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RÉDEI'S BAND OF PERIODIC π-GROUPS

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ABSTRACT: In this paper we consider GU_{n+1}-semigroups i.e. semigroups with the following condition:

$$(\forall x_1,x_2,\dots,x_{n+1})(\exists m)\quad (x_1x_2\dots x_{n+1})^m\in\langle x_1\rangle\cup\langle x_2\rangle\cup\dots\cup\langle x_{n+1}\rangle$$

and we prove that S <u>is a π-regular GU_{n+1}-semigroup if and only if</u> S <u>is a Rédei's band of periodic nil-extensions of groups (π-groups)</u> .

1. INTRODUCTION AND PRELIMINARIES

A semigroup S is a <u>U-semigroup</u> if the union of every two subsemigroups of S is a subsemigroup of S, which is equivalent with $xy\in\langle x\rangle\cup\langle y\rangle$ for all $x,y\in S$. These semigroups have been considered more a time in conection with a study of lattices of subsemigroups of some semigroup . S is a <u>GU-semigroup</u> if for every $x,y\in S$ there exists $m\in Z^+$ such that $(xy)^m\in\langle x\rangle\cup\langle y\rangle$,[2] . An other generalization for U-semigroups is the notion of U_{n+1}-semigroup: S is a U_{n+1}-<u>semigroup</u> if $x_1x_2\dots x_{n+1}\in\langle x_1\rangle\cup\langle x_2\rangle\cup\dots\cup\langle x_{n+1}\rangle$ for every $x_1,x_2,\dots,x_{n+1}\in S$,[6] . In this paper we consider a semigroup for which the following condition holds:

$$(\forall x_1,x_2,\dots,x_{n+1})(\exists m)\ (x_1x_2\dots x_{n+1})^m\in\langle x_1\rangle\cup\langle x_2\rangle\cup\dots\cup\langle x_{n+1}\rangle.$$

Such a semigroup we call GU_{n+1}-<u>semigroup</u> (<u>generalized</u> U_{n+1}-<u>semigroup</u>) .

A semigroup S is a <u>Rédei's band</u> if $xy\in\{x,y\}$

AMS Subject Classification (1980) : Primary 20 M for all $x,y \in S$. S is a *π-regular* semigroup if for every $a \in S$ there exists $m \in Z^+$ such that $a^m \in a^m S a^m$. S is a *π-group* if S is a nil-extension of a group . The main result of this paper is the following : *S is a (π-)regular GU_{n+1}-semigroup if and only if S is a Rédei's band of periodic (π-)groups* .

For non defined notions and notations we refer to [1] and [8] .

2. A GU_{n+1}-SEMIGROUP

DEFINITION 2.1. *Let $n \in Z^+$. A semigroup S is a generalized U_{n+1}-semigroup or simply GU_{n+1}-semigroup if S satisfies the following condition :*

$$(\forall x_1,x_2,\ldots,x_{n+1})(\exists m)\ (x_1x_2\ldots x_{n+1})^m \in \langle x_1\rangle \cup \langle x_2\rangle \cup \ldots \cup \langle x_{n+1}\rangle .$$

A group G is a GU_{n+1}-*group* if G is a GU_{n+1}-semigroup .

A GU_2-semigroup (GU_2-group) we call simply GU--semigroup (GU-group) .

LEMMA 2.1. *Every subsemigroup and every homomorphic image of a GU_{n+1}-semigroup is a GU_{n+1}-semigroup* . □

THEOREM 2.1. *The following conditions are equivalent:*

(i) *G is a GU_{n+1}-group* ;

(ii) *G is a GU-group* ;

(iii) *G is a periodic group* .

Proof. (i)$\Rightarrow$(iii). Let G be a GU_{n+1}-group with the identity element e . Let $x \in G$. If $n+1=2k$, $k \in Z^+$,then

$$e = ((xx^{-1})^k)^m \in \langle x\rangle \cup \langle x^{-1}\rangle$$

for some $m \in Z^+$, so $e \in \langle x \rangle$. If $n+1=2k+1$, $k \in Z^+$, then there exists $m \in Z^+$ such that

$$e = ((xx^{-1})^{k-1}x^{-1}x^2x^{-1})^m \in \langle x\rangle \cup \langle x^2\rangle \cup \langle x^{-1}\rangle = \langle x\rangle \cup \langle x^{-1}\rangle$$

whence $e \in \langle x \rangle$. Thus , in any case G is periodic .

