha-1)(\alpha_1+1)\ldots(\alpha_n+1)+34$ homology groups which generate $(19\alpha-7)(\alpha_1+1)\ldots(\alpha_n+1)+94$ antihomology groups of the M^1-type, $6(11\alpha-5)(\alpha_1+1)\ldots(\alpha_n+1)+264$ multiple antihomology groups of the M^2-type and $84(2\alpha-1)(\alpha_1+1)\ldots(\alpha_n+1)+588$ multiple antihomology groups of the M^3-type.

As a partial result, we can conclude that there are $N_0=132+83=215$ crystallographic point groups of the category H_{30} which generate $N_1=317+115=432$ antisymmetry groups of the M^1-type [7,8,9,10], $N_2=820+126=946$ multiple antihomology groups of the M^2-type and $N_3=1596+0=1596$ multiple antihomology groups of the M^3-type, where the first number corresponds to the crystallographic groups included into 7 infinite classes, and the second to the crystallographic polyhedral groups.

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PUNKTUALNE GRUPE PROSTE I VIŠESTRUKE ANTIHOMOLOGIJE H_{30}^{l}

Primenom metode antisimetrijskih karakteristika (*AK*-metode), izvedene su punktualne grupe proste i višestruke antihomologije H_{30}^{l}.

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