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A QUADRATIC SPLINE COLLOCATION METHOD FOR SINGULAR TWO-POINT BOUNDARY VALUE PROBLEMS

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Abstract. The spline collocation method for solving boundary value problems is adapted for problems with singular perturbation. Quadratic spline is used as an approximating function. The exponential features of the exact solution are transfered to spline coefficients by the fitting factor. In this way a uniform stability of the system of matrix is achieved, while the convergence for fixed ε is preserved.

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We consider two-point boundary value problem

(1) $$Ly = \varepsilon y'' + p(x)y' + q(x)y = f(x), \quad x \in [0,1]$$

(2) $$y(0) = y(1) = 0$$

where the function p, q, f are sufficiently smooth, so that we have a unique solution $y(x) \in C^4\,[0,1]$ of (1) and (2).

We first impose a uniform partition of the interval $[0,1]$ as $x_i = ih$, $i = 0(1)n$, $h = 1/n$ with additional mesh points $x_{-2} < x_{-1} < x_0 = 0$ and $x_{n+2} > x_{n+1} > x_n = 1$. The quadratic spline has single knots at the points x_i, i.e. their second derivates suffer discontinuity at these points. We take the collocation points to be the mid-points $t_i = x_i + \frac{h}{2}$, $i = 0(1)n-1$. The quadratic B-splines which satisfy our choice of knots x_i are

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$$\tilde{B}_i(x) = \frac{1}{h^2}\begin{cases} (x_i - x_{i-1})^2, & x \in [x_{i-1}, x_i], \\ 2h^2 - (x_{i+1} - x)^2 - (x - x_i)^2, & x \in [x_i, x_{i+1}], \\ (x_{i+2} - x)^2, & x \in [x_{i+1}, x_{i+2}], \\ 0 & x \in \text{elsewhere}. \end{cases}$$

We will modify these splines to give a basis for the space of quadratic splines satisfying equation (1) and the zero boundary condition (2), as follows

$$(3) \qquad \begin{cases} B_0(x) = \tilde{B}_0(x) - \tilde{B}_{-1}(x), \\ B_i(x) = \tilde{B}_i(x), & i = 1(1)n-2. \\ B_{n-1}(x) = \tilde{B}_{n-1}(x) - \tilde{B}_n(x) \end{cases}$$

We define $\bar{y}(x)$ to be the quadratic spline of interpolation to $y(x)$. It is expanded as linear combination of basic functions

$$(4) \qquad \bar{y}(x) = \sum_{i=0}^{n-1} \bar{C}_i B_i(x).$$

Let $\tilde{y}(x) = \sum_{i=0}^{n-1} C_i B_i(x)$ be the solution of the collocation equation

$$(5) \qquad \tilde{L}_i \tilde{y}(t_i) = \delta_i \tilde{y}''(t_i) + p(t_i)\tilde{y}'(t_i) + q(t_i)\tilde{y}(t_i) = f(t_i)$$

where δ_i is a fitting factor which will be determined subsequently. We can see that

$$(6) \qquad \tilde{y}(t_i) = C_{i-1}B_{i-1}(t_i) + C_iB_i(t_i) + C_{i+1}B_{i+1}(t_i).$$

By replacing (6) into (5) we obtain the following system of the linear equations:

$$(7) \qquad \begin{cases} k_iC_{i-1} + \ell_iC_i + m_iC_{i+1} = f_i, \\ \ell_0C_0 + m_0C_1 = f_0, & i = 1(1)n-1, \\ k_{n-1}C_{n-2} + \ell_{n-1}C_{n-1} = f_{n-1}, \end{cases}$$

where

$$k_i = \frac{2}{h^2}\left(\delta_i - \frac{p_ih}{2} + \frac{q_ih^2}{8}\right), \quad i = 1(1)n-1$$

$$\ell_i = -\frac{4}{h^2}(\delta_i - \frac{3}{8} q_i h^2), \quad i = 1(1)n-2$$

$$m_i = \frac{2}{h^2}(\delta_i + \frac{p_i h}{2} + q_i \frac{h^2}{8}), \; i = 0(1)n-2$$

$$\ell_0 = -\frac{6}{h^2}(\delta_0 - \frac{p_0 h}{6} - \frac{5}{4} q_0 h^2)$$

$$\ell_{n-1} = -\frac{6}{h^2}(\delta_{n-1} + \frac{p_{n-1} h}{6} - \frac{5}{4} q_{n-1} h^2)$$

$$p_i = p(t_i), \; q_i = q(t_i), \; f_i = f(t_i).$$

Collecting this equations we obtain

$$(8) \qquad A\,C = F$$

where A is an $n \times n$ matrix and C is an n-dimensional vector with components C_i, $i = 0(1)n-1$.