(ii)$\Rightarrow$(iii) . This is similar with (i)$\Rightarrow$(iii).

(iii)$\Rightarrow$(ii) and (iii)$\Rightarrow$(i) follows immediately. □

Also we obtain the following :

LEMMA 2.2. Let S be a GU_{n+1}-semigroup and let $E(S) \neq \emptyset$. Then $E(S)$ is a Rédei's band .

Proof. Let $e,f \in E(S)$. Then there exists $m \in Z^+$ such that

$$(ef)^m = (e\ldots ef)^m \in \langle e \rangle \cup \langle f \rangle = \{e,f\} .$$

If $(ef)^m = e$, then $ef = (ef)^m f = (ef)^m = e$. Similarly , from $(ef)^m = f$ it follows that $ef = f$. Thus , $E(S)$ is a Rédei's band . □

DEFINITION 2.2. A band Y of semigroups $S_\alpha, \alpha \in Y$, is a GU_{n+1}-band of semigroups if for all $x_i \in S_{\alpha_i}$, $\alpha_i \in Y$, $i = 1,2,\ldots,n+1$, there exists $m \in Z^+$ such that

$$(x_1 x_2 \ldots x_{n+1})^m \in \langle x_1 \rangle \cup \langle x_2 \rangle \cup \ldots \cup \langle x_{n+1} \rangle$$

for all $\alpha_1,\ldots,\alpha_{n+1} \in Y$ such that $\alpha_i \neq \alpha_j$ for some $i,j \in \{1,2,\ldots,n+1\}$.

In a similar way we define a GU_{n+1}-semilattice and GU_{n+1}-chain of semigroups .

DEFINITION 2.3. A band Y of semigroups $S_\alpha, \alpha \in Y$, is a Rédei's band of semigroups if Y is a Rédei's band .

3. A REGULAR GU_{n+1}-SEMIGROUP

LEMMA 3.1. The following conditions on a semigroup S are equivalent:

(i) S is a completely simple GU_{n+1}-semigroup ;

(ii) S is a periodic left or right group ;

(iii) S is a left or a right zero band of perodic groups .

Proof. (i)$\Rightarrow$(ii). Let S be a completely simple GU_{n+1}-semigroup . Then $E(S) \neq \emptyset$ and by Lemma 2.2. $E(S)$ is a Rédei's band , whence S is a rectangular group (Theorem IV 3.3. [8]) . Now by Lemma 2.1. [5] and Theorem IV 3.9.[8] we have that S is a left or a right group . By Theorem 2.1. S is periodic .

(ii)$\Rightarrow$(iii). This follows immediately .

(iii)$\Rightarrow$(i). Let S be a left zero band Y of periodic groups G_α , $\alpha \in Y$. Then $E(S)$ is a left zero band . By Theorem IV 3.9. [8] it follows that S is a left group , so S is completely simple . Let $x_i \in G_{\alpha_i}$, $\alpha_i \in Y$,

$i = 1,2,\ldots,n+1$. Then $x_1x_2\ldots x_{n+1} \in G_{\alpha_1}$ so there exists $m \in Z^+$ such that

$$(x_1x_2\ldots x_{n+1})^m = e \in \langle x_1 \rangle ,$$

where e is the identity element of the group G_{α_1} . Therefore, S is a completely simple GU_{n+1}-semigroup . □

THEOREM 3.1. <u>The following conditions on a semigroup</u> S <u>are equivalent:</u>

(i) S <u>is a regular</u> GU_{n+1}<u>-semigroup</u> ;

(ii) S <u>is a</u> GU_{n+1}<u>-chain of periodic left and right groups</u> ;

(iii) S <u>is a Rédei's band of periodic groups</u> ;

(iv) S <u>is a regular GU-semigroup</u> ;

(v) S <u>is a GU-chain of periodic left and right groups</u> .

