Zbornik radova Filozofskog fakulteta u Nišu
Serija Matematika 4(1990), 115-120

Angelo Bella

TIGHTNESS AND SPLITTABILITY

(Received 26.07.1990.)

Abstract. We prove that tightness, T-tightness and set tightness coincide with their closed splittable versions. It is also shown that these results are no longer true in the case of splittability with respect to continuous maps. In additional we give a partial answer to the question whether a space which is closed splittable over the class of sequential spaces is itself sequential.

Given a class of maps $\mathcal{M}$ and a class of topological spaces $\mathcal{P}$, we say that a space X is **$\mathcal{M}$-splittable over $\mathcal{P}$** provided that for any subset B of X there exist $f \in \mathcal{M}$ and $Y \in \mathcal{P}$ such that $f:X \to Y$ and $B = f^{-1}f(B)$. When $\mathcal{M}$ is the class of all continuous maps or the class of all closed maps we simply use the expression **splittable** or **closed splittable.** Several papers related to the above notion have recently appeared (see the references of this paper). It should be recalled, however, that the idea of splittability is due to A.V. Arhangel'skii.

One problem to be investigated arises in connection with the notion of cardinal function. Given a class of maps $\mathcal{M}$, a cardinal function φ and a space X, the **$\mathcal{M}$-splittable version of φ** is the cardinal function $\varphi_{\mathcal{M},S}(X)$ defined as the smallest cardinal τ such that X is $\mathcal{M}$-splittable over the class of spaces having φ not exceeding τ. Clearly, $\varphi_{\mathcal{M},S} \le \varphi$ always holds and in many cases the inequality is proper.

In [3] (Theorem 5.2) it is proved that if $t_S(X)$ is countable for a compact space X, then t(X) is also countable and in [13] it was remarked that the same proof (mantaining the assumption X compact) can be carried out for any cardinal.

The aim of the present note is to prove the previous result in the general case, without any hypothesis on X. It turns out that the only thing we need is splittability with respect to closed continuous maps;

AMS Subject Classification (1980): 54A25

this assumption cannot be weakened to splittability with respect to continuous maps. We also prove an analogous result for the T-tightness and for the set tightness. Yet in connection with the notion of closed splittability, we in additional give a partial answer to the problem whether a space which is closed splittable over the class of almost radial (sequential) spaces must be itself almost radial (sequential).

Henceforth τ and ρ denote infinite cardinal numbers and $cf(\rho)$ denotes the cofinality of ρ. α,β,ν,ξ are ordinal numbers. P(A) denotes the power set of A. If X is a topological space and γ a family of subsets of X, then we write $\bar{\gamma}$ to indicate the family $\{\bar{C}:C \in \gamma\}$. All spaces are assumed to be T_1.

To begin with we recall the definitions og tightness and two of its variations which will be discussed here. These two variations of the classical notion of tightness were introduced in [10] by I. Juhász.

DEFINITION. *The* **tightness** *of X, denoted with t(X), is the smallest cardinal number* τ *such that for any subset B of X and any point* $x \in \bar{B}$ *there exists a subset* A *of B satisfying* $|A| \le \tau$ *and* $x \in \bar{A}$.

The **T-tightness** *of X, denoted by T(X), is the smallest cardinal number* τ *such that whenever* $\{F_\alpha:\alpha \in \rho\}$ *is an increasing sequence of closed subsets of X with* $cf(\rho) > \tau$ *it follows that* $\cup\{F_\alpha:\alpha \in \rho\}$ *is closed.*

The **set tightness** *of X, denoted by* $t_s(X)$, *is the smallest cardinal number* τ *such that for any subset B of X and any point* $x \in B'$ *there exists a family* $\gamma \subset P(B)$ *such that* $|\gamma| \le \tau$, $x \in \overline{\cup\gamma}$ *and* $x \notin \cup\bar{\gamma}$.

In [10], it is shown that the relation $t_s(X) \le T(X) \le t(X)$ always holds and the inequalities can be proper.

