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ABOUT THE COINCIDENCE OF WEIGHT AND NETWORK WEIHGT FOR MAPPINGS

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Abstract. We give sufficient conditions for the coincidence of weight and network weight for a mapping. In particular, we prove the equality $w(f) = nw(f)$ for each Čech-complete mapping f.

All spaces in this paper are topological spaces and all mappings are continuous. If $f':X' \to Y$, $X \subset X'$ and $f'|X=f$, then f is a **submapping** of f'. The following definitions for mappings were introduced by B. A. Pasynkov [1].

Let $f:X \to Y$ be a mapping from a space X into a space Y. Then:

1) f is called a **T_2-mapping** (or a **Hausdorff mapping**) if for every pair of different points $x, x' \in X$ with $f(x) = f(x')$, there exist open neighbourhoods U of x and V of x' such that $U \cap V = \emptyset$.

2) f is called **regular** if for every closed set $F \subset X$ and every point $x \in X \setminus F$ there exists an open set $O=O(f(x)) \subset Y$ such that x and the set $F \cap f^{-1}(O)$ have disjoint neighbourhoods.

3) f is **normal** if for every pair of disjoint closed subsets $A.B \subset X$ and every point $y \in Y$ there exists a neighbourhood O of y for which the sets $A \cap f^{-1}(O)$ and $B \cap f^{-1}(O)$ have disjoint neighbourhoods.

4) f is **bicompact** if it is a closed mapping and all the fibers $f^{-1}(y)$, $y \in Y$, are compact subsets of X.

5) A family $\mathcal{N}$ of subsets of X is a **network for f** if the family $\{A \cap f^{-1}(O)$: $A \in \mathcal{N}$ and O is open in Y is a network in X (in the sense of Arhangel'skii). If all the sets $A \in \mathcal{N}$ are open, then $\mathcal{N}$ is called a **base of f**. The **network weight of f** [**weight of f**] , denoted nw(f) [w(f)], is the smallest cardinal number of the form $|\mathcal{N}|$, where $\mathcal{N}$ is a network for f [a base for f].

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6) An **external base** for a submapping $f:X \to Y$ of a mapping $f':X' \to Y$ is a family $\mathcal{B}$ of open subsets of X' with the property: for every $x \in X$ and every its X'-neighbourhood U there exist $V \in \mathcal{B}$ and an open set $O \subset Y$ for which $x \in V \cap f'^{-1}(O) \subset U$.

LEMMA. Let $f':X' \to Y$ be a Hausdorff bicompact mapping and $f:X \to Y$ a submapping of f'. If Y has the following property

(*) for every $Z \subset Y$, every open cover of Z has a τ-disjoint open refinement, τ a cardinal,

then the following conditions are equivalent:

(1) $nw(f) \le \tau$ and there exists a family λ of open subsets of X' with $|\lambda| \le \tau$ such that for every $x \in X$ and every $x' \in f'^{-1}f(x)$, $x' \ne x$, one can find a set $U \in \lambda$ for which $x \in U$ and $x' \notin U$.

(2) There exists a family $\mathcal{F}$ of closed subsets of X' with $|\mathcal{F}| \le \tau$ such that for every $x \in X$ and every $x' \in f'^{-1}f(x)$, $x' \ne x$, there exist $F_1, F_2 \in \mathcal{F}$ and an open set $O \subset Y$ for which $x \in F_1 \cap f'^{-1}(O)$, $x' \in F_2 \cap f'^{-1}(O)$ and $F_1 \cap F_2 \cap f'^{-1}(O) = \varnothing$.

(3) There exists a family μ of open subsets of X' with $|\mu| \le \tau$ such that for every $x \in X$ and every $x' \in f'^{-1}f(x)$, $x' \ne x$, there are sets V_1, $V_2 \in \mu$ for which $x \in V_1$, $x' \in V_2$ and $V_1 \cap V_2 = \varnothing$.

(4) There exists an external base $\mathcal{B}$ for f having cardinality $\le\tau$.

Proof. (1) $\Rightarrow$ (2). Let $\mathcal{N}$ be a network for f with $|\mathcal{N}| \le \tau$. Put

$$\mathcal{F} = \{F = [A] : A \in \mathcal{N}\} \cup \{X' \setminus U : U \in \lambda\}$$

and prove that $\mathcal{F}$ satisfies (2). Take $U \in \lambda$ with $x \in U$ and $x' \notin U$. Let $F_2 = X' \setminus U$. then $x' \in F_2 \in \mathcal{F}$. The mapping f' is regular and thus there is a neighbourhood O of $f(x)$ such that in the set $f'^{-1}(O)$ the point x and the set $F_2' = F_2 \cap f'^{-1}(O)$ have disjoint neighbourhoods V and W, respectively. Take $A \in \mathcal{N}$ and a neighbourhood O_1 of $f(x)$ such that $x \in A \cap f'^{-1}(O_1) \subset V$. From $A \subset X$ we obtain $A \cap f^{-1}(O_1) = A \cap f'^{-1}(O_1) \equiv A' \subset V$. Let us denote $O_2 = O \cap O_1$, $A'' = A' \cap f'^{-1}(O_2)$, $V' = V \cap f'^{-1}(O_2)$, $F_2'' = F_2' \cap f'^{-1}(O_2)$, $W' = W \cap f'^{-1}(O_2)$. Then $x \in A'' \subset V'$, $x' \in F_2'' \subset W'$ and $[V'] \cap W' = \varnothing$. From $A = A'' \cup (A \setminus A'')$ and $A \setminus A'' \subset X' \setminus f'^{-1}(O_2)$ we obtain $[A \setminus A''] \cap f'^{-1}(O_2) = \varnothing$. Then $[A] \cap f'^{-1}(O_2) = [A''] \cap f'^{-1}(O_2) \subset [A''] \subset [V']$, $[A''] \cap W' = \varnothing$ and $[A''] \cap F_2'' = \varnothing$. Let $F_1 = [A]$. Then $x \in F_1 \cap f'^{-1}(O_2)$, $x' \in F_2 \cap f'^{-1}(O_2)$ and $F_1 \cap F_2 \cap f'^{-1}(O_2) = \varnothing$.