<u>Proof.</u> (i) ⇒ (ii). Let S be a regular GU_{n+1}-semigroup . For $a \in S$ there exists $x \in S$ such that $a = axa$ and $x = xax$. By Lemma 2.2. it follows that

$$(ax)(xa) = ax \quad \text{or} \quad (ax)(xa) = xa .$$

Assume that $ax^2a = xa$. If $n+1=2k$, $k \in Z^+$, then

$$\begin{aligned} xa &= ((xa)^k)^m && \text{, for every } m \in Z^+ , \\ &\in \langle x \rangle \cup \langle a \rangle && \text{, for some } m \in Z^+ . \end{aligned}$$

If $xa = x^p$ for some $p \in Z^+$, then $x = xax = x^{p+1}$ and $x^2a = x^{p+1} = x$, whence $ax = ax^2a = xa$, so $a = axa = ax^2a^2$, so $a \in aSa^2$. If $xa = a^p$ for some $p \in Z^+$, then $a = axa = a^{p+1} \in aSa^2$. Let $n+1=2k+1$, $k \in Z^+$. Then

$$\begin{aligned} xa &= (ax)(xa) = ((ax)(xa)^k)^m && \text{, for every } m \in Z^+, \\ &\in \langle ax \rangle \cup \langle a \rangle \cup \langle x \rangle && \text{, for some } m \in Z^+ , \\ &= \{ax\} \cup \langle a \rangle \cup \langle x \rangle . \end{aligned}$$

If $xa = ax$, then $a = axa = xaa = ax^2a^2 \in aSa^2$. If $xa = x^p$ for some $p \in Z^+$, then $x = xax = x^{p+1}$ and $x^2a = x^{p+1} = x$, whence $ax = ax^2a$, so $a = axa = ax^2a^2 \in aSa^2$. If $xa = a^p$ for some $p \in Z^+$, then $a = axa = a^{p+1} \in aSa^2$. Assume that $ax^2a = ax$. Then $a = axa = ax^2a^2 \in aSa^2$. Thus , in any cases $a \in aSa^2$, so by Theorem $_{IV}$ 1.6. [8] it follows that S is completely regular and so S is a semilattice Y of completely simple semigroups S_α, $\alpha \in Y$. Then by Lemma 2.1. and by

Lemma 3.1. we have that S_α is a periodic left or right group , for every $\alpha \in Y$. It is clear that Y is a chain , so S is a GU_{n+1}-chain of periodic left and right groups S_α, $\alpha \in Y$.

(ii)$\Rightarrow$(i). This implication follows by Lemma 3.1. and by the definition of the GU_{n+1}-chain of semigroups .

(ii)$\Rightarrow$(iii). Let S be a GU_{n+1}-chain Y of periodic left and right groups S_α, $\alpha \in Y$. Then S is periodic and by (i)$\Rightarrow$(ii) it follows that S is a GU_{n+1}-semigroup. Thus , $S = \bigcup_{e \in E(S)} K_e$, where K_e is a $\mathcal{K}$-class containing an idempotent e and $\mathcal{K}$ is an equivalence relation defined on S by:

$$a \mathcal{K} b \iff (\exists p,q \in Z^+) \ a^p = b^q$$

Let $\alpha \in Y$. Then by Lemma 3.1. it follows that S_α is a left or a right zero band of periodic groups , whence S_α is a band $E(S_\alpha)$ of periodic groups K_e , $e \in E(S_\alpha)$.

