Zbornik radova Filozofskog fakulteta u Nišu
Serija Matematika 3(1989), 73-82

Svetislav M. Minčić

FRENET FORMULAS FOR CURVES IN A GENERALIZED RIEMANNIAN SPACE

SUMMARY. In a generalized Riemannian space GR_N, as defined by L.P.Eisenhart [1],[2], the basical tensor is nonsymmetric and it is posible to define two kinds of absolute derivative of a vector and, of this base, in the present worktwo kinds of Frenet formulas of a curve are obtained. Further, Frenet formulas for curves in the associated Riemannian space R_N are observed, and then relations between two kinds of curvatures of a curve in the GR_N and the curvatures of the same curve in the R_N are established.

0. INTRODUCTION

L.P.Eisenhart has defined in the works [1],[2] a generalized Riemannian space of N dimension (GR_N), as a space with nonsymmetric basical tensor $g_{ij} \neq g_{ji}$.

Let us denote by $g_{\underline{ij}}$ and $g_{\underset{\vee}{ij}}$ the symmetric, respectively antisymmetric part of g_{ij} (and analogically in other cases). Letting down and getting up of indices one performs by $g_{\underline{ij}}$ and $g^{\underline{ij}}$ respectively, where

$$g_{\underline{ij}} g^{\underline{kj}} = \delta^k_i . \tag{0.1}$$

If by the comma one denotes usual partial derivative, for example, $g_{ij,k} = \partial g_{ij}/\partial x^k$, one can define generalized Cristoffel symboles:

$$\Gamma_{i.jk} = \tfrac{1}{2}(g_{ji,k} - g_{jk,i} + g_{ik,j}), \quad \Gamma^{\ i}_{jk} = g^{\underline{ip}}\Gamma_{p.jk}, \tag{0.2a,b}$$

which are nonsymmetric on the indices j,k too.

Because of nonsymmetry of the generalized Cristoffel symbols, one can define in GR_N two kinds of covariant and absolute derivative of a vector. For example, denoting by $\underset{1}{|}$, $\underset{2}{|}$ covariant derivatives of the first and the second kind, by D/Ds and $\delta/\delta s$ absolute derivatives with regard to the arc along some curve, we have for a vector u^i:

AMS Subject Classification (1980):53B05

(0.3a) $$\frac{Du^i}{Ds} = u^i_{\underset{1}{|}m}\frac{dx^m}{ds} = \frac{du^i}{ds} + \Gamma^i_{pm}u^p\frac{dx^m}{ds},$$

(0.3b) $$\frac{\delta u^i}{\delta s} = u^i_{\underset{2}{|}m}\frac{dx^m}{ds} = \frac{du^i}{ds} + \Gamma^i_{mp}u^p\frac{dx^m}{ds}$$

It is easy to prove (see [3]) that

(0.4a, b) $$g_{\underline{ij}\underset{1}{|}m} = g_{\underline{ij}\underset{2}{|}m} = 0, \quad Dg_{\underline{ij}}/Ds = 0, \quad \delta g_{\underline{ij}}/\delta s = 0,$$

and the same is for $g^{\underline{ij}}$.

1. FRENET FORMULAS OF THE FIRST KIND

If a curve C is given in the GR_N by the equations

(1.1) $$x^i = x^i(s) \qquad (s=1,\dots N),$$

then

(1.2) $$t^i = dx^i/ds$$

is a unit tangent vector of this curve, i.e. it is

(1.3) $$g_{\underline{ij}}t^i t^j = e = \pm 1.$$

Hence, taking into consideration (0.4), by absolute differentiation of the first kind one obtains

$$g_{\underline{ij}}\left(\frac{Dt^i}{Ds}t^j + t^i\frac{Dt^j}{Ds}\right) = 0,$$

i.e.

(1.4) $$g_{\underline{ij}}\frac{Dt^i}{Ds}t^j = 0 \text{ i.e. } \frac{Dt^i}{Ds} \perp t^i,$$

and we can write

(1.5) $$Dt^i/Ds = k_{(1)}t^i_{(1)},$$

where $t^i_{(1)}$ is *a unit vector of the first normal* of the curve C, and $k_{(1)} > 0$ *the first curvature* of the curve. (We shall prove later that one obtains the same by the derivative of the second kind too.). Accordingly,

(1.6a, b) $$g_{\underline{ij}}t^i_{(1)}t^j_{(1)} = e_{(1)} = \pm 1, \quad g_{\underline{ij}}t^i t^j_{(1)} = 0.$$

Taking absolute derivative of the first kind of (1.6a), we conclude

(1.7) $$Dt^i_{(1)}/Ds \perp t^i_{(1)},$$

and from here and (1.6b) we see that the vector $t^i_{(1)}$ is orthogonal to t^i and to $Dt^i_{(1)}/Ds$. Now, we can choose a unit vector $t^i_{(2)}$ in the two-dimensional space, which is determined by t^i and $Dt^i_{(1)}/Ds$, so as to be