A subset B of a space X is said to be **τ-closed** provided that for any subset A of B with $|A| \le \tau$ we have $\bar{A} \subset B$. A useful fact concerning the tightness is the following: (see e.g. [1]) $t(X) \le \tau$ if and only if any τ-closed subset of X is closed.

THEOREM 1. *If X is closed splittable over a class of spaces having tightness not exceeding* τ, *then* $t(X) \le \tau$.

Proof. It is enough to check that any τ-closed subset B of X is closed. To this end select a space Y with $t(Y) \le \tau$ and a closed map $f:X \to Y$ such that $B = f^{-1}f(B)$. We claim that f(B) is τ-closed in Y. Indeed, for any $C \subset f(B)$ with $|C| \le \tau$ there exists a subset D of X such that $f(D) = C$ and $|D| \le \tau$. Therefore, we have $\bar{D} \subset B$ and $f(\bar{D}) = \overline{f(D)} = \bar{C}$ and consequently $\bar{C} \subset f(B)$. Since $t(Y) \le \tau$ it follows that f(B) is closed in Y and therefore B is closed in X. This completes the proof.

Since any continuous map defined on a compact space is closed, we immediately obtain:

COROLLARY 1. *If a compact space X is splittable over a class of spaces having tightness not exceeding* τ*, then* $t(X) \le \tau$.

THEOREM 2. *If a space X is closed splittable over a class of spaces having T-tightness not exceeding* τ*, then* $T(X) \le \tau$.

Proof. Let $\{F_\alpha : \alpha \in \rho\}$ be an increasing sequence of closed subsets of X and suppose that $cf(\rho) > \tau$. Let $B = \cup\{F_\alpha : \alpha \in \rho\}$ and select a space Y with $T(Y) \le \tau$ and a closed map $f : X \to Y$ such that $B = f^{-1}f(B)$. The family $\{f(F_\alpha) : \alpha \in \rho\}$ is an increasing sequence of closed subsets of Y and consequently $\cup\{f(F_\alpha) : \alpha \in \rho\}$ is closed in Y. The latter union is equal to $f(B)$ and therefore $B = f^{-1}f(B)$ is closed in X. This shows that $T(X) \le \tau$.

THEOREM 3. *If a space X is closed splittable over a class of spaces having set tightness not exceeding* τ*, then* $t_s(X) \le \tau$.

Proof. Let A be a subset of X and let $B = A \cup(\cup\{\overline{\cup\gamma} : \gamma \subset P(A)$ and $|\gamma| \le \tau\})$. It is clear that $A \subset B \subset \bar{B}$ and to prove that $t_s(X) \le \tau$ it is enough to check that B is closed. To this end take a space Y satisfying $t_s(Y) \le \tau$ and a closed continuous map $f : X \to Y$ such that $B = f^{-1}f(B)$. In order to prove that B is closed it is sufficient to verify that $f(B)$ is closed in Y. Thus let $y \in \overline{f(B)}$ and notice that, because $B \subset \bar{A}$, we also have $y \in \overline{f(A)}$. Since $t_s(Y) \le \tau$, there exists a family γ of subsets of $f(A)$ such that $|\gamma| \le \tau$, $y \in \overline{\cup\gamma}$ and $y \notin \cup\bar{\gamma}$. Put $\gamma' = \{f^{-1}(C) : C \in \gamma\}$ and to begin with observe that $\gamma' \subset P(A)$ and $|\gamma| \le \tau$. Moreover, as $y \in \overline{\cup\gamma}$ and f is closed, we must have $f^{-1}(y) \cap \overline{\cup\gamma'} \ne \emptyset$ and consequently there exists a point $x \in \overline{\cup\gamma'}$ such that $f(x) = y$. On the other hand, since $y \notin \cup\bar{\gamma}$, we have $f^{-1}(y) \cap \overline{f^{-1}(C)} = \emptyset$ for any $C \in \gamma$ and hence $x \notin \cup\overline{\gamma'}$. This means that $x \in B$ and therefore $y = f(x) \in f(B)$. $f(B)$ is then closed and the proof is complete.