Let $x \in K_e$, $y \in K_f$ for some $e,f \in E(S)$. If $x,y \in S_\alpha$ for some $\alpha \in Y$, then by Lemma 3.1. we have that $xy \in K_e = K_{ef}$, if S_α is a left group , and $xy \in K_f = K_{ef}$, if S_α is a right group . Let $x \in S_\alpha$, $y \in S_\beta$, $\alpha,\beta \in Y$. If $\alpha < \beta$ then by Lemma 2.2. it follows that $ef = fe = e$ and $xy \in S_{\alpha\beta} = S_\alpha$, and by

$$\begin{aligned} (xy)^m &= (e\ldots exy)^m && \text{, for all } m \in Z^+ , \\ &\in \langle e \rangle \cup \langle x \rangle \cup \langle y \rangle && \text{, for some } m \in Z^+ , \\ &= \langle x \rangle \cup \langle y \rangle && , \end{aligned}$$

it follows that $(xy)^m \in \langle x \rangle \subseteq K_e$, so $xy \in K_e = K_{ef}$. The similar proof we have if $\beta < \alpha$. Therefore , S is a Rédei's band $E(S)$ of periodic groups K_e , $e \in E(S)$.

(iii)$\Rightarrow$(i). Let S be a Rédei's band Y of periodic groups G_α, $\alpha \in Y$. Let $x_i \in G_{\alpha_i}$ for some $\alpha_i \in Y$, $i = 1,2,\ldots,n+1$. Then

$$x_1x_2\ldots x_{n+1} \in G_{\alpha_1}G_{\alpha_2}\ldots G_{\alpha_{n+1}} \subseteq G_{\alpha_1\alpha_2\ldots\alpha_{n+1}} = G_{\alpha_k}$$

for some $k \in \{1,2,\ldots,n+1\}$, whence there exists $m \in Z^+$ such that

$$(x_1x_2\ldots x_{n+1})^m = e \in \langle x_k \rangle ,$$

where e is the identity element of a group G_{α_k} . Therefore , S is a GU_{n+1}-semigroup .

In a similar way we prove that (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v). □

4. A π-REGULAR GU_{n+1}-SEMIGROUP

THEOREM 4.1. The following conditions on a semigroup S are equivalent :

(i) S is a π-regular GU_{n+1}-semigroup and $E(S)$ is a left zero band ;

(ii) S is a retractive nil-extension of a periodic left group ;

(iii) S is a left zero band of periodic π-groups ;

(iv) S is a π-regular GU-semigroup and $E(S)$ is a left zero band ;

(v) S is π-regular and

(4.1) $(\forall x_1, x_2, \ldots, x_{n+1} \in S)(\exists m \in Z^+)\ (x_1 x_2 \ldots x_{n+1})^m \in \langle x_1 \rangle$;

(vi) S is π-regular and

(4.2) $(\forall x, y \in S)(\exists m \in Z^+)\ (xy)^m \in \langle x \rangle$.

Proof. (i) $\Rightarrow$ (iii). Let S be a π-regular GU_{n+1}-semigroup and $E(S)$ be a left zero band . Then by Theorem VI 3.2.1. [1] and by Lemma 3.1. we have that S is a nil-extension of a periodic left group T . Thus , S is periodic so $S = \bigcup_{e \in E(S)} K_e$ and by Theorem X 1. [1] we have that $\mathcal{K} = \tau = \mathcal{H}^*$ and K_e , $e \in E(S)$ are π-groups . Let $x \in K_e$, $y \in K_f$, $e, f \in E(S)$, $e \neq f$. Assume that $xy \in K_g$ for some $g \in E(S)$, i.e. $(xy)^s = g$ for some $s \in Z^+$. Then we have that there exists $k, m \in Z^+$ such that

$$(ye)^k = (ye\ldots e)^k \in \langle y \rangle \cup \langle e \rangle ,$$

and

$$(xf)^m = (xf\ldots f)^m \in \langle x \rangle \cup \langle f \rangle .$$

Let $(ye)^k = e$. Moreover , there exists $p \in Z^+$ such that

$$(ey)^p = (e\ldots ey)^p \in \langle e \rangle \cup \langle y \rangle .$$

Assume that $(ey)^p \in \langle y \rangle$, i.e. $(ey)^r = f$ for some $r \in Z^+$. Then

$$e = ef = e(ey)^r = (ey)^r = f$$

which is not possible . Therefore , $(ey)^p = e$. Let $y^t = f$, $t \in Z^+$. By $(ey)^p = e$ it follows that $ey \in K_e$ and

