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A SHORT NOTE ON PERFECT SPLITTABILITY

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Abstract. We give a general theorem about perfect splittability of topological spaces. We also answer the question whether a space that is perfectly splittable over the class of strongly Frechet-Urysohn spaces is itself strongly Frechet-Urysohn.

A topological space X is said to be **[perfectly] splittable over a class** $\mathcal{P}$ of topological spaces if for every $A \subset X$ there are a space $Y \in \mathcal{P}$ and a [perfect] continuous mapping $f: X \to Y = f(X)$, such that $f^{-1}f(A) = A$ (see [1], [3], [5]). All spaces will be Hausdorff. τ is a cardinal.

In [5], it was shown the following result:

PROPOSITION 1. *If a space X is perfectly splittable over the class $\mathcal{P}$, then $X \in \mathcal{P}$ for the following classes $\mathcal{P}$ of spaces Y: (i) Y is a Moore space, (ii) Y is a σ-space, (iii) Y is metrizable, (iv) $w(Y) \le \tau$, (v) $nw(Y) \le \tau$.*

Here we prove a theorem which provides direct proofs of all these assertions and has other intersting consequences (see also [3]).

THEOREM 1. *If X is perfectly splittable over the class of all spaces of cardinality $\le\tau$, then $|X| \le \tau$.*

Proof. Let $A \subset X$. Choose a perfect mapping f from X onto a space Y with $|Y| \le \tau$ such that $f^{-1}f(A) = A$. So, $A = \cup\{f^{-1}(y): y \in f(A)\}$ is the union of $\le\tau$ compact sets. Now we use the following result recently shown by J. Gerlits, A. Hajnal and Z. Szentmiklóssy: if any subset of a space X is the union of $\le\tau$ compact sets, then $|X| \le \tau$. (In [4], it was shown: if $\tau^{\omega} = \tau$, then $|X| \le \tau$ iff every $A \subset X$ is the union of $\le\tau$ compact sets.)

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NOTE. In terms of splittable versions of cardinal functions (see [3], [5]) the above result can be reformulated as follows: for every space X we have $|X| = |X|_{p,s}$.

In the proof of the following theorem we will use a known lemma.

LEMMA. *If S is a set of cardinality* $\le 2^{\tau}$, *then there exists a point separating family* γ *of subsets of X such that* $|\gamma| \le \tau$.

THEOREM 2. *Let* $\mathcal{P}$ *be a class of topological spaces which is hereditary,* τ*-multiplicative and with each compact member (of* $\mathcal{P}$*) has cardinality* $\le 2^{\tau}$. *If a space X is perfectly splittable over* $\mathcal{P}$, *then* $X \in \mathcal{P}$.

Proof. In fact, we will prove that $X \in \mathcal{P}$ if X is splittable over $\mathcal{P}$ and there exists a perfect mapping $f: X \to Y$ from X onto some member $Y \in \mathcal{P}$. For every $y \in Y$ the set $A_y = f^{-1}(y)$ is compact and since it is also perfectly splittable over spaces having cardinality $\le 2^{\tau}$, by Theorem 1 we have $|A_y| \le 2^{\tau}$. By Lemma, in every A_y one can find a point separating family γ_y of subsets of A_y with $|\gamma_y| \le \tau$. Let $\gamma_y = \{A_{y,\alpha} : \alpha \in \tau\}$. For every $\alpha \in \tau$, put $A_\alpha = \cup \{A_{y,\alpha} : y \in Y\}$ and fix a space $Y_\alpha \in \mathcal{P}$ and a mapping $f_\alpha : X \to Y_\alpha$ such that $f_\alpha^{-1} f_\alpha(A_\alpha) = A_\alpha$. Let φ be the diagonal product of f and all f_α, $\alpha \in \tau$. Then φ is a perfect mapping from X onto a space in $\mathcal{P}$. We are going to prove that φ is one-to-one. Take a.b $\in$ X, $a \ne b$. There exist two possibilities:

(1) $f(a) \ne f(b)$. Clearly, then $\varphi(a) \ne \varphi(b)$.

(2) $f(a) = f(b) = y \in Y$. Then $a, b \in A_y$. Since γ_y is a point separating family, there is some $A_{y,\alpha} \in \gamma_y$ with $a \in A_{y,\alpha}$, $b \notin A_{y,\alpha}$, so that $a \in A_\alpha$ and $b \notin A_\alpha$. This means $\varphi(a) \ne \varphi(b)$. Hence, φ is a homeomorphism between X and $\varphi(X) \in \mathcal{P}$, i.e. $X \in \mathcal{P}$. The theorem is proved.

REMARK. In [3], it was shown an analogous result using the assumption "$\psi(Y) \le \tau$ for every $Y \in \mathcal{P}$" instead of "every compact member of $\mathcal{P}$ has cardinality $\le 2^{\tau}$". The countable version of Theorem 2 was announced by A. Arhangel'skii and B. Šapirovskii (we do not know their proof).

In [5], it was proved: if a space X is perfectly splittable over the class of Frechét-Urysohn spaces, then X is also Frechét-Urysohn. In [2], it is proved that a regular scattered space is sequential if it is closed splittable over he class of sequential spaces. Here we give a similar result for the strongly Frechét-Urysohn case.

Recall that a space X is said to be **strongly Fréchet-Urysohn** if, whenever $\{A_n : n \in \omega\}$ is a decreasing sequence of subsets of X and $x \in X$ a point with $x \in \cap\{\bar{A}_n : n \in \omega\}$, then there exist $x_n \in A_n$ such that the sequence $(x_n : n \in \omega)$ converges to x. A space X is strongly Frechét-Urysohn if and only if $X \times I$ is Frechét-Urysohn [6] (where I is the closed unit interval).

THEOREM 3. *If a Tychonoff space X is perfectly splittable over the class $\mathcal{P}$ of strongly Frechét-Urysohn spaces and $\psi(X) \leq \omega$, then $X \in \mathcal{P}$.*

Proof. Let $(A_n : n \in \omega)$ be a decreasing sequence of subsets of X accumulating at $x \in X$. Put $A = \cup\{A_n : n \in \omega\}$ and take a space $Y \in \mathcal{P}$ and a perfect mapping $f: X \to Y$ such that $f^{-1}f(A) = A$. Since X is a Tychonoff space and $\psi(X) \leq \omega$, it follows that each singleton is a zero-set and so there exists a continuous mapping $g: X \to I$ such that $\{x\} = g^{-1}(0)$. Let us put $Z = Y \times I$ and let h be the diagonal product $f \Delta g$. Then $h: X \to Z$ is perfect, Z is strongly Frechét-Urysohn (see [6; 4.D.4]) and we have $A = h^{-1}h(A)$ and $\{x\} = h^{-1}h(x)$. Clearly $h(x) \in h(\bar{A}_n)$ for every $n \in \omega$, and consequently there exist $y_n \in h(A_n)$ such that the sequence $(y_n : n \in \omega)$ converges to $h(x)$. For every $n \in \omega$ pick a point $x_n \in h^{-1}(y_n) \cap A_n$. We claim that $(x_n : n \in \omega)$ converges to x and this is obviously enough to show that X is strongly Frechét-Urysohn. So let U be a neighbourhood of x. By the closedness of h and the fact $h^{-1}h(x) = \{x\}$, there exists a neighbourhood V of h(x) such that $h^{-1}(V) \subset U$. Because (y_n) converges to h(x) there is $n^* \in \omega$ such that for every $n > n^*$ one has $y_n \in V$ and thus $x_n \in U$ as well. The claim is proved and so the proof of the theorem is complete.

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