(1.8a-c) $$g_{\underline{ij}}t^i_{(2)}t^j_{(2)} = e_{(2)} = \pm 1, \quad g_{\underline{ij}}t^i t^j_{(2)} = 0, \quad g_{\underline{ij}}t^i_{(1)}t^j_{(2)} = 0,$$

and we can express

(1.9) $$Dt^i_{(1)}/Ds = at^i + k_{(2)}t^i_{(2)},$$

where we ought to determine the scalar a . Compunding the preceding equation by $g_{\underline{ij}}t^j$ and using (1.8b),one obtains

(1.10) $$g_{\underline{ij}}\frac{Dt^i_{(1)}}{Ds}\, t^j = ae.$$

On the other hand,by absolute differentiation of the equation (1.6b) and using (1.5):

$$g_{\underline{ij}}\left(\frac{Dt^i}{Ds}t^j_{(1)} + t^i\frac{Dt^j_{(1)}}{Ds}\right) = 0 \Rightarrow g_{\underline{ij}}\frac{Dt^i_{(1)}}{Ds}t^j = -g_{\underline{ij}}\frac{Dt^i}{Ds}t^j_{(1)} =$$

$$= -g_{\underline{ij}}k_{(1)}t^i_{(1)}t^j_{(1)} = -e_{(1)}k_{(1)},$$

and herefrom and from (1.10):

$$a = -e_{(1)}k_{(1)}/e = -ee_{(1)}k_{(1)},$$

and the equation (1.9) becomes

(1.9´) $$Dt^i_{(1)}/Ds = -ee_{(1)}k_{(1)}t^i + k_{(2)}t^i_{(2)}.$$

The vector $t^i_{(2)}$ is *a unit vector of the second normal of the first kind* of the curve C,and the scalar $k_{(2)}$ is *the second curvature of the first kind* of the curve.

From (1.8a) one concludes

(1.11a) $$g_{\underline{ij}}\frac{Dt^i_{(2)}}{Ds}t^j_{(2)} = 0 \text{ i.e. } Dt^i_{(2)}/Ds \perp t^i_{(2)},$$

and from (1.8b,5,8c) it is

$$g_{\underline{ij}}\left(\frac{Dt^i}{Ds}t^j_{(2)} + t^i\frac{Dt^j_{(2)}}{Ds}\right) = 0 \Rightarrow g_{\underline{ij}}t^i\frac{Dt^j_{(2)}}{Ds} =$$

$$= -g_{\underline{ij}}\frac{Dt^i}{Ds}t^j_{(2)} = -g_{\underline{ij}}k_{(1)}t^i_{(1)}t^j_{(2)} = 0,$$

i.e.

(1.11b) $$Dt^i_{(2)}/Ds \perp t^i.$$

Hence and from (1.11a) we see that the vector $Dt^i_{(2)}/Ds$ is normalto t^i and to $t^i_{(2)}$.Because $t^i_{(1)}$ is also normal to t^i and to$t^i_{(2)}$,the vectors $t^i_{(1)}$ and $Dt^i_{(2)}/Ds$ for $N > 3$ in a general case determine a twodimensional space, which is normal to t^i and $t^i_{(2)}$. Let us consider in this twodimensional space the unit vector $t^i_{(3)}$ to be normal to $t^i_{(1)}$. So, it will be

$$(1.12a\text{-}d)\quad g_{\underline{ij}}t^i_{(3)}t^j_{(3)}=e_{(3)}=\pm 1,\ g_{\underline{ij}}t^it^j_{(3)}=0,\ g_{\underline{ij}}t^i_{(1)}t^j_{(3)}=0,\ g_{\underline{ij}}t^i_{(2)}t^j_{(3)}=0.$$

The vector $Dt^i_{(2)}/Ds$ can be writen

$$(1.13)\qquad Dt^i_{(2)}/Ds = bt^i_{(1)}+k_{(3)}t^i_{(3)},$$

where it needs determine the scalar b. Compounding by $g_{\underline{ij}}t^j_{(1)}$, from the previous equation one gets

$$(1.14)\qquad g_{\underline{ij}}\frac{Dt^i_{(2)}}{Ds}t^j_{(1)} = e_{(1)}b,$$

and from (1.8c, 9′, 8b):

$$g_{\underline{ij}}\Big(\frac{Dt^i_{(1)}}{Ds}t^j_{(2)}+t^i_{(1)}\frac{Dt^j_{(2)}}{Ds}\Big)=0 \Rightarrow g_{\underline{ij}}\frac{Dt^i_{(2)}}{Ds}t^j_{(1)}=-g_{\underline{ij}}\frac{Dt^i_{(1)}}{Ds}t^j_{(2)} =$$

$$=-g_{\underline{ij}}(-ee_{(1)}k_{(1)}t^i+k_{(2)}t^i_{(2)})t^j_{(2)}=0-e_{(2)}k_{(2)}.$$

From here and from (1.14) it is $b=-e_{(1)}e_{(2)}k_{(2)}$, and (1.13) becomes

$$(1.13')\qquad Dt^i_{(2)}/Ds = -e_{(1)}e_{(2)}k_{(2)}t^i_{(1)}+k_{(3)}t^i_{(3)},$$

where $t^i_{(3)}$ is *a unit vector of third normal of the first kind* of the curve C, and $k_{(3)}$ *the third curvature of the first kind* of the curve C.