$$z = (ey)^{k-1}e \in K_e \ , \quad yz = e \ \text{ and } \ z \in G_e \ ,$$

where G_e is the maximal subgroup of K_e . Now we have that

$$ye = yee = yyze = y^2z = y^2ez = y^2yzz = y^3z^2 = \dots = y^qz^{q-1}$$

for every $q \in Z^+$. Hence

$$ye = y^tz^{t-1} = fz^{t-1}$$

whence

$$e = yz = yez = fz^{t-1}z = fz^t = f(fz^t) = f(yez) = f(yz) = \\ = fe = f$$

which is not possible . Hence , $(ye)^k \in \langle y \rangle \subseteq K_f$. In a similar way we obtain that $(xf)^m \in \langle x \rangle \subseteq K_e$. Thus , we have that $ye \in K_f \cap T = G_f$ and $xf \in K_e \cap T = G_e$. Now it follows that $xfye \in G_eG_f \subseteq G_e$ (Lemma 3.1.) so

$$g = ge = (xy)^se = (xy)^{s-1}xye = (xy)^{s-1}xfye = (xy)^{s-1}e(xfye) = \\ = \dots = (xfye)^s \in G_e \ .$$

Thus , $g = e$, i.e. $xy \in K_e$, so S is a left zero band $E(S)$ of periodic π-groups K_e , $e \in E(S)$.

(iii)$\Rightarrow$(ii). Let S be a left zero band Y of periodic π-groups S_α , $\alpha \in Y$. Then S is periodic , $E(S)$ is a left zero band isomorphic to Y and $S_\alpha = K_e$ if $e \in S_\alpha \cap E(S)$. Thus , S is a left zero band $E(S)$ of periodic π-groups K_e , $e \in E(S)$. By Theorem VI 3.2.1. [1] we have that S is a nil-extension of a periodic left group T . Define a function $\varphi : S \to T$ by

$$\varphi(x) = ex \quad \text{if} \quad x \in K_e \ , \ e \in E(S) \ .$$

Let $x \in K_e$, $y \in K_f$. Then $xy \in K_{ef} = K_e$, so

$$\varphi(xy) = e(xy) = (ex)y = (ex)ey = (ex)efy = (ex)e(fy) = \\ = (ex)(fy) = \varphi(x)\varphi(y) \ .$$

Therefore , φ is a retraction , so S is a retractive nil-extension of a periodic left group .

(ii)$\Rightarrow$(v). Let S be a retractive nil-extension of a periodic left group T with the retraction φ . Let $x_1, x_2, \dots, x_{n+1} \in S$. Then there exists $m \in Z^+$ such that $(x_1x_2\dots x_{n+1})^m \in T$, so

$$(x_1x_2\dots x_{n+1})^m = \varphi((x_1x_2\dots x_{n+1})^m) = (\varphi(x_1)\dots\varphi(x_{n+1}))^m \ .$$

By Lemma 3.1. it follows that there exists $k \in Z^+$ such that

$$(\varphi(x_1)\varphi(x_2)\dots\varphi(x_{n+1}))^k \in \langle\varphi(x_1)\rangle$$

whence

$$(x_1x_2\dots x_{n+1})^{km} = (\varphi(x_1)\varphi(x_2)\dots\varphi(x_{n+1}))^{km} \in \langle\varphi(x_1)\rangle .$$

Let $x_1^p = e$ for some $e\in E(S)$ and some $p\in Z^+$. By Lemma 3.1. [7] we have that

$$\varphi(x_1) = ex_1 = x_1^{p+1} \in \langle x_1 \rangle$$

so (4.1) holds .

(v)$\Rightarrow$(i). This follows immediately .

In a similar way we prove that (iv) $\Rightarrow$(iii) $\Rightarrow$(vi) $\Rightarrow$(iv) . □

A subsemigroup T of a semigroup S is a <u>retract</u> of a semigroup S if there exists a retraction of S onto T .

