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SOME SEPARATION AXIOMS IN FUZZY SPACE

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Abstract. We introduce and study axioms weaker than FT_1 and FR_0 in fuzzy topology and their inclusion relations. We continue to investigate fuzzy almost regularity introduced by N.Ajmal and B.K.Tyagi.

1. Introduction. Unlike classic general topology where spaces are usually at least Hausdorff, in fuzzy topology even the axioms T_1 is often too restrictive .For example, the fuzzy unit interval and fuzzy real line (see [12]) itself are not FT_1. Therefore, the main aim of the first part of this article is to introduce and study the separation aximos weaker than FT_1 and extend the definitions and results from general topology (see [3], [7]) to fuzzy sets. Introduction of the new axioms is based on the observation that in FT_0 spaces the derived fuzzy set of a fuzzy point - $d\{x_t\}$ - is the union of fuzzy closed sets, while in FT_1 spaces $d\{x_t\} = 0$, and hence closed for all $\{x_t\}$.

In the second part of this paper we introduce new axioms weaker than FR_0, and analyse their inclusion relations. All definitions are in a natural way analogous to that in general topology (see [10], [11]). N.Ajmal and B.K.Tyagi have introduced [2] the notions of fuzzy almost ragularity and fuzzy weakly regularity. They have shown that these two notions coincide for T_1 - fuzzy topological spaces.

In chapter 4 we shall show that these two notion coincide for R_0 - fuzzy topological spaces, and that the product ofan arbitrary family of fuzzy topolofical spaces is fuzzy almost regular iff each factor space is fuzzy almost regular. We refere to [1], [5], [8], [9] as general references on fuzzy topological spaces.

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2. Separation axioms betwen FT_0 and FT_1

A definition of a derived fuzzy set using the concept of a fuzzy limit point was first given in [13]. Pu Pao Ming and Liu Ying Ming in their paper [9] introduced the definition of a accumulation point of an accumulation point of a fuzzy set using the concept of a Q-relation i.e. the quasi-coincident relation and the coresponding neighbourhood system, called the Q-neighbourhood system. They called the union of all the accumulatioin points of fuzzy set by the derived set of a fuzzy set. In this paper we shall take lemma 5.1. from [9] as the definition of the derived set of a fuzzy point and introduce some new separation axioms in fuzzy spaces.

DEFINITION 2.1. *Let (X,δ) be a fuzzy topological space (fts-in short) and let $x_t \in I^X$ be a fuzzy point. We define the derived fuzzy set of a fuzzy point x_t (denote by $d\{x_t\}$) as follows:*

$$d\{x_t\}(y) \begin{cases} cl\{x_t\}(y) & \text{if } \ y \neq x \\ 0 & \text{if } \ y = x \ \text{ and } \ cl\{x_t\}(x) = t \\ cl\{x_t\}(x) & \text{if } \ y = x \ \text{ and } \ cl\{x_t\}(x) > t, \ t \in (0,1] \end{cases}$$

This definition essentialy agrees with that in the crisp case.

DEFINITION 2.2. *By the covering of a fuzy point x_t we call the fuzzy set*

$$\langle x\rangle = cl\{x_t\} \wedge Ker\{x_t\}$$

where $Ker\{x_t\} = \wedge\{G|G$ open fuzzy set and $x_t \in G\}$ is called the kernel of x_t.

PROPOSITION 2.1. *Let x_t, y_t be two distinct points of a fts (X,δ). Then*

(a) $y_r \in cl\{x_t\}$ *implies* $cl\{y_r\} \leq cl\{x_t\}$

(b) $y_r \in Ker\{x_t\}$ *implies* $Ker\{y_r\} \leq Ker\{x_t\}$

(c) $y_r \in \langle x_t\rangle$ *implies* $\langle y_r\rangle \leq \langle x_t\rangle$

DEFINITION 2.3. *(See [1]) A fts (X,δ) is called an FT_1 - space if each point is closed.*

DEFINITION 2.4. *A fts (X,δ) is an FT_D - space if $d\{x_t\}$ is closed for every point x_t.*

DEFINITION 2.5. *A fts (X,δ) is an $FT_{\overline{UD}}$ space if $d\{x_t\}$ is the union of disjoint closed fuzzy sets for every point x_t.*

DEFINITION 2.6. *A fts (X,δ) is an FT_0 - space if $d\{x_t\}$ is the union of closed fuzzy sets for every point x_t.*

PROPOSITION 2.2. $FT_1 \Rightarrow FT_D \Rightarrow FT_{UD} \Rightarrow FT_0$

Proof. It is sufficient to compare the respective definitions. All the implications are strict.

