Zbornik radova Filozofskog fakulteta u Nišu
Serija Matematika 4(1990), 3-5

V.V. Fedorchuk

COMPLETIONS OF FUNCTIONAL SPACES AND MULTIVALUED MAPPINGS

(Received 05.10.1989.)

Abstract. Let $(X,\mathcal{U})$ be a compact uniform space and $C(X)$ the space of all uniformly continuous real-valued functions on X with the topology of uniform convergence. We consider a completion $C_V(X)$ of $C(X)$.

1. Let X be a metrizable compact space with a given metric d. This metric implies a metric ρ on the product $X \times \mathbb{R}$:

$$\rho((x_1,t_1),(x_2,t_2)) = \max\{d(x_1,x_2),\ |t_1-t_2|\}.$$

Let $\exp_c(X \times \mathbb{R})$ be the set of all non-empty compact subsets of $X \times \mathbb{R}$. The metric space $(\exp_c(X\times\mathbb{R}),\rho_H)$, where ρ_H is the Hausdorff metric for the metric ρ, is complete and locally compact.

Let C(X) be the set of all continuous real-valued functions on X. If we identify every function $\varphi \in C(X)$ with its graph $Gr(\varphi)$, then we can consider the set C(X) as a subset of $\exp_c(X\times\mathbb{R})$. The topology of the Hausdorff metric on C(X) coincides with the sup-norm topology. But the metric space $(C(X),\rho_H)$ is never complete if X is infinite.

For a uniform space $(X,\mathcal{U})$, where the uniformity $\mathcal{U}$ consists of uniform coverings, we denote by $\exp\mathcal{U}$ the uniformity on the hyper-space expX such that its base of entourages consists of sets

$$E_u = \{(A,B)\in\exp X \times \exp X: A \subset st_u B,\ B \subset st_u A\},$$

where $u \in \mathcal{U}$.

Let $\mathcal{M}_{\mathbb{R}}$ be the natural metric uniformity on $\mathbb{R}$. Then for a uniform space $(X,\mathcal{U})$ we denote by $\mathcal{V}(\mathcal{U})$ the uniformity $\exp(\mathcal{U}\times\mathcal{M}_{\mathbb{R}})$ on $\exp(X\times\mathbb{R})$. The uniformity $\mathcal{V}(\mathcal{U})$ generates on the set $C(X,\mathcal{U})$ of all uniformly continuous real-valued functions functions on X the topology of uniform convergence (see [1]). We denote by $C_V(X,\mathcal{U})$ a completion of the space $C(X,\mathcal{U})$ with respect to the uniformity $\mathcal{V}(\mathcal{U})$. For a compact space X there is the unique uniformity $\mathcal{U}$. So in this case we denote the uniformity $\mathcal{V}(\mathcal{U})$ by $\mathcal{V}$

AMS Subject Classification (1980): 54B20, 54C35, 54C60, 54E15

and the space $C_V(X, \mathcal{U})$ by $C_V(X)$. If X is a metrizable compact space, the uniformity $\mathcal{V}$ coincides with the uniformity generated by the Hausdorff metric ρ_H on $\exp_c(X\times\mathbb{R})$ for any metric d on X.

2. For any topological spaces X and Y we denote by USC(X,Y) (respectively by USCC(X,Y)) the set of all compact-valued upper semi-continuous mappings $F: X \to Y$ (respectively with connected fibers F(x)). If we identify every mapping $F \in USC(X,Y)$ with its graph Gr(F), we can consider the set USC(X,Y) and its subset USCC(X,Y) as subsets of $\exp(X\times Y)$. Below we assume that the sets USC(X,Y) and USCC(X,Y) are equipped by the Vietoris topology. For $Y = \mathbb{R}$ we denote these spaces by USC(X) and USCC(X).

PROPOSITION 1. *For any Hausdorff compact space X we have*

$$C_V(X) = [C(X)]_{USC(X)}.$$

PROPOSITION 2. *For any locally connected Hausdorff compact space X,*

$$C_V(X) \subset USCC(X).$$

THEOREM 1. *For any locally connected Hausdorff compact space X with no isolated points we have* $C_V(X) = USCC(X)$.

Since every locally connected Hausdorff compact space X for any its open cover u has a monotone u-mapping onto a metrizable compact space, it is easy to reduce the proof of Theorem 1 to the metrizable case. In this case the proof is based on Proposition 2, on the existence of a locally convex metric on X [2] and on Michael's selection theorem for a lower semi-continuous convex-valued mapping into a Banach space [3].

THEOREM 2. *For any zero-dimensional Hausdorff compact space X with no isolated points ,* $C_V(X) = USC(X)$.

A proof of this theorem is based on zero-dimensional Michael's selection theorem [4].

3. **THEOREM 3.** *For any non-degenerate Peano continuum X the space* $C_V(X)$ *is homeomorphic to* $Q \times [0,1)$, *where Q is the Hilbert cube.*

By Theorem 1 and Chapman's theorem on complements of Z-sets in Q [5] it is enough to prove that $Y = USCC(X,I)$ is homeomorphic to Q. For this first of all we need Y to be an absolute retract. But Y has even stronger property: there is a base of closed neighborhoods in Y such that any non-empty intersection of finitely many members of this base is contractible. After this one needs only to check that Y satisfies

Torunczyk's general position characterization of the Hilbert cube [6].

COROLLARY 1. *For any infinite locally connected metrizable compact space X the spaces $C_V(X)$ and USCC(X) are homeomorphic to* $Q \times [0,1)$.

4. For a non-metrizable Hausdorff compact space X the space $C_V(X)$ is never an absolute retract. Moreover, its Alexandroff's compactification $\alpha(C_V(X))$ is not a dyadic compact. Indeed, a dense subset C(X) in $\alpha(C_V(X))$ consists of points of countable character in $\alpha(C_V(X))$. So being dyadic $\alpha(C_V(X))$ is metrizable by Efimov's theorem [7]. But it is easy to verify that for a non-metrizable Hausdorff compact space X the space $C_V(X)$ is non-metrizable.

REFERENCES

[1] J.R. ISBELL, *Uniform Spaces,* AMS, 1964.

[2] R.H. BING, *Partitioning continuous curves,* Bull. Amer. Math. Soc. 58(1952), 536-556.

[3] E. MICHAEL, *Continuous selections I,* Ann. Math. 63(1956), 361-382.

[4] E. MICHAEL, *Continuous selections II,* Ann. Math. 64(1956), 562-580.

[5] T.A. CHAPMAN, *Lectures on Hilbert cube manifolds,* AMS, 1976.

[6] H. TORUNCZYK, *On CE-images of the Hilbert cube and characterization of Q-manifolds,* Fund. Math. 106(1980), 31-40.

[7] B.A. EFIMOV, *Dyadic bicompacta,* Trudy Mosk. Mat. Obshch. 14(1965), 211-247.

V.V. Fedorchuk

KOMPLETIRANJE FUNKCIONALNIH PROSTORA I VIŠEZNAČNA PRESLIKAVANJA

Za kompaktan uniformni prostor X neka C(X) označava prostor svih uniformno neprekidnih realnoznačnih preslikavanja sa topologijom uniformne konvergencije. Razmatra se kompletiranje $C_V(X)$ prostora C(X).

Chair of General Topology and Geometry
Mech.-Math. Faculty
Moscow State University
Moscow, USSR