THEOREM 4.2. *The following conditions on a semigroup S are equivalent :*

(i) S *is a* π*-regular* GU_{n+1}*-semigroup* ;

(ii) S *is a periodic* GU_{n+1}*-semigroup* ;

(iii) S *is a* GU_{n+1}*-chain of retractive nil-extensions of periodic left and right groups* ;

(iv) S *is a Rédei's band of periodic* π*-groups* ;

(v) S *is a* π*-regular GU-semigroup* ;

(vi) S *is a periodic GU-semigroup* ;

(vii) S *is a GU-chain of retractive nil-extensions of periodic left and right groups* ;

(viii) S *contains a retract* T *which is a regular GU-semigroup and some power of each element of* S *lies in* T .

Proof. (i)$\Rightarrow$(iii). Let S be a π-regular GU_{n+1}-semigroup . Then $E(S) \neq \emptyset$ and by Lemma 2.2. E(S) is a Rédei's band , so by Proposition 1. [3] it follows that Reg(S) = T is a subsemigroup of S . Now , by Theorem 3.1. we have that Reg(S) = Gr(S) , so by Theorem X 1. [1] we have that S is a semilattice Y of semigroups S_α, $\alpha\in Y$ and S_α is a nil-extension of a completely simple semigroup T_α for every $\alpha\in Y$. By Lemmas 2.1 and 3.1. we have that T_α is a periodic left or a right group . Since E(S) is a Rédei's band , then Y is a chain , so by Theorem 4.1. it follows that (iii) holds .

(iii)$\Rightarrow$(iv). Let S be a GU_{n+1}-chain Y of semigroups S_α , $\alpha \in Y$ and S_α is a retractive nil-extension of a periodic left or right group T_α, $\alpha \in Y$. By Theorem 4.1. S_α is a left or a right zero band $E(S_\alpha)$ of π-groups K_e , $e \in E(S_\alpha)$. Since S is periodic , then $S = \bigcup_{e \in E(S)} K_e$. Let $x \in K_e$, $y \in K_f$, $e \in S_\alpha$, $f \in S_\beta$, $\alpha \neq \beta$. Assume that $\alpha < \beta$. Since S is a GU_{n+1}-chain of semigroups S_α , $\alpha \in Y$, then $ef = fe = e$. Also , $xy \in S_{\alpha\beta} = S_\alpha$. Let $xy \in K_g$ for some $g \in E(S_\alpha)$, i.e. $(xy)^k = g$ for some $k \in Z^+$. Assume that $E(S_\alpha)$ is a left zero band . Let $(ey)^t \in E(S_\alpha)$ for some $t \in Z^+$ (since $ey \in S_{\alpha\beta} = S_\alpha$) . Then $(ey)^t = e(ey)^t = e$, so $ey \in G_e$. Moreover , there exists $s \in Z^+$ such that

$$(ye)^s = (ye\ldots e)^s \in \langle y \rangle \cup \langle e \rangle ,$$

and since $ye \in S_{\beta\alpha} = S_\alpha$, $\langle y \rangle \subseteq S_\beta$, $\alpha \neq \beta$, then $(ye)^s = e$ and $ye \in G_e$. Now we have that

$$ey = (ey)e = e(ye) = ye ,$$

whence , by this and by Theorem I 4.3. [1] , it follows that

$$g = ge = (xy)^k e = e(xy)^k = eg = e .$$

Thus $xy \in K_e = K_{ef}$. The similar proof we have if $E(S_\alpha)$ is a right zero band and the similar proof we have if $\beta < \alpha$. Therefore S is a Rédei's band $E(S)$ of π-groups K_e , $e \in E(S)$.

(iv) $\Rightarrow$ (ii). Let S be a Rédei's band Y of periodic π-groups S_α , $\alpha \in Y$. Then S is periodic . Let $x_i \in S_{\alpha_i}$, $\alpha_i \in Y$, $i = 1,2,\ldots,n+1$. Then $\alpha_1\alpha_2\ldots\alpha_{n+1} = \alpha_k$ for some $k \in \{1,2,\ldots,n+1\}$, so

$$x_1x_2\ldots x_{n+1} \in S_{\ k} .$$

Since S_{α_k} is power joined , then there exist $t,s \in Z^+$ such that

$$(x_1x_2\ldots x_{n+1})^t = x_k^s \in \langle x_k \rangle .$$

Therefore S is a periodic GU_{n+1}-semigroup .