EXAMPLE 2.1. Let X = [0,1]. Let $\mu_{x\alpha}$, $x \in X$, $\alpha \in [0,1]$ defined by

$$\mu_{x\alpha}(y) = \begin{cases} \alpha & y \in [x,1] \\ 0 & \text{otherwise} \end{cases}$$

be the closed fuzzy sets and their finite unions. This space is FT_0 but not FT_{UD}.

EXAMPLE 2.2. Let X = [0,1]. Let the closed sets be 1_X, 0_X, $\{x_t\}$, $x \in I$, $t \in (0,1)$, $\{x_1\}$, $x \neq 0$ and their finite unions. This space is FT_{UD} but not FT_D.

EXAMPLE 2.3. Let X = [0,1]. Let the closed sets be 1_X, 0_X, $\mu_{\alpha x}$, $x \in [0,1]$ defined by

$$\mu_{\alpha x}(y) = \begin{cases} \alpha & y \in [x,0] \\ 0 & \text{otherwise} \end{cases} \qquad \text{and their finite unions.}$$

This space is FT_D but not FT_1.

DEFINITION 2.7. *A fuzzy point* x_t *is said to be fuzzy separated from* y_r, $x \neq y$, $r,t \in I$ *if there exists an open fuzzy set G,* $x_t \in G$ *such that* $G \wedge y_r = 0$. *If* $x = y$ *we say that* x_t *is separated from* x_r, $r<t$ *if there exists an open f.set G such that* $x_r \in G$ *and* $x_t \notin G$. *We shall write* $\{x_t\} \vdash \{y_r\}$.

THEOREM 2.1. *If a fts* (X,δ) *is* FT_1 *- space then the following statements are equivalent:*

(a) For every point x_t, $cl\{x_t\} = \{x_t\}$.

(b) For every point x_t, y_r, $\{x_t\} \vdash \{y_r\}$.

(c) For every x_t, $d\{x_t\} = 0$, *and hence it is closed.*

Proof. (a) ⇔ (b) See proposition 2.2. in [1].

(a) ⇔ (c) The proof follows directly from the definition of the derived set.

PROPOSITION 2.3. *If a fts* (X,δ) *is an* FT_1 *- space, then*

(a) For every x_t, $\langle x_t \rangle = \{x_t\}$.

(b) For every x_t, $Ker\{x_t\} = \{x_t\}$.

Proof. (a) It is trivial.

(b) To clarity the condition (b) we might point out that FT_1 implies FR_0 (see th.2.6. in [11]) which implies $Ker\{x_t\} = cl\{x_t\}$, for each x_t (see [11]), hence $Ker\{x_t\} = x_t$.

THEOREM 2.2. *If a fts (X,δ) is an FT_0 - space, then the following statements are equivalent:*

(a) $y_r \in cl\{x_t\}$ implies $x_t \notin cl\{y_r\}$, i.e. distinct points have distinct closures.

(b) For every x_t, $d\{x_t\}$ is the union of closed sets.

Proof. (a) $\Leftrightarrow$ (b) From theorem 6.2. in [9] follows: (X,δ) is an FT_0 - space iff for every $x \in X$ and $0<t<r\leq 1$, $x_r \notin cl\{x_t\}$ and for any two distinct points x, y in X and every $r, t \in [0,1)$ there exists $U \in \delta$ such that $U(x) = r$ and $U(y) > t$ or $U(x) > r$ and $U(y) = t$. By taking complement this fact can be stated as follows: (X,δ) is an FT_0 - space iff for every $x \in X$ and $0<t<r\leq 1$, $x_r \notin cl\{x_t\}$ and for any two distinct points x, y in X and every $r,t \in [0,1)$ there exists a closed fuzzy set H such that $x_r \in H$ and $H(y)=0$ or $y_t \in H$ and $H(x)=0$. Therefore $cl\{x_t\}(x)=t$, for each $x \in X$ and for every $y_r \in cl\{x_t\}$, $y \neq x$ there exists a closed fuzzy set H_{y_r} such that $y_r \in H_{y_r} < cl\{x_t\}$ and $H_{y_r}(x)=0$.