LEMMA 1. Let $q(x) < 0$ and

$$(9) \qquad \delta_i \pm \frac{p_i h}{2} + q_i \frac{h^2}{8} \geqq 0$$

then A is an inverse monotone matrix, i.e.

$$A^{-1} \leqq 0.$$

Proof. Under these conditions A is a strictly dominant L matrix and according to [3] it is inverse monotone.

LEMMA 1. (uniform stability). Let $\delta_i = \varepsilon \delta_i(t_1) * \delta_2(t_i)$ where

$$\delta_1(t_i) = \frac{q_i h^2}{8\varepsilon}\left(1 + \frac{2}{sh^2(\sqrt{(q_i/\varepsilon)}h/2)}\right) \text{ (see [3])}$$

$$\delta_2(t_i) = \begin{cases} \frac{p_i h}{2\varepsilon} \operatorname{cth} \frac{p_i h}{2\varepsilon} & \text{for } p_i \neq 0 \\ 1 & \text{for } p_i = 0 \end{cases}$$

then $\|A^{-1}\|_\infty \leqq M$, where M is a constant independent of ε and h .

Proof. Since $\delta_1(t_i) = 1 + (q_i h^2/\delta\varepsilon) + \theta_i$, $\theta_i > 0$,

$\theta_i = O(h^4/\varepsilon^2)$ and xcthx $\geqq 1$ we have that

$$\delta_i \pm \frac{p_i h}{2} + \frac{q_i h^2}{8} = \frac{hp_i}{2}\left(\mathrm{cth}\,\frac{hp_i}{2\varepsilon} \pm 1\right) - \frac{q_i h^2}{8}\left(\frac{hp_i}{2\varepsilon}\,\mathrm{ch}\,\frac{hp_i}{2\varepsilon} - 1\right) +$$

$$+ Q_i\delta_2 \geqq 0.$$

Thus $k_i \geqq 0$, $m_i \geqq 0$, $\ell_i < 0$ and

$$\|A^{-1}\|_\infty < \max_i \left|\frac{1}{\ell_i + k_i + m_i}\right| = \max_i \left(-\frac{1}{2q_i}\right) \leqq M.$$

THEOREM 1. Let $\varepsilon = 1$ in (1). Let $\tilde{y}(x)$ be the quadratic spline which satisfies the collocation equations (5). Let $q(x) < 0$ and $y(x) \in C^4[0,1]$. Then

$$\|y(x) - \tilde{y}(x)\| \leqq Mh^2.$$

M is a constant independent of h, $\|\cdot\|$ is the usual, maximum norm.

P r o o f. The interpolatory spline $\bar{y}(x)$ is defined to be that quadratic spline which is equal to $y(x)$ at n collocation points $t_0,\ldots,t_{n-1}$ and in addition interpolates $y(x)$ at the two non-collocation points x_0, x_n. This information is sufficient to define $\bar{y}(x)$ uniquely. According to [1]

$$\|y^{(k)}(x) - \bar{y}^{(k)}(x)\| \leqq Mh^2, \quad k = 0,1,2; \quad x \in [0,1]$$

$$\tilde{L}\bar{y}_i - Ly_i = (\delta_i - 1)\bar{y}_i'' + \bar{y}_i'' - y_i'' + p_i(\bar{y}_i' - y_i') + q_i(\bar{y}_i - y_i).$$

Since $|\delta_i - 1| \leqq Mh^2$ ([4]) we have

$$\tilde{L}\bar{y}_i - Ly_i = O(h^2) \quad \text{and} \quad \tilde{L}\bar{y}_i = f_i + O(h^2).$$

Let $\bar{y}(x) = \sum_{i=0}^{n-1} \bar{C}_i B_i(x)$. In the same way as before we can see that constants $\bar{C}_i$ satisfy the system

$$A\bar{C} = F + O(h^2).$$

From Lemma 2 and (8) we have

$$\max_i |\bar{C}_i - C_i| \leqq Mh^2.$$

Thus,

$$\|y(x) - \tilde{y}(x)\| \leq \|y(x) - \bar{y}(x)\| + \|\bar{y}(x) - \tilde{y}(x)\| \leq Mh^2 +$$

$$+ \left\| \sum_{i=0}^{n-1} (\bar{C}_i - C_i)B_i(x) \right\| \leq Mh^2.$$

REMARK 1. In the case $p(x) = 0$ the corresponding difference scheme is derived in [3] and it has the second order of the uniform convergence.

