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NEW GENERALIZATIONS OF ORDERING RELATIONS

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Abstract. In this note we give a new generalized ordering relation and its connections with the binary ordering and total binary ordering.

1. Definitions

We use terminology from [1]-[3]. Let S be a non-empty set and R an (n+1)-ary relation on S. Then:

1. R is **(n+1)-reflexive** iff $(\overset{n+1}{a}) \in S$ for each $a \in S$ [1].

2. R is **2-antisymmetric** iff for each $a, b \in S$ the following is satisfied: if all permutations of a,b are included in (n+1)-tuples of R, then a=b [1].

3. R is **iA_1-transitive**, $i \in N_n$, iff for each $a_0, \dots, a_{n+1} \in S$
$((a_0^{i-1}, a_i, a_{i+1}^n) \in R \ \& \ (a_1^{i-1}, a_i, a_{i+1}^{n+1}) \in R) \Rightarrow (a_0^{i-1}, a_{i+1}^{n+1}) \in R$ [2].

4. R is **compressible** iff for all $a_1, \dots, a_k \in S$ the following holds: if $(\overset{i_1}{a_1}, \dots, \overset{i_k}{a_k}) \in R$, $i_1 + \dots + i_k = n+1$, $i_1, \dots, i_k \in N$, then for all $j_1, \dots j_k \in N \cup \{0\}$ with $j_1 + \dots + j_k = n+1$, $(\overset{j_1}{a_1}, \dots, \overset{j_k}{a_k}) \in R$ [1].

If $(\overset{r}{a}, \overset{s}{b}) \in R$, $r+s=n+1$, $r, s \in N$, then $(\overset{i}{a}, \overset{j}{b}) \in R$ for all $i, j \in N \cup \{0\}$ with $i+j=n+1$ [4].

5. R is **2-complete** iff for all $a, b \in S$ the following is satisfied: $((\exists\ a_1, \dots, a_{n+1} \in S$ such that $(a_1^{n+1}) \in R$ and $a_i = a$, $a_j = b$, $1 \le i < j \le n+1)$ or $((\exists\ b_1, \dots, b_{n+1} \in S$ such that $(b_1^{n+1}) \in R$ and $b_k = b$, $b_m = a$, $1 \le k < m \le n+1)$ [4].

6. R is **strongly 2-complete** iff for all $a, b \in S$ the following holds $(\exists\ r, s \in N$, $r+s=n+1$ with $(a,b) \in R)$ or $(\exists\ k, m \in N$, $k+m=n+1$ with $(b,a) \in R)$ [4].

7. R is **p-transitive** iff for each $a_1, \dots, a_{n+1}, b_1, \dots, b_{n+1} \in S$ such that $(a_1^{n+1}) \in R$, $(b_1^{n+1}) \in R$ and $a_i = b_j$ $(1 \le i, j \le n+1)$, one has: for all $x \in \{a_1, \dots, a_{i-1}, b_1, \dots, b_{j-1}\}$ and all $y \in \{a_{i+1}, \dots, a_{n+1}, b_{j+1}, \dots, b_{n+1}\}$ there exist $c_1, \dots, c_{n+1} \in S$ such that $(c_1^{n+1}) \in R$ and $x = c_k$, $y = c_r$, $1 \le k < r \le n+1$.

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2. On generalizations of ordering relations

PROPOSITION 1. *If an (n+1)-ary relation R on S is compressible and iA_1-transitive, then it is p-transitive.*

Proof. Let $(a_1^{n+1}) \in R$, $(b_1^{n+1}) \in R$ and let $a_i=b_j=a$. Let $x=a_k$ and $y=b_s$, $k<i$, $s>j$. The compressibility of R implies $(x,\overset{n}{a}) \in R$ and $(\overset{n}{a},y) \in R$. From this fact and iA_1-transitivity $(i \in N_n)$ of R it follows $(x,{}^{n}\overset{-1}{a}{}^{1}) \in R$ so that R is a p-transitive relation. If $x=b_k$, $k<j$, and $y=a_s$, $s>i$, the proof is analogous.

The converse of this proposition is not valid in general, as the following example shows.

EXAMPLE 1. Let $S = \{a,b,c,d,e\}$, $n=2$. The relation

$$R = \{(\overset{3}{x}):x \in S\} \cup \{(a,c,e),(a,c,d),(b,c,d),(b,c,e)\}$$

is p-transitive but it is neither compressible nor weakly compressible.