COROLLARY 2.1. *If a fts (X,δ) is an FT_0 - space, then*

(a) For every x_t, $d\{x_t\}(x) = 0$, i.e. $d\{x_t\} = cl\{x_t\} - \{x_t\}$.

(b) $y_r \in Ker\{x_t\}$ implies $x_t \notin Ker\{y_r\}$.

(c) $\langle x_t \rangle = cl\{x_t\} \wedge Ker\{x_t\} = \{x_t\}$.

3. Axioms weaker then FR_0

DEFINITION 3.1. *(See [11]) A fts (X,δ) is called an FR_0 - space if*

(a) For any fuzzy points x_t, x_r, $t<r$ implies $x_r \notin cl\{x_t\}$.

(b) For any fuzzy points x_t, y_r, $x \neq y$, $x_t \in cl\{x_r\}$.

iff $y_r \in cl\{x_t\}$.

PROPOSITION 3.1. *In an FR_0 - space (X,δ) if for any point x_t, $\langle x_t \rangle = \{x_t\}$ then $cl\{x_t\} = \{x_t\}$ and $d\{x_t\} = 0$.*

Proof. In a FR_0 - space $cl\{x_t\} = Ker\{x_t\}$ for every fuzzy point x_t, so the proof is trivial. Proposition 3.1. suggests the introduction of the following axioms:

DEFINITION 3.2. *A fts $(X\ \delta)$ is called an FR_{UD} space if for all fuzzy points x_t, $\langle x_t \rangle = \{x_t\}$ implies $d\{x_t\}$ is closed.*

DEFINITION 3.3. *A fts (X,δ) is called an FR_{UD} - space if for all x_t, $\langle x_t \rangle = \{x_t\}$ implies $d\{x_t\}$ is the union of disjoint closed fuzzy sets.*

The space in example 2.3. is FR_{UD} but not FR_D and the space in example 2.4. is FR_D but not FR_0.

PROPOSITION 3.2.

$$\begin{array}{ccccccc} FT_1 & \Rightarrow & FT_D & \Rightarrow & FT_{UD} & \Rightarrow & FT_0 \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ FR_0 & \Rightarrow & FR_D & \Rightarrow & FR_{UD} & \Rightarrow & \textit{eny fts.} \end{array}$$

The proof is straightforward.

PROPOSITION 3.3.

$$FT_0 + FR_D = FT_D$$
$$FT_0 + FR_{UD} = FT_{UD}$$

Proof. Since in a FT_0 - space $\langle x_t\rangle = \{x_t\}$ for all points x_t, then the proof is an immediate consequence of the respective definition.

4. Some further results on FR_0-spaces

DEFINITION 4.1. *(See [8]) A fts (X,δ) is said to be fuzzy regular if for each open fuzzy set U there exist open fuzzy sets U_i such that*

$$U = V_i U_i \quad \text{and} \quad clU_i \leq U \quad \text{for each } i.$$

PPROPOSIION 4.1. *If a fts (X,δ) is fuzzy regular then it is an FR_0 space.*

Proof. Let U be any open fuzzy set in δ and $x_t \in U$. By fuzzy regularity $x_t \in U = V_i U_i$ implies that there exists an open fuzzy set U_{i_0} such that $x_t \in U_{i_0} \leq clU_{i_0} \leq U$. Hence $clx_t \leq clU_{i_0} \leq U$, so (X,δ) is an FR_0-space (see Theorem 2.1.(6) in [11]).

DEFINITION 4.2.([2]) *Fuzzy sets μ and η in a fts (X,δ) are said to be* **weakly separated** *if there exist open fuzzy sets U and V such that $\mu \leq U$, $\eta \leq V$, $U + \eta \leq 1_X$ and $V + \mu \leq 1_X$.*

DEFINITION 4.3.([2])*Fuzzy sets μ and η in a fts (X,δ) are said to be* **strongly separated** *if there exist open fuzzy sets U and V such that $\mu \leq U$, $\eta \leq V$ and $U + V \leq 1_X$.*

For the definitions of regular open and regular closed fuzzy sets and related results we refer to Azad [4].