(ii) $\Rightarrow$ (i) . This follows immediately .

In a similar way we prove that (v) $\Rightarrow$ (vi) $\Rightarrow$ (vii) $\Rightarrow$ (iv) .

(iv) $\Rightarrow$ (viii). Let S be a Rédei's band Y of periodic π-groups S_α, $\alpha \in Y$. Then it is clear that S is periodic , $E(S)$ is a Rédei's band isomorphic to Y and that

$S_\alpha = K_e$ if $e \in S_\alpha \cap E(S)$. Thus S is a Rédei's band $E(S)$ of periodic π-groups K_e , $e \in E(S)$. Define a function $\varphi: S \to T$, $T = Reg(S)$ by

$$\varphi(x) = ex \quad \text{if} \quad x \in K_e \ , \ e \in E(S) \ .$$

Let $x \in K_e$, $y \in K_f$. Then $xy \in K_{ef}$. If $ef = e$, then

$$\varphi(xy) = efxy = exy = (ex)ey = (ex)efy = (ex)(fy) = \\ = \varphi(x)\varphi(y) \ .$$

If $ef = f$, then by Theorem I 4.3. [1] we have that

$$\varphi(xy) = efxy = xyef = xyf = xfyf = xefyf = (ex)(fy) = \\ = \varphi(x)\varphi(y) \ .$$

Hence , φ is a retraction . By Proposition 1. [3] , Lemma 2.1. we have that $T = Reg(S)$ is a regular GU-semigroup , and by π-regularity it follows that some power of each element of S lies in T .

(viii)$\Rightarrow$(iv). Let S contains a retract T which is a regular GU-semigroup and some power of each element of S lies in T . By Theorem 3.1. it follows that S is periodic . Let $\varphi: S \to T$ be a retraction and let T be a Rédei's band Y of periodic groups G_α, $\alpha \in Y$ (this follows by Theorem 3.1.). Let we denote

$$S_\alpha = \varphi^{-1}(G_\alpha) \quad , \quad \alpha \in Y \ .$$

Then S_α is periodic semigroup with exactly one idempotent , so S_α is a nil-extension of a periodic group , for every $\alpha \in Y$. Also , S is a Rédei's band Y of semigroups S_α, $\alpha \in Y$.□

REMARK. In Theorem 4.2. (viii) the retract T and the retraction φ are not uniquely determined . For example , it is not hard to see that $E(S)$ is also a retract with the retraction ψ with the following representation :

$$\psi(x) = e \quad \text{if} \quad x \in K_e \ , \ e \in E(S) \ .$$

By the proof of Theorem 4.2. we have that statements of this theorem holds for $E(S)$ and for this retraction .

EXAMPLE 4.1. The semigroup S given by the following table

	x	e	f
x	e	e	x
e	e	e	e
f	x	e	f

is a U-semigroup but $Reg(S) = \{e,f\}$ is not an ideal of S since $xf = x \notin Reg(S)$.

EXAMPLE 4.2. The semigroup S given by the following table

	x	e	f	g
x	e	e	g	e
e	e	e	e	e
f	f	f	f	f
g	g	g	g	g

is a nil-extension of a left zero band , but S is not a GU-semigroup .

EXAMPLE 4.3. The semigroup S given by the following table

	e	f	g	x
e	e	e	e	e
f	f	f	f	f
g	e	f	g	x
x	f	e	x	g

is a chain of GU-semigroups , but S is not a GU-semigroup .

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RÉDEI-JEVA TRAKA PERIODIČKIH π-GRUPA

U ovom radu se razmatraju GU_{n+1}-polugrupe , t.j. polugrupe sa sledećom osobinom :

$$(\forall x_1,x_2,\dots,x_{n+1})(\exists m)\ (x_1x_2\dots x_{n+1})^m \in \langle x_1\rangle\cup\langle x_2\rangle\cup\dots\cup\langle x_{n+1}\rangle$$

i pokazujemo da S jeste π-regularna GU_{n+1}-polugrupa ako i samo ako S jeste Rédei-jeva traka peridičkih nil-ekstenzija grupa (π-grupa) .

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