NUMERICAL RESULTS

The calculation reported in this section have been performed on Delta 340 (PDP 11/34) computer in double precision (16 significant figures). The algorithm was written in FORTRAN IV plus and described in [2]. The test of uniform convergence is also taken from [2].

TABLE 1. contains the test of uniform convergence when scheme (8) is applied to the problem

$$-\varepsilon u'' + u = -(\cos\pi x)^2 - 2\varepsilon\pi^2(\cos 2\pi x)$$

with homogeneous conditions $u(0) = u(1) = 0$.

$\varepsilon \backslash k$	0	1	2	3	4	5	p_ε
1	2.07	2.02	2.01	2.00	2.00	2.00	2.02
2^{-1}	2.08	2.02	2.01	2.00	2.00	2.00	2.02
2^{-2}	2.08	2.02	2.05	2.00	2.00	2.00	2.03
2^{-3}	2.09	2.02	2.05	2.00	2.00	2.00	2.03
2^{-4}	2.10	2.03	2.06	2.01	2.00	2.00	2.03
2^{-5}	2.13	2.04	2.01	2.00	2.00	2.00	2.03
2^{-6}	2.18	2.05	2.01	2.00	2.00	2.00	2.04
2^{-7}	2.27	2.08	2.02	2.00	2.00	2.00	2.21
2^{-8}	2.39	2.14	2.04	2.01	2.00	2.00	2.10
2^{-9}	2.53	2.23	2.07	2.02	2.00	2.00	2.14
2^{-10}	2.61	2.36	2.12	2.03	2.01	2.00	2.19
10^{-5}	1.98	2.00	2.00	2.08	2.42	2.60	2.18
10^{-7}	1.98	2.00	2.00	2.00	2.00	2.00	2.00

In TABLE 2. is given the maximum errors $Zs = |y(x_i) - y_i|$, where y_i is approximate solution. N is a number of subintervals of interval [0,1].

$\varepsilon \backslash N$	1	2	3	4	5	6
2^{-6}	0.346E-02	0.841E-03	0.209E-03	0.521E-04	0.103E-04	0.525E-05
2^{-9}	0.385E-02	0.845E-03	0.204E-03	0.504E-04	0.126E-04	0.314E-05
2^{-10}	0.436E-02	0.883E-03	0.206E-03	0.505E-04	0.126E-04	0.314E-05
10^{-5}	0.959E-02	0.234E-02	0.583E-03	0.131E-03	0.242E-02	0.386E-05

In TABLE 3. is given corresponding test but with the problem with the first derivate term:

$$\varepsilon u'' + u' = x \qquad u(0) = u(1) = 0.$$

$\varepsilon \backslash K$	1	2	3	4	5	p_ε
1	1.99	2.00	2.00	2.00	2.00	2.00
2^{-1}	2.00	2.00	2.00	2.00	2.00	2.00
2^{-2}	1.99	2.00	2.00	2.00	2.00	2.00
2^{-3}	1.98	1.99	2.00	2.00	2.00	1.99
2^{-4}	1.87	1.98	2.00	2.00	2.00	1.97
2^{-5}	1.72	1.92	1.99	1.99	2.00	1.92
2^{-6}	1.27	1.92	1.98	1.99	1.99	1.78
2^{-7}	0.953	1.72	1.91	1.98	1.98	1.58
2^{-8}	0.901	1.32	1.71	1.92	1.92	1.38
2^{-9}	0.901	1.01	1.34	1.72	1.72	1.19
10^{-5}	0.901	0.954	0.989	0.994	0.994	0.963
10^{-7}	1.98	0.953	0.989	0.994	0.994	0.963

$\varepsilon \backslash N$	16	32	64	128	256	512
2^{-6}	0.154E-01	0.450E-02	0.218E-02	0.298E-03	0.747E-04	0.187E-04
2^{-9}	0.275E-01	0.128E-01	0.550E-02	0.197E-02	0.573E-03	0.150E-03
10^{-4}	0.293E-01	0.146E-01	0.732E-02	0.365E-02	0.182E-02	0.906E-03
10^{-6}	0.293E-01	0.147E-01	0.732E-02	0.366E-02	0.183E-02	0.915E-03

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KVADRATNI SPLAJN KOLOKACIONI METOD ZA SINGULARNI DVOTAČKASTI GRANIČNI PROBLEM

Splajn kolokacioni metod za rešavanje graničnih problema je prilagođen za singularno perturbovane probleme. Kao aproksimirajuća funkcija korišćen je kvadratni splajn. Eksponencijalna osobina tačnog rešenja je preneta na splajn koeficijente pomoću faktora fitovanja. Na taj način je postignuta uniformna stabilnost matrice, a red konvergencije je za fiksno ε očuvan.

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