(2) $\Rightarrow$ (3). Let $F_a, F_b \in \mathcal{F}$. Take the greatest open set $O_{ab} \subset Y$ with the property $F_a \cap F_b \cap f'^{-1}(O_{ab}) = \varnothing$. The mapping f' is normal, and thus every point $y \in O_{ab}$ has a neighbourhood $O(y) \subset O_{ab}$ such that in the set

$f'^{-1}(O(y))$ the sets $F_a \cap f'^{-1}(O(y))$ and $F_b \cap f'^{-1}(O(y))$ have disjoint neighbourhoods. The open cover $\{O(y): y \in O_{ab}\}$ of O_{ab}, according to (*) has an open τ-disjoint refinement $\Omega_{ab} = \cup \{ \Omega_{abc}: c \in C, |C| \le \tau \}$, where all Ω_{abc} are disjoint systems. For each Ω_{abc} take two X'-open disjoint sets U'_{abc} and U''_{abc} as follows: if $V \in \Omega_{abc}$, then in its inverse image $f'^{-1}(V)$, the sets $F_a \cap f'^{-1}(V)$ and $F_b \cap f'^{-1}(V)$ have disjoint neighbourhoods $U'(V)$ and $U''(V)$; we put $U'_{abc} = \cup \{U'(V): V \in \Omega_{abc}\}$ and $U''_{abc} = \cup \{U''(V): V \in \Omega_{abc}\}$. Let $\mu_{ab} = \{U'_{abc}: c \in C\} \cup \{U''_{abc}: c \in C\}$ and $\mu = \cup_{a,b} \mu_{ab}$. Then μ satisfies (3).

(3) $\Rightarrow$ (4). Let $\mathcal{B}$ consist of all different members of μ and all their (different) finite unions and intersections. We will show that $\mathcal{B}$ is an external base for f. Take a point $x \in X$ and a X'-neighbourhood U of x. The set $M = f'^{-1}f(x) \setminus U$ is bicompact, so that from this and from (3) it follows that we may choose a finite number $V_1, \dots, V_n$ of neighbourhoods of some points $x_1, \dots, x_n \in M$ and neighbourhoods $W_1, \dots, W_n$ of x such that $V_i \cap W_i = \emptyset$, $1 \le i \le n$, and $V = V_1 \cup \dots \cup V_n$ is a neighbourhood of M. Then $U^* = U \cup V$ is a neighbourhood of $f'^{-1}f(x)$. Let $W = W_1 \cap \dots \cap W_n$. Then $V \cap W = \emptyset$. Since f' is a closed mapping, there is a neighbourhood O of f(x) for which $f'^{-1}(O) \subset U^*$. Then $W \cap f'^{-1}(O) \subset U$, i.e. $\mathcal{B}$ is an external base of f such that $|\mathcal{B}| \le \tau$.

(4) $\Rightarrow$ (1). Let $\mathcal{B}$ be as in (4). We are going to prove that $\mathcal{B}$ may be λ from (1). Let $x \in X$ and $x' \in f'^{-1}f(x)$, $x' \ne x$. As f' is a T_1-mapping we can choose a neighbourhood U of x with $x' \notin U$. By (4), there exist a set $V \in \mathcal{B}$ and an open set $O \subset Y$ such that $x \in f'^{-1}(O) \cap V \subset U$. Then $x \in V$ and $x' \notin V$. On the other hand, it is easily seen that $\mathcal{B}_X = \{B \cap X : B \in \mathcal{B}\}$ is a base for the mapping f and $|\mathcal{B}_X| \le \tau$.

The lemma is proved.

A mapping $f: X \to Y$ is **Čech-complete** if it has a Hausdorff bicompactification $uf: uX \to Y$ ($X \subset uX$, $uf|X = f$) such that X is a G_δ-set in uX.

THEOREM. *Let $f: X \to Y$ be a Čech-complete mapping and $nw(f) \le \tau$. If the space Y has property (*), then $w(f) \le \tau$ (and thus $w(f) = nw(f)$).*

NOTE. If in the above theorem Y is the one-point space, we get the following known result of Arhangel'skii: for every Čech-complete space X we have $w(X) = nw(X)$.

REFERENCES

[1] B.A. PASYNKOV, *The extension of some concepts and statements concerning spaces to mappings,* In: Mappings and functors, MGU, Moscow, 1984, 72-102.

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O JEDNAKOSTI TEŽINE I MREŽNE TEŽINE PRESLIKAVANJA

Teorema Arhangel'skog o jednakosti težine i mrežne težine Čech-komletnih topoloških prostora prenosi se na slučaj neprekidnih Čech-kompletnih preslikavanja.

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