Continuing the preceding procedure, we conclude that in the end

$$(1.15)\qquad Dt^i_{(N-1)}/Ds = -e_{(N-2)}e_{(N-1)}k_{(N-1)}t^i_{(N-2)}.$$

The equations (1.5, 9′, 13′, ..., 15) are *Frenet formulas of the first kind* for curves in GR_N. One can write these formulas by one equation

$$(1.16)\quad Dt^i_{(r)}/Ds = -e_{(r-1)}e_{(r)}k_{(r)}t^i_{(r-1)}+k_{(r+1)}t^i_{(r+1)},$$

for $r=0,1,\dots,N-1$, where it is $g_{\underline{ij}}t^i_{(r)}t^j_{(q)}=e_{(r)}\delta_{rq}$, $t^i_{(o)}=t^i=dx^i/ds$, $k_{(o)}=k_{(N)}=0$. The vector $t^i_{(r)}$ is *the unit vector of the r-th normal of the first kind* of the curve C, and the scalar $k_{(r)}$ is *the r-thcurvature of the first kind* of C.

2. FRENET FORMULAS OF THE SECOND KIND

We can repeat the previous consideration using the second kind of absolute derivative, on the base of (0.3b), in the place of the first.

However, since for the unit tangent vector $t^i=dx^i/ds$ it is

$$(2.1)\qquad \frac{\delta t^i}{\delta s} = \frac{d^2x^i}{ds^2} + \Gamma^{\ i}_{mp}\frac{dx^p}{ds}\frac{dx^m}{ds} = \frac{Dt^i}{Ds}\ ,$$

the first Frenet formula of the second kind is identical with (1.5) i.e.

$$(2.2)\qquad \delta t^i/\delta s = k_{(1)}t^i_{(1)},$$

where $k_{(1)}, t^i_{(1)}$ have the previous meaning, i.e. (1.6) is in force. In the same way as (1.7), one proves that

$$(2.3)\qquad \delta t^i_{(1)}/\delta s \perp t^i_{(1)},$$

and herefrom and from (1.6b) we see that the vector $t^i_{(1)}$ is normal to t^i and to $\delta t^i/\delta s$. We have seen previously that $t^i_{(1)}$ is normal to t^i and Dt^i/Ds too. But, since in a general case it is $N > 3$, the vector $\delta t^i_{(1)}/\delta s$ need not be linearly dependent of the vectors t^i and $Dt^i_{(1)}/Ds$. In the twodimensional space determined by t^i and $\delta t^i_{(1)}/\delta s$ we notice the unit vector $\tau^i_{(2)}$, which is normal to t^i and we have

$$(2.4a\text{-}c)\qquad g_{\underline{ij}}\tau^i_{(2)}\tau^j_{(2)} = \varepsilon_{(2)} = \pm 1,\quad g_{\underline{ij}}t^i\tau^j_{(2)} = 0,\quad g_{\underline{ij}}t^i_{(1)}\tau^j_{(2)} = 0.$$

If one writes

$$\delta t^i_{(1)}/\delta s = \alpha t^i + \chi_{(2)}\tau^i_{(2)},$$

we can determine α by the same procedure, as we have determined a in (1.9). So, one obtains

$$(2.5)\qquad \delta t^i_{(1)}/\delta s = -ee_{(1)}k_{(1)}t^i + \chi_{(2)}\tau^i_{(2)}.$$

Continuing this procedure, we get

$$(2.6)\qquad \delta\tau^i_{(2)}/\delta s = -e_{(1)}\varepsilon_{(2)}\chi_{(2)}t^i_{(1)} + \chi_{(3)}\tau^i_{(3)},$$

$$(2.7)\qquad \delta\tau^i_{(3)}/\delta s = -\varepsilon_{(2)}\varepsilon_{(3)}\tau^i_{(2)}\chi_{(3)} + \chi_{(4)}\tau^i_{(4)},$$

. . . .

$$(2.8)\qquad \delta\tau^i_{(N-1)}/\delta s = -\varepsilon_{(N-2)}\varepsilon_{(N-1)}\chi_{(N-1)}\tau^i_{(n-2)},$$

and generally

$$(2.9)\qquad \delta\tau^i_{(r)}/\delta s = -\varepsilon_{(r-1)}\varepsilon_{(r)}\chi_{(r)}\tau^i_{(r-1)} + \chi_{(r+1)}\tau^i_{(r+1)},$$

where

$$r = 0,1,\dots,N-1;\quad \tau^i_{(o)} = t^i_{(o)} = t^i;\quad \tau^i_{(1)} = t^i_{(1)};\quad \chi_{(o)} = \chi_{(N)} = o,$$

$$\chi_{(1)} = k_{(1)},\ g_{\underline{ij}}\tau^i_{(r)}\tau^j_{(q)} \equiv \varepsilon_{(r)}\delta_{rq},\quad \varepsilon_{(o)} = \varepsilon = e.$$

The vector $\tau^i_{(r)}$ is *the unit vector of the r-th normal of the second kind* of the curve C, the scalar $\chi_{(r)}$ is *the r-th curvature of the second kind* of the curve C in GR_N, and the equations (2.2, 5-9) are *Frenet formulas of the second kind* of the curve C in GR_N.