DEFINITION 4.4.([2]) *A fts (X,δ) is said to be* **fuzzy almost regular***if any regular closed fuzzy set μ and any fuzzy point x_t in $1_X - \mu$ can be strongly separated.*

DEFINITION 4.5. *A fts (X,δ) is said to be* **fuzzy weakly regular** *if every weakly separated pair consisting of a regular closed fuzzy set μ and a fuzzy point $x_t \in I^X$ can be strongly separated.*

PROPOSITION 4.2. *Fuzzy regular implies fuzzy almost regular.*

Proof. Let $x_t \in U$ - where U is a regular open fuzzy set. It is fuzzy open and by fuzzy regularity $U = \vee_i U_i$, $clU_i \le U$. Since $U_i \le Int\ clU_i \le U$ then $U = \vee_i Int\ clU_i$ and there exists i_0 such that $x_t \in Int\ clU_{i_0} = V$, V is regular open fuzzy set such that $x_t \in V \le clV < U$. Hence by theorem 4.2.(b) from [2] (X,δ) is fuzzy almost regular. We know that the product of an arbitrary family of fuzzy topological spaces is fuzzy regular iff each factor space is fuzzy regular. The same is true for fuzzy almost regular spaces.

THEOREM 4.1. *The product of an arbitrary of fuzzy topological spaces is fuzzy almost regular iff each factor space is fuzy almost regular.*

Proof. Let $X = \prod_{\alpha\in I} X_\alpha$, where $\{(X_\alpha,\delta_\alpha):\alpha\in I\}$ is any family of fuzzy almost regular spaces and δ is the product fuzzy topology. Let λ be any open set from δ containing a point, $x_t=(x_\alpha)\in I^X$. Then there exist an open fuzzy set μ which belongs to the standard base of the product fuzzy topology i.e. $\mu = \bigwedge_{i=1}^{n} p_{\alpha i}^{-1}(\mu_{\alpha i})$, $\mu_{\alpha i} \in \delta$ such that $x_t \in \mu \le \lambda$. Thus $(x_{\alpha i})_t \in \mu_{\alpha i}$. Since $(X_{\alpha i},\delta_{\alpha i})$ is fuzzy almost regular for each $i = 1,\dots,n$, therefore there exist an open fuzzy subset $\eta_{\alpha i}$ in $\delta_{\alpha i}$ such that

$$(x_{\alpha i t}) \in \eta_{\alpha i} \le cl\eta_{\alpha i} \le Int\ cl\mu_{\alpha i}.\ \text{Now let } \eta = \bigwedge_i p_{\alpha i}^{-1}(\eta_{\alpha i})$$

Then $x_t \in \eta$ and η is an open fuzzy set. Also,

$$\eta \le cl\eta = cl(\bigwedge_{i=1}^{n} p_{\alpha i}^{-1}(\eta_{\alpha i})) \le \bigwedge_i cl\ p_{\alpha i}^{-1}(\eta_{\alpha i}) \le \bigwedge_{i=1}^{n} p_{\alpha i}^{-1}(Int\ cl\mu_{\alpha i})$$

Since $p_{\alpha i}^{-1}(Int\ cl\mu_{\alpha i})$ belongs to the standard base of the product topology we may asume that $\bigwedge_{i=1}^{n} p_{\alpha i}^{-1}(Int\ cl\mu_{\alpha i}) \le \lambda \le Int\ cl\lambda$. Hence, $x_t \in \eta \le cl\eta \le Int\ cl\lambda$, and by theorem 4.2.(d) in [2] (X,δ) is fuzzy almost regular.

Conversely, let $(X=\prod_\alpha X_\alpha,\delta)$ be a fuzzy almost regular product space. Consider a factor space (X_α,δ_α). Let α_β be an open fuzzy set from (X_β,δ_β) containing a point $(x_\beta)_t \in \lambda_\beta$. Let $\lambda = p_\beta^{-1}(\lambda_\beta)$. It is an open fuzzy set in (X,δ). Let $x_t = \{(x_\alpha)_t\}$. where $(x_\alpha)_t$ is arbitraly chosen fuzzy point from (X_α,δ_α) for each $\alpha \ne \beta$. Then $\{(x_{\alpha t})\} \in \lambda \in \delta$. Since (X,δ)

is fuzzy almost regular, there exists an open subset $\mu \in \delta$ such that $x_t \in \mu \leq cl\mu \leq Int\, cl\lambda$. Without any loss of generality we can take μ belongs to the standard base of the product fuzzy topology, i.e. $\mu = \wedge_{i=1}^{n} p_{\alpha i}^{-1}(\mu_{\alpha i})$ and $\mu_{\alpha i}$ ia an open fuzzy set in $\delta_{\alpha i}$. Then $(x_\beta)_t \in \mu_\beta \leq cl\mu_\beta \leq Int\, cl\lambda_\beta$. Thus (X_β, δ_β) is fuzzy almost regular. Hence each $(X_\alpha, \delta_\alpha)$ is fuzzy almot regular.