It is important to observe that in the statements of Theorem 1, Theorem 2 and Theorem 3 the word "closed" cannot be omitted; in other terms tightness, T-tightness and set tightness are generally different than their splittable versions. This is easily seen by the next example.

Let A be a set of cardinality ω_1 and $p \in A$. Let X be the topological space obtained from the set A declaring a set $U \subset A$ to be open either if $p \notin U$ or if $p \in U$ and $|A \setminus U| \le \omega$. In addition let Y be the one-point compactification of the discrete space $A \setminus \{p\}$ having p as the accumulation point. It is obvious that the identity map $i:X \to Y$ is continuous and injective and we have $t(Y) = T(Y) = t_s(Y) = \omega$. This is enough to say that X is splittable over a class of spaces having countable tightness as well as countable T-tightness and countable set tightness, but clearly we have $t(X) = T(X) = t_s(X) = \omega_1$.

To finish the paper we present a result concerning the invariance of sequentiality with respect to closed splittability (compare with Theorems 2.1 and 2.2 in [12]). This result holds for a particular class of spaces and it is the consequence of a more general statement.

Recall (see [4]) that a space X is said to be **almost radial** provided that for any non-closed subset B of X there exist a regular cardinal τ and a transfinite sequence $\{x_\alpha:\alpha \in \tau\}$ in B which converges to some point $x \in \bar{B} \setminus B$ and satisfies $x \notin \overline{\{x_\beta:\beta \in \alpha\}}$ for any $\alpha \in \tau$. The class of almost radial spaces contains all radial and all sequential spaces.

THEOREM 4. *If a regular scattered space X is closed splittable over the class of almost radial spaces, then X is almost radial.*

Proof. Let B a non-closed subset of X and select an almost radial space Y and a closed map $f:X \to Y$ such that $B = f^{-1}f(B)$. Clearly f(B) is not closed in Y and hence there exist a regular cardinal τ and a sequence $\{y_\alpha:\alpha \in \tau\}$ in f(B) which converges to some $y \in \overline{f(B)} \setminus f(B)$ and satisfies $y \notin \overline{\{y_\beta:\beta \in \alpha\}}$ for any $\alpha \in \tau$. For any $\alpha \in \tau$ pick a point x_α in $f^{-1}(y_\alpha)$ and let S be the set so obtained. Because $y \in \overline{f(S)}$ and f is closed it follows that $f^{-1}(y) \cap \bar{S} \neq \emptyset$. Let x be an isolated point in the subspace $f^{-1}(y) \cap \bar{S}$ and, by virtue of the regularity of X, choose two disjoint open subsets U and V of X such that $x \in U$ and $f^{-1}(y) \cap \bar{S} \setminus \{x\} \subset V$. Since x is in the closure of $U \cap S$ and consequently f(x) is in the closure of $f(U \cap S)$, it follows that $U \cap S$ has cardinality τ. Therefore $U \cap S$ is a cofinal subsequence of S and we can write $U \cap S = \{x_{\alpha(\nu)}:\nu \in \tau\}$. The proof of the theorem will be complete showing that the sequence $\{x_{\alpha(\nu)}:\nu \in \tau\}$ converges to x and for any $\nu \in \tau$ we have $x \notin \overline{\{x_{\alpha(\xi)}:\xi \in \nu\}}$. The latter assertion follows easily from the continuity of f and the analogous property of the sequence $\{y_\alpha:\alpha \in \tau\}$. To check the convergence of $\{x_{\alpha(\nu)}:\nu \in \tau\}$ let W be an arbitrary neighborhood of x which, without loss of generality can be supposed to be contained in U. The set $(X \setminus \bar{S}) \cup W \cup V$ is a neighborhood of $f^{-1}(y)$ and consequently, by the closedness of f, there exists a neighborhood R of y such that $f^{-1}(R) \subset (X \setminus \bar{S}) \cup W \cup V$. Fix $\alpha^* \in \tau$ in such a way that for any $\alpha > \alpha^*$ we have $y_\alpha \in R$ and choose