3. FRENET FORMULAS IN THE ASSOCIATED RIEMANNIAN SPACE

On the same manifold, on which generalized Riemannian space GR_N is defined, one can define usual Riemannian space R_N, using as thebasical tensor $g_{\underline{ij}}$. In this case the symmetric part of the connexion Γ^i_{jk} i.e. $\Gamma^i_{\underline{jk}}$ one forms by $g_{\underline{ij}}$ and this is the connexion in R_N. We can consider the curve (1.1) as a curve in R_N and write its Frenet formulas with respect to R_N. If one denotes absolute derivative along a curve in R_N by $\bar{D}/\bar{D}s$, instead (0.3a,b) we have

$$\frac{\bar{D}u^i}{\bar{D}s} = \frac{du^i}{ds} + \Gamma^i_{\underline{pm}} u^p \frac{dx^m}{ds}, \tag{3.1}$$

and (0.3a,b) one can write respectively

$$\frac{Du^i}{Ds} = \frac{\bar{D}u^i}{\bar{D}s} + \Gamma^i_{\overset{\vee}{pm}} u^p \frac{dx^m}{ds}, \quad \frac{\delta u^i}{\delta s} = \frac{\bar{D}u^i}{\bar{D}s} - \Gamma^i_{\overset{\vee}{pm}} u^p \frac{dx^m}{ds}. \tag{3.2a,b}$$

For the curve C in R_N it is valid (1.3), and since, based on (2.1), it is

$$Dt^i/Ds = \delta t^i/\delta s = \bar{D}t^i/\bar{D}s,$$

then the first Frenet formula in R_N is identical with (1.5) i.e. (2.2):

$$\bar{D}t^i/\bar{D}s = k_{(1)} t^i_{(1)}, \tag{3.3}$$

where $k_{(1)}, t^i_{(1)}$ have the previous sence, i.e. (1.6a,b) is valid. In the same way, as in the case of the first and the second kind of derivative, we obtain the following *Frenet formulas for a curve C in the associated* R_N:

$$\bar{D}t^i_{(1)}/\bar{D}s = -ee_{(1)}k_{(1)}t^i + \bar{k}_{(2)}\bar{t}^i_{(2)}, \tag{3.4}$$

$$\bar{D}\bar{t}^i_{(2)}/\bar{D}s = -e_{(1)}e_{(2)}\bar{k}_{(2)}t^i_{(1)} + \bar{k}_{(3)}\bar{t}^i_{(3)}, \tag{3.5}$$

$$\bar{D}\bar{t}^i_{(3)}/\bar{D}s = -\bar{e}_{(2)}\bar{e}_{(3)}\bar{k}_{(3)}\bar{t}^i_{(2)} + \bar{k}_{(4)}\bar{t}^i_{(4)}, \tag{3.6}$$

. . . .

$$\bar{D}\bar{t}^i_{(N-1)}/\bar{D}s = -\bar{e}_{(N-2)}\bar{e}_{(N-1)}\bar{k}_{(N-1)}\bar{t}^i_{(N-2)}, \tag{3.7}$$

or generally

$$\bar{D}\bar{t}^i_{(r)}/\bar{D}s = -\bar{e}_{(r-1)}\bar{e}_{(r)}\bar{k}_{(r)}\bar{t}^i_{(r-1)} + \bar{k}_{(r+1)}\bar{t}^i_{(r+1)}, \tag{3.8}$$

where

$$r=o,1,\dots,N-1; \bar{t}^i_{(o)} = t^i_{(o)} = t^i, \bar{t}^i_{(1)} = t^i_{(1)},$$

$$\bar{k}_{(o)} = \bar{k}_{(N)} = 0, \bar{k}_{(1)} = k_{(1)}, g_{\underline{ij}}\bar{t}^i_{(r)}\bar{t}^j_{(q)} = \bar{e}_{(r)}\delta_{rq}, \bar{e}_{(o)} = e.$$

The vector $\bar{t}^i_{(r)}$ is *the unit vector of the r-th normal* of the curve C in the R_N, the scalar $\bar{k}_{(r)}$ is *the r-th curvature* of C in R_N.