REMARK. By Proposition 1, the (n+1)-ary relation ω in Theorem 1 in [1] is a p-transitive relation.

The following theorem is a generalization of Theorem 12 from [1]. Example 2 shows that this generalization is independent of the generalization of the same theorem given in [4].

THEOREM 2. *Let R be an (n+1,2,p)-RAT relation on S. The binary relation $\le$ on S defined by: $a\le b$ iff there exist $a_1,\dots,a_{n+1} \in S$ such that $(a_1^{n+1}) \in R$ and $a_i=a$, $b_j=b$, $1\le i<j\le n+1$, is an ordering relation.*

Proof. Reflexivity and antisymmetry of $\le$ are immediate consequences of (n+1)-reflexivity and 2-antisymmetry of R. Let us prove that $\le$ is a transitive relation. Let $a\le b$ and $b\le c$. Then there are $a_1,\dots,a_{n+1}$, $b_1,\dots,b_{n+1} \in S$ so that $(a_1^{n+1}) \in R$ and $(b_1^{n+1}) \in R$, $a_i=a$, $a_j=b$, $1\le i<j\le n+1$ and $b_k=b$, $b_s=c$, $1\le k<s\le n+1$. Since $a_j=b_k=b$, $a \in \{a_1,\dots,a_{j-1},b_1,\dots,b_{k-1}\}$, $c \in \{b_{k+1},\dots,b_{n+1},a_{j+1},\dots,a_{n+1}\}$, by p-transitivity of R there are $c_1,\dots,c_{n+1} \in S$ such that $(c_1^{n+1}) \in R$, $c_p=a$, $c_q=c$, $1\le p<q\le n+1$, which means that $a\le c$.

EXAMPLE 2. Let $S = \{a,b,c,d,e\}$, $n=2$ and

$$R = \{(\overset{3}{x}):x \in S\} \cup \{(a,a,b),(a,b,b),(a,a,c),(a,c,c),(a,b,c),(b,d,e)\}.$$

The relation R is 3-reflexive, 2-antisymmetric, $2A_1$-transitive and weakly compressible but it is not p-transitive.

The following theorem is given without proof (compare [4;Th. 4]).

THEOREM 3. *Let R be an $(n+1)$-reflexive, 2-antisymmetric, 2-complete and p-transitive $(n+1)$-ary relation on S. Then the (binary) relation $\leq$ defined by: $a \leq b$ iff there exist $a_1, \ldots, a_{n+1} \in S$ such that $(a_1^{n+1}) \in R$ and $a_i = a$, $a_j = b$, $1 \leq i < j \leq n+1$, is a total order in S.*

The converse need not be valid as the following example shows.

EXAMPLE 3. Let $S = \{a,b,c,d\}$, $n=2$ and

$R = \{(\overset{3}{x}) : x \in S\} \cup \{(a,b,c),(b,c,d),(a,b,d),(a,a,b),(a,a,c),(a,a,d)\}$.

R is a 3-reflexive, 2-antisymmetric, 2-complete and p-transitive relation but it is neither compressible nor weakly compressible; namely, it does not satisfy both conditions of Theorem 4 and Theorem 5 from [4].

The following proposition provides a proof that Theorem 3 is a generalization of Theorem 5 from [4].

PROPOSITION 4. *If an $(n+1)$-ary relation R on S is weakly compressible, strongly 2-complete and nA_1-transitive, then it is p-transitive.*

Proof. Let $(a_1^{n+1}) \in R$, $(b_1^{n+1}) \in R$ and $a_i = b_j = a$. Suppose that

$x \in \{a_1, \ldots, a_{i-1}, b_1, \ldots, b_{j-1}\}$, $y \in \{a_{i+1}, \ldots, a_{n+1}, b_{j+1}, \ldots, b_{n+1}\}$.

From the strong 2-completeness and weak compressibility of R it follows $(x, \overset{n}{a}) \in R$ and $(\overset{n}{a}, y) \in R$. By nA_1-transitivity we have $(x, \overset{n-1}{a}, y) \in R$, which means that R is p-transitive.

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NOVE GENERALIZACIJE RELACIJA PORETKA

U radu je data definicija p-tranzitivnosti generalisane relacije i pomoću nje nova generalisana relacija poretka kao i njene veze sa binarnom relacijom. Poopšteni su neki raniji rezultati.

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