$\nu^* \in \tau$ so that $\alpha(\nu^*) > \alpha^*$. For any $\nu > \nu^*$ we have $f(x_{\alpha(\nu)}) \in R$ and hence $x_{\alpha(\nu)} \in (X\setminus\bar{S}) \cup W \cup V$. Now the fact that $(U \cap S) \cap ((X\setminus\bar{S}) \cup V) = \emptyset$ obviously implies $x_{\alpha(\nu)} \in W$ and this is sufficient to prove that the sequence $\{x_{\alpha(\nu)}: \nu \in \tau\}$ converges to x.

Replacing in the above proof transfinite sequences with countable ones, we immediately get the following

COROLLARY 2. *If a regular scattered space X is closed splittable over the class of sequential spaces, then X is sequential.*

REFERENCES

[1] A.V. ARHANGEL'SKII, *Structure and classification of topological spaces and cardinal invariants,* Russ. Math. Surveys 33(1978),33-95.

[2] A.V. ARHANGEL'SKII, *A general conception of splittability of topological spaces over a class of spaces,* Abstracts of the Tirasp. Symp. 1985, 8-10 (in Russian).

[3] A.V. ARHANGEL'SKII, *Some new trends in the theory of continuous mappings,* In: Continuous functions on topological spaces, LGU, Riga 1986, 5-35 (in Russian).

[4] A.V. ARHANGEL'SKII, R. ISLER & G. TIRONI, *On pseudo-radial spaces and another generalization of sequential spaces,* Proc. Conference on Convergence, Bechyne (1984), Math. Research 24, Akademie-Verlag, Berlin (1985), 33-37.

[5] A.V. ARHANGEL'SKII & Lj. KOČINAC, *Concerning splittability and perfect mappings,* Publ. Inst. Math. (Belgrade) 47(61)(1990).

[6] A.V. ARHANGEL'SKII & D.B. SHAKHMATOV, *On pointwise approximation of arbitrary functions by countable collections of continuous functions,* Trudy Sem. I.G. Petrovskogo 13(1988), 206-227 (in Russian).

[7] A. BELLA, *Free sequences in pseudo-radial spaces,* Comm. Math. Univ. Carolinae 27(1986), 163-170.

[8] A. BELLA, *Some remarks on set tightness,* Ricerche di Matematica 35 (1986), 317-323.

[9] A. BELLA, F. CAMMAROTO & Lj. KOČINAC, *Remarks on splittability of topological spaces,* (to appear).

[10] I. JUHASZ, *Variations on tightness,* Studia Sci. Math. Hungarica 24 (1989), 179-186.

[11] Lj. KOČINAC, *On (M,P)-splittability of topological spaces,* Proc. IV Meeting on Topology, Sorrento 1988 (to appear).

[12] Lj. KOČINAC, *Perfect $\mathcal{P}$-splittability of topological spaces,* Zbornik rad. Fil. fak. (Niš), Ser. Mat. 3(1989), 19-24.

[13] Lj. KOČINAC, *Metrizability and cardinal invariants using splittability,* Comp. Rend. Acad. Bulg. Sci. 43(1990), 9-12.

[14] Lj. KOČINAC, F. CAMMAROTO & A. BELLA, *A survey on splittability of topological spaces,* Atti Accademia Pelor. dei Pericolanti 68(1990).

Angelo Bella

TESNOĆA I RASCEPLJENOST

Dokazuje se da su tesno a, T-tesno a i skupovna tesno a topološkog prostora jednake svojim zatvoreno rascepljenim verzijama. Primerom je pokazano da se ovi rezultati ne mogu proširiti na slučaj rascepljivosti posredstvom (samo) neprekidnih preslikavanja. Daje se i parcijalan odgovor na pitanje je li prostor koji je zatvoreno rascepljiv nad klasom sekvencijalnih prostora i sam sekvencijalan.

Dipartimento di Matematica
Università di Messina
Salita Sperone - Contrada Papardo
98166 Messina, Italy