4. RELATIONS OF THE CURVATURES OF THE FIRST AND THE SECOND KIND WITH THE CURVATURES OF THE SAME CURVE IN THE ASSOCIATED RIEMANNIAN SPACE

We shall see that one can express the curvatures $k_{(r)}, \chi_{(r)}$ by $\bar{k}_{(r)}$. On the base of (1.9′) and (3.4) it is

$$Dt^i_{(1)}/Ds-\bar{D}t^i_{(1)}/\bar{D}s = k_{(2)}t^i_{(2)}-\bar{k}_{(2)}\bar{t}^i_{(2)},$$

and from (3.2a)

$$Dt^i_{(1)}/Ds - \bar{D}t^i_{(1)}/\bar{D}s = \Gamma^{\,i}_{\underset{\vee}{pm}}t^p_{(1)}t^m,$$

i.e.

$$(4.1)\quad k_{(2)}t^i_{(2)} = \bar{k}_{(2)}\bar{t}^i_{(2)}+\Gamma^{\,i}_{\underset{\vee}{pm}}t^p_{(1)}t^m.$$

In order to find $k_{(2)}$ from here, we shall decompose $t^i_{(2)}$ by $\bar{t}^i_{(r)}$. How it is

$$t^i_{(2)}\perp t^i=t^i_{(o)}=\bar{t}^i_{(o)},\quad t^i_{(2)}\perp t^i_{(1)}=\bar{t}^i_{(1)},$$

it will be

$$(4.2)\qquad t^i_{(2)} = \sum_{p=0}^{N-1}A_{(2,p)}\bar{t}^i_{(p)} = \sum_{p=2}^{N-1}A_{(2,p)}\bar{t}^i_{(p)}\quad (A_{(2,o)}=A_{(2,1)}=o),$$

which, if we put into (4.1) and effect the composition by $g_{\underline{ij}}\bar{t}^j_{(2)}$ yields

$$k_{(2)}A_{(2,2)}\bar{e}_{(2)} = \bar{k}_{(2)}\bar{e}_{(2)}+ \Gamma_{j.\underset{\vee}{pm}}t^m t^p_{(1)}t^j_{(2)},$$

from where, considering (0.2a), the tensor $\Gamma_{i.\underset{\vee}{jk}}$ is antisymmetric on all pairs of indices, we obtain

$$(4.3)\qquad k_{(2)} =\{ \bar{k}_{(2)}-\bar{e}_{(2)}\Gamma_{i.\underset{\vee}{jk}}t^i t^j_{(1)}\bar{t}^k_{(2)}\} /A_{(2,2)}.$$

Now, let us find $k_{(3)}$. If we use (4.2) and

$$(4.4)\qquad t^i_{(3)} = \sum_{p=0}^{N-1}A_{(3,p)}\bar{t}^i_{(p)} = \sum_{p=2}^{N-1}A_{(3,p)}\bar{t}^i_{(p)}$$

and make a substitution in (1.13′) and use (3.2a), we obtain

$$k_{(3)}\sum_{p=2}^{N-1}A_{(3,p)}\bar{t}^i_{(p)}-e_{(1)}e_{(2)}k_{(2)}t^i_{(1)} = \frac{D}{Ds}\{\sum_{p=2}^{N-1}A_{(2,p)}\bar{t}^i_{(p)}\} =$$

$$= \sum_{p=2}^{N-1}\{\frac{dA_{(2,p)}}{ds}\bar{t}^i_{(p)}+A_{(2,p)}\frac{D\bar{t}^i_{(p)}}{Ds}\} = \sum_{p=2}^{N-1}\{\frac{dA_{(2,p)}}{ds}\bar{t}^i_{(p)}+A_{(2,p)}\frac{\bar{D}\bar{t}^i_{(p)}}{\bar{D}s} +$$

$$+A_{(2,p)}\Gamma^{\,i}_{\underset{\vee}{qm}}\bar{t}^q_{(p)}t^m\} = \sum_{p=2}^{N-1}\{ \frac{dA_{(2,p)}}{ds}\bar{t}^i_{(p)}+A_{(2,p)}[-\bar{e}_{(p-1)}\bar{e}_{(p)}\bar{k}_{(p)}\bar{t}^i_{(p-1)}+$$

$$+\bar{k}_{(p+1)}\bar{t}^i_{(p+1)}] + A_{(2,p)}\Gamma^i_{\underset{\vee}{qm}}\bar{t}^q_{(p)}t^m \}.$$

From here, by composition with $g_{\underline{ij}}\bar{t}^j_{(3)}$:

$$k_{(3)}A_{(3,3)}\bar{e}_{(3)} = \frac{dA_{(2,3)}}{ds}\bar{e}_{(3)} - A_{(2,4)}\bar{e}_{(3)}\bar{e}_{(4)}\bar{k}_{(4)}\bar{e}_{(3)} +$$

$$+A_{(2,2)}\bar{k}_{(3)}\bar{e}_{(3)} + \sum_{p=2}^{N-1} A_{(2,p)}\Gamma_{j.\underset{\vee}{qm}}\bar{t}^j_{(3)}\bar{t}^q_{(p)}t^m,$$

i.e.

