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$\mathcal{E}_B$-CONNECTEDNESS IN BITOPOLOGICAL SPACES

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Abstract. In this paper the concept of $\mathcal{E}$-connectedness of bitopological spaces is introduced and studied.

1. Introduction

In [4] Perwin has given a characterization of pairwise connected bitopological spaces in terms of bicontinuous functions: a bitopological space (X,T_1,T_2) is pairwise connected if every bicontinuous function from X into $(D,\mathcal{L}/D,\mathcal{R}/D)$ is constant. In this paper D stands for the set $\{0,1\}$ endowed with the left-hand topology $\mathcal{L}/D = \{\emptyset,D,\{0\}\}$ and the right-hand topology $\mathcal{R}/D = \{\emptyset,D,\{1\}\}$. The class of topological (bitopological) spaces which contain more than one point will be denoted by $\mathcal{E}$ $(\mathcal{E}_B)$. The set of all bicontinuous functions from (X,T_1,T_2) into (E,U_1,U_2) will be denoted by $BC(X,E)$.

The bitopological quasicomponent of a point x in (X,T_1,T_2) is the set of all points y in X such that x and y cannot be separated by a separation of X [8]. This quasicomponent will be denoted by Q_x.

A subset A of (X,T_1,T_2) is called p-open if $A=H_1\cup H_2$, where $H_i\in T_i$ for $i=1,2$. It is obvious that each open set in topology T_i is p-open, but the converse is not necessarily true [1].

Preuss introduced and studied $\mathcal{E}$-connectednedness oftopological spaces. A topological space X is $\mathcal{E}$-connected if every continuous function from X into Y is constant, for every $Y\in\mathcal{E}$ [6].

2. Results

DEFINITION 2.1. A bitopological space (X,T_1,T_2) is pairwise $\mathcal{E}$-connected

AMS Subject Classification (1980): 54E55 (briefly $\mathcal{E}_B$-connected) if every bicontinuous function from X into (E,U_1,U_2) is constant for each $E \in \mathcal{E}_B$.

REMARK. Every $\mathcal{E}_B$-connected bitopological space is pairwise connected, but the converse is not necessarily true. If $\mathcal{E}_B = \{D\}$ then each pairwise connected bitopological space is $\mathcal{E}_B$-connected.

DEFINITION 2.2. A subset G of bitopological space (X,T_1,T_2) is $\mathcal{E}_B$-open if there exist $E \in \mathcal{E}_B$, $f \in BC(X,E)$ and a p-open set V in E such that $f^{-1}(V)=U$. The complement of an $\mathcal{E}_B$-open set is $\mathcal{E}_B$-closed.

REMARK. Every $\mathcal{E}_B$-open set is p-open, but the converse is not necessarily true. If $\mathcal{E}_B = \{D\}$ then each p-open set is $\mathcal{E}_B$-open.

DEFINITION 2.3. A bitopological space X is $\mathcal{E}_B$-connected between (non - empty) subsets A and B if $U_j\text{-cl } f(A) \cap U_i\text{-cl } f(B) \neq \emptyset$ for each $E \in \mathcal{E}_B$, each f in BC(X,E) and $i \neq j$, $i,j=1,2$. If one of the sets A and B (or both) consists of a single point, then X is $\mathcal{E}_B$-connected between a point and a subset, respectively between a subset and a point (or between two points).

PROPOSITION 2.4. Let $\mathcal{E}_B$ be the class of PT_1 spaces. Then the following statements are equivalent:

(i) X is $\mathcal{E}_B$-connected,

(ii) X is $\mathcal{E}_B$-connected between every two subsets,

(iii) X is $\mathcal{E}_B$-connected between every two disjoint subsets,

(iv) X is $\mathcal{E}_B$-connected between each T_i-closed and each T_j-closed subset such that they are disjoint for $i \neq j$, $i,j=1,2$,

(v) X is $\mathcal{E}_B$-connected between every T_k-closed set A and every point p in $X \setminus A$, $k=1,2$,

(vi) X is $\mathcal{E}_B$-connected between every two distinct points.