THEOREM 4.2. *An FR_0 - fts (X,δ) is fuzzy weakly regular iff for every fuzzy point $x_t \in I^X$ and every regular open fuzzy set λ containing x_t there is an open fuzzy set μ such that*

$x_t \in \mu \leq cl\mu \leq \lambda$.

Proof. Let (X,δ) be a fuzzy weakly regular space. Let $x_t \in \lambda$, where λ is a regular open fuzzy set, so it is open. By FR_0 $x_t \in \lambda$ implies $cl\{x_t\} \leq \lambda$. The idea of the rest of the proof is the same as in the proof of the theorem 4.1. from [2].

COROLLARY 4.1. *An R_0-fuzzy topological space is fuzzy almost regular iff is fuzzy weakly regular.*

DEFINITION 4.6. *A fts (X,δ) is said to be a D_1-fuzzy space if every closed fuzzy set has a contable open base the open fuzzy set containing it.*

THEOREM 4.3. *If (X,δ) is FR_0, D_1- fuzzy space then*

(a) Every closed fuzzy set is a G_δ- set, i.e. countable intersection of fuzzy open sets.

(b) (X,δ) is first countable fuzzy space.

Proof. Since (X,δ) is FR_0- space, so any closed fuzzy set F can be expressed as $F = \wedge \{G \text{ open} : F \subset G\}$ (see th 2.1.(3) in [11]). In a D_1-fuzzy space for the closed set F there must exist a countable base, say $\{G_n\}$ consisting of open sets for open sets containing F. Therefore $F = \wedge_n \{G_n : F \subset G_n\}$.

(b) For any point x_t in (X,δ) we have to show that the set $\{x_t\}$ has a countable base for the open fuzzy set containing it. Since (X,δ) is a D_1- fuzzy space there must exist a countable base say $\{G_n\}_n$ for the open sets containing teh closed set $cl\{x_t\}$. Now if x_t belongs to an open set G, then by FR_0- property $cl\{x_t\} \leq G$ and hence there exist a numer n such that $x_t \in G_n \leq G$. Therefore the family $\{G_n\}_n$ is the reguired countable base for the open fuzzy set containing x_t.

COROLLARY 4.2. *A fuzzy normal, FR_0- space satisfying D_1- property is perfectly normal.*

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NEKE AKSIOME SEPARACIJE U FAZI PROSTORIMA

U ovom radu uvedeni su novi aksiomi separacije FT_D i FT_{UD} u fazi topologiji, slabiji od FT_1- aksiome. Slično kao u topološkom slucaju uvodjenje se zasniva na činjenici da je izvedeni skup fazi tačke unija zatvorenih fazi skupova, dok je u FT_1- prostorima izvedini fazi skup tačke prazan skup.

U drugom poglavlju uvedeni su aksiomi separacije FR_D, FR_{UD} slabiji od aksiome FR_0 i dokazano da vrijedi:

$$\begin{array}{ccccccc} FT_1 & \Rightarrow & FT_D & \Rightarrow & FT_{UD} & \Rightarrow & FT_0 \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ FR_0 & \Rightarrow & FR_D & \Rightarrow & FR_{UD} & \Rightarrow & \text{proizvoljan fazi topološki prostor.} \end{array}$$

Na kraju rada, koristeći rezultate i definicije od N. Ajmal i B. K. Tyagi [2] kao i od K. K. Azad [4] pokazali smo da se fazi gotovo regularan i fazi skoro regularan podudaraju u fazi R_0- prostorima kao i teoreme: Produkt proizvoljne familije fazi topoloških prostora je fazi gotovo regularan ako je svaki faktor prostor fazi gotovo regularan, te nadalje da fazi normalan, FR_0 prostor koji zadovoljava D_1- fazi svojstvo je savršeno normalan.

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