$$(4.5)\quad k_{(3)} = \frac{1}{A_{(3,3)}}\{\frac{dA_{(2,3)}}{ds} - \bar{e}_{(3)}\bar{e}_{(4)}A_{(2,4)}\bar{k}_{(4)} + A_{(2,2)}\bar{k}_{(3)} -$$

$$-\bar{e}_{(3)}\Gamma_{i.\underset{\vee}{jk}}\sum_{p=2}^{N-1} A_{(2,p)}t^i\bar{t}^j_{(p)}\bar{t}^k_{(3)}\}.$$

Further, let us determine $k_{(r+1)}$ by $\bar{k}_{(p)}$, if we have determined $k_{(r)}$. From (1.16), by means of decomposition

$$(4.6)\qquad t^i_{(r)} = \sum_{p=o}^{N-1} A_{(r,p)}\bar{t}^i_{(p)},$$

we have

$$k_{(r+1)}\sum_{p=o}^{N-1} A_{(r+1,p)}\bar{t}^i_{(p)} - e_{(r-1)}e_{(r)}k_{(r)}\sum_{p=o}^{N-1} A_{(r-1,p)}\bar{t}^i_{(p)} =$$

$$=\frac{D}{Ds}\{\sum_{p=o}^{N-1} A_{(r,p)}\bar{t}^i_{(p)}\} = \sum_p\{\frac{dA_{(r,p)}}{ds}\bar{t}^i_{(p)} + A_{(r,p)}\frac{D\bar{t}^i_{(p)}}{Ds}A_{(r,p)}\Gamma^i_{\underset{\vee}{qm}}\bar{t}^q_{(p)}t^m\} =$$

$$=\sum_p\{\frac{dA_{(r,p)}}{ds}\bar{t}^i_{(p)} + A_{(r,p)}[-\bar{e}_{(p-1)}\bar{e}_{(p)}\bar{k}_{(p)}\bar{t}^i_{(p-1)} + \bar{k}_{(p+1)}\bar{t}^i_{(p+1)}] +$$

$$+ A_{(r,p)}\Gamma^i_{\underset{\vee}{qm}}\bar{t}^q_{(p)}t^m\}.$$

Composing by $g_{\underline{ij}}\bar{t}^j_{(r+1)}$, we obtain from here

$$k_{(r+1)}A_{(r+1,r+1)}\bar{e}_{(r+1)} - e_{(r-1}e_{(r)}k_{(r)}A_{(r-1,r+1)}\bar{e}_{(r+1)} =$$

$$=\frac{dA_{(r,r+1)}}{ds}\bar{e}_{(r+1)} - A_{(r,r+2)}\bar{e}_{(r+1)}\bar{e}_{(r+2)}\bar{k}_{(r+2)}\bar{e}_{(r+1)} +$$

$$+\bar{k}_{(r+1)}A_{(r,r)}\bar{e}_{(r+1)} + \sum_{p=o}^{N-1} A_{(r,p)}\Gamma_{j.\underset{\vee}{qm}}\bar{t}^j_{(r+1)}\bar{t}^q_{(p)}t^m,$$

whence

$$(4.7)\quad k_{(r+1)} = \{\frac{dA_{(r,r+1)}}{ds} + e_{(r-1)}e_{(r)}A_{(r-1,r+1)}k_{(r)} -$$

$$-\bar{e}_{(r+1)}\bar{e}_{(r+2)}A_{(r,r+2)}\bar{k}_{(r+2)} + A_{(r,r)}\bar{k}_{(r+1)} -$$

$$-\bar{e}_{(r+1)}\Gamma_{i.\underset{\vee}{jk}}\sum_{p=o}^{N-1} A_{(r,p)}t^i\bar{t}^j_{(p)}\bar{t}^k_{(r+1)}\}/A_{(r+1,r+1)}.$$

$$(r=0,1,\ldots,N-2)$$

This equation includes the eauations (4.3,5) as particular cases.

In (4.7), besides the magnitudes which relate to the associated space R_N, enter the torsion tensor $\Gamma_{i.\underset{\vee}{jk}}$, the coefficients $A_{(r,p)}$ of the decomposition (4.6), and the curvature $\bar{k}_{(r)}$. It remains over the determination of $e_{(r)}$ (and $e_{(r-1)}$) in (4.7). Since $e_{(o)}=\bar{e}_{(o)}=e=\pm1, e_{(1)}=\bar{e}_{(1)}=\pm1$, then for $r=2,\dots,N-1$ we have

$$e_{(r)}=g_{\underline{ij}}t^i_{(r)}t^j_{(r)}=g_{\underline{ij}}\{\sum_{p=2}^{N-1}A_{(r,p)}\bar{t}^i_{(p)}\}\{\sum_{q=2}^{N-1}A_{(r,q)}\bar{t}^j_{(q)}\}=$$

$$=\sum_{p,q}A_{(r,p)}A_{(r,q)}\bar{e}_{(p)}\delta_{pq}=\sum_p\{A_{(r,p)}\}^2\bar{e}_{(p)}.$$

We can apply the same procedure to study how to express the curvatures of the second kind by the curvatures of the same curve in the associated Riemannian space R_N. Using the decomposition

$$\tau^i_{(r)}=\sum_{p=o}^{N-1}\alpha_{(r,p)}\bar{t}^i_{(p)}, \tag{4.8}$$

we obtain

$$\chi_{(r+1)}=\{\frac{d\alpha_{(r,r+1)}}{ds}+\varepsilon_{(r-1)}\varepsilon_{(r)}\alpha_{(r-1,r+1)}\chi_{(r)}-$$

$$-\bar{e}_{(r+1)}\bar{e}_{(r+2)}\alpha_{(r,r+2)}\bar{k}_{(r+2)}+\alpha_{(r,r)}\bar{k}_{(r+1)}+$$

$$+\bar{e}_{(r+1)}\Gamma_{i.\underset{\vee}{jk}}\sum_{p=o}^{N-1}\alpha_{(r,p)}t^i\bar{t}^j_{(p)}\bar{t}^k_{(r+1)}\}/\alpha_{(r+1,r+1)}\quad (r=o,1,\dots,N-2). \tag{4.9}$$

From here, for $r=1$:

$$\chi_{(2)}=\{\bar{k}_{(2)}+\bar{e}_{(2)}\Gamma_{i.\underset{\vee}{jk}}t^it^j_{(1)}\bar{t}^k_{(k)}\}/\alpha_{(2,2)}. \tag{4.10}$$

In the case $N=3$ the vectors $t^i, \overset{\vee}{D}t^i_{(1)}/Ds, \delta t^i_{(1)}/\delta s, \bar{D}\bar{t}^i_{(1)}/\bar{D}s$ are normal to $t^i_{(1)}$ and must be linear dependent (but different), from where it is $t^i_{(2)}=\tau^i_{(2)}=\bar{t}^i_{(2)}$ and then $\alpha_{(2,2)}=A_{(2,2)}=1, e_{(2)}=\varepsilon_{(2)}=\bar{e}_{(2)}$ and from (4.3,10) we obtain

$$k_{(2)}=\bar{k}_{(2)}-e_{(2)}\Gamma_{i.\underset{\vee}{jk}}t^it^j_{(1)}t^k_{(2)},\ \chi_{(2)}=\bar{k}_{(2)}+e_{(2)}\Gamma_{i.\underset{\vee}{jk}}t^it^j_{(1)}t^k_{(2)}. \tag{4.11a,b}$$

As we see, in this case $k_{(2)}$ and $\chi_{(2)}$ are expressed by $\bar{k}_{(2)}$ and torsion tensor $\Gamma_{i.\underset{\vee}{jk}}$.

REMARK 1. It is easy to see, that in the case when GR_N reduces to R_N, the Frenet formulas of the first kind and the second kind reduce to classical formulas and $k_{(r)}=\chi_{(r)}=\bar{k}_{(r)}$ $(r=1,\dots,N-1)$.

REMARK 2. In the work [4] Saxena and Behari have studied the problem of Frenet formulas for curve in GR_N, using only one kind ofabsolute derivative (the first) and taking $\bar{e}_{(r)}=1$ $(r=0,1,\dots,N-1)$. In this work are observed only normals corresponding to the symmetric connexion and to the vectors of these normals one applies absolute derivative (of the first kind) corresponding to nonsymmetric connexion.

REFERENCES

[1] L.P.Eisenhart, Generalized Riemannian spaces I, Proc.Nat.Acad.Sci. USA, 37(1951), 311-315.

[2] ________, II, ____, 38(1952), 505-508.

[3] S.M.Mincic, Derivational formulas of a subspace of a generalized Riemannian space, Publ.Inst.Math.(Beograd), NS, t.34(48), 1983, 125-135.

[4] S.C.Saxena, Ram Behari, Some properties of generalized Riemann spaces, Proc.Nat.Inst.of Sci. India, 1960, A 26, No 2, 95-103.

S.M.Mincic

FRENEOVE FORMULE ZA KRIVE U GENERALISANOM RIMANOVOM PROSTORU

U generalisanom Rimanovom prostoru GR_N, kako ga je definisao Eisenhart [1], [2], osnovni tenzor je nesimetričan, pa je moguće definisati dve vrste apsolutnog izvoda vektora i na osnovu toga se u ovom radu dobijaju dve vrste Freneovih formula. Dalje se razmatraju Freneove formule krive u pridruženom Rimanovom prostoru R_N, a zatim su uspostavljene veze između dveju vrsta krivina krive u GR_N i krivina iste krive u R_N.

Filozofski fakultet
18 000 Niš
Jugoslavija

Аналогично могут быть получены значения тензора $R_{IJ,KL}$ для симметрических пространств, образующих реализации пространств $(\mathbb{A}\otimes\mathbb{B})\widetilde{S}_n$ в случае, когда одна из алгебр $\mathbb{A}$ и $\mathbb{B}$ или обе эти алгебры являются алгебрами $'\mathbb{C}, '\mathbb{H}$ или $'\mathbb{O}$ и значения тензора $R_{IJ,K}^{\cdot\cdot\cdot L}$ для симметрических пространств, образующих реализации пространств $(\mathbb{A}\otimes\mathbb{B})\widetilde{S}_n$ в случае, когда одна из алгебр $\mathbb{A}$ и $\mathbb{B}$ является алгеброй $°\mathbb{C}$, $°\mathbb{H}$, $°\mathbb{O}$.

ЛИТЕРАТУРА

[1] Э. КАРТАН, Риманова геометрия в ортогональном репере, Изд-во Моск унив., 1960, с.307.

[2] П.А. ШИРОКОВ, Об одном типе симметрических пространств, Матем. сборник 41(83)(1957), 361-372 [в книге: П.А. Широков, Избранные работы по геометрии, Казань, 1966].

[3] В.В. ВИШНЕВСКИЙ, А.П. ШИРОКОВ, В.В. ШУРЫГИН, Пространства над алгебрами, Изд-во Казанского унив., Казань, 1985, с. 264.

[4] Б.А. РОЗЕНФЕЛЬД, Неевклидовы геометрии, ГИТТЛ, Москва, 1956, с. 744.

[5] Б.А. РОЗЕНФЕЛЬД, Л.М. КАРПОВА, Симметрические полуримановы пространства, Изв. вузов, Математика 1(1964), 100-116.

[6] Л.М. МАРКИНА, Дуальные эрмитовы неевклидовы пространства, Уч. зап. МОНИ им. Н.К. Крупской 272(1969), 147-165.

[7] И.Г. ЛУЩИЦКАЯ, Полукватернионные пространства, Автореферат диссертации на соискание уч. степени кандидата физ.-мат. наук, Ташкент, 1968.

[8] Д.Б. ПЕРСИЦ, Геометрия над вырожденными октавами, Докл. АН СССР 173(1967), 1010-1013.

[9] Б.А. РОЗЕНФЕЛЬД, Неевклидовы пространства, Наука, Москва, 1969, с. 547.

[10] Ш. КОБАЯСИ, К. НОМИДЗУ, Основы дифференциальной геометрии, Наука, Москва, 1981, с. 414.

[11] В.Ф. КИРИЧЕНКО, О геометрии однородных К-пространств, Матем. заметки 30 (1981), 569- 582.

[12] M. PRVANOVIĆ, Holomorphically projective transformations in a locally product space, Math. Balkanica 1(1971), 195-213.

[13] Н.В. ТАЛАНТОВА, А.П. ШИРОКОВ, Проективные модели унитарных пространств постоянной кривизны над алгеброй дуальных чисел, Труды геом. семинара, вып. 16, Казань, 1984, 103-110.

[14] Б.А. РОЗЕНФЕЛЬД, С.Л. АТАНАСЯН, Т.А. ТИМОШЕНКО, Геометрия расслоений Хопфа, Изв. вузов, Математика 6(1987), 52-57.

[15] Г. ФРЕЙДЕНТАЛЬ, Октавы, особые группы Ли и октавная геометрия, Математика, сборник переводов 1(1957), 117-153.

[16] Э.Б. ВИНБЕРГ, Конструкци особых простых алгебр Ли , Труды сем. по векторному и тензорному анализу, вып. 13, МГУ, Москва, 1966, 7-9.

[17] Н.Т. АББАСОВ, Бикомплексные эллиптические пространства, Ученые записки Азерб. гос. унив., серия физ.-мат. наук 2(1962), 3-9.

[18] Н.Т. АББАСОВ, Бикватернионные эллиптичиские пространства, Ученые записки Азерб. гос. унив., серия физ.-мат. наук 2 (1963), 3-9.

[19] Л.В. РУМЯНЦЕВА, Квадрикватернионные эллиптические пространства, Ученые записки Азерб. гос. унив., серия физ.-мат. наук 3 (1963), 35-38.

[20] Б.А. РОЗЕНФЕЛЬД, Геометрические интерпретации компактных простых групп Ли класса Е, Докл. АН СССР 106 (1956), 600-603.

B.A. Rozenfel'd, T.A. Burceva, N.V. Dušina,
L.P. Kostrikina, V.V. Maljutin, T.I. Juhtina

TENZORI KRIVINE ERMITOVIH ELIPTIČKIH PROSTORA

U odnosu na adaptirani ortonormirani reper odreden je tenzor krivine sledećih prostora: simetričnih rimanovih prostora V_{2n}, V_{4n} i V_{16} ranga 1 koji su izometrični ermitovim eliptičkim prostorima $\mathbb{C}\bar{S}_n$, $\mathbb{H}\bar{S}_n$ i ravni $\mathbb{O}\bar{S}_2$; simetričnih pseudorimanovih prostora ${}^{n}V_{2n}$, ${}^{2n}V_{4n}$ i ${}^{8}V_{16}$ ranga 1. Daje se i oblik tenzora krivine simetričnih prostora višeg ranga koji su realna interpretacija ermitovih eliptičkih prostora nad tenzorskim proizvodom algebri.

Б.А. Розенфельд, ул. Удальцова 10,кв.11, Москва 117 415, СССР