PROOF. (i)$\Rightarrow$(vi) Let $x,y \in X$, $x \neq y$, $E \in \mathcal{E}_B$ and $f \in BC(X,E)$. Since X is $\mathcal{E}_B$-connected we have $f(x)=f(y)$. Hence $U_j\text{-cl}\{f(x)\} \cap U_i\text{-cl}\{f(y)\} \neq \emptyset$ for $i \neq j$, $i,j=1,2$. This means that X is $\mathcal{E}_B$-connected between any two points.

(vi)$\Rightarrow$(i) Suppose that X is not $\mathcal{E}_B$-connected. Then there exist $E \in \mathcal{E}_B$, $f \in BC(X,E)$ and two distinct points $x,y \in X$ such that $f(x) \neq f(y)$. Since E is a PT_1 space, $U_i\text{-cl}\{f(x)\} \cap U_j\text{-cl}\{f(y)\} = \{f(x)\} \cap \{f(y)\} = \emptyset$, i.e. X is not $\mathcal{E}_B$-connected between x and y.

In a similar way we prove (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(v)$\Rightarrow$(vi). [Note that this implication chain holds for any class $\mathcal{E}_B$.]

PROPOSITION 2.5. Let X be a bitopological space. Consider the following statements:

(i) X is $\mathcal{E}_B$-connected,

(ii) The $\mathcal{E}_B$-open sets in X are only X and $\emptyset$,

(iii) If $X = A \cup B$, where A and B are $\mathcal{E}_B$-open sets, then X=A or X=B,

(iv) If $X = A \cup B$, with A and B $\mathcal{E}_B$-closed sets, then X=A or X=B.

Then the implication chain (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv) holds for any class $\mathcal{E}_B$. Moreover

(a) If $\mathcal{E}_B$ is the class of PT_o spaces, then (ii)$\Rightarrow$(i),

(b) If $\mathcal{E}_B$ is the class of spaces with the property $bicl\{x\} \cap bicl\{y\} = \emptyset$ for every two distinct points x and y (in each $E \in \mathcal{E}_B$), then (iii)$\Rightarrow$(i),

(c) If $\mathcal{E}_B$ is the class of PR_1 spaces such that $U_i\text{-}cl\{x\} \cap U_j\text{-}cl\{y\} = \emptyset$ for every two distinct points x and y, $i \neq j$, $i,j=1,2$, then (iv)$\Rightarrow$(i).

PROOF. (i)$\Rightarrow$(ii) Let G be an $\mathcal{E}_B$-open set in X such that $\emptyset \neq G \neq X$. Then there exist $E \in \mathcal{E}_B$, $f \in BC(X,E)$ and a p-open set $W \subset E$ such that $G = f^{-1}(W)$. Let $x \in G$ and $y \in X \setminus G$. Then $f(x) \neq f(y)$ and f is not constant. Therefore X is not $\mathcal{E}_B$-connected, which is a contradiction.

(a) Let X is not $\mathcal{E}_B$-connected. Then there exist $E \in \mathcal{E}_B$, $f \in BC(X,E)$ and points x,y in X such that $f(x) \neq f(y)$. Since E is a PT_o space, there is a set $G \in U_1$ with $f(x) \in G$, $f(y) \notin G$ or a set $V \in U_2$ with $f(x) \notin V$, $f(y) \in V$. Then $f^{-1}(G) \neq \emptyset$, $f^{-1}(G) \neq X$, $f^{-1}(V) \neq \emptyset$, $f^{-1}(V) \neq X$. But $f^{-1}(G)$ and $f^{-1}(V)$ are $\mathcal{E}_B$-open and we have a contradiction.

(i)$\Rightarrow$(iii) is evident so that we shall prove (b). Suppose that X is not $\mathcal{E}_B$-connected. Then there are $E \in \mathcal{E}_B$, $f \in BC(X,E)$ and two distinct points $x,y \in X$ such that $f(x) \neq f(y)$. Since $bicl\{f(x)\} \cap bicl\{f(y)\} = \emptyset$ (in E), the sets $A = f^{-1}(E \setminus bicl\{f(x)\})$ and $B = f^{-1}(E \setminus bicl\{f(y)\}$ are $\mathcal{E}_B$-open, because one has $A = f^{-1}(E \setminus (U_1\text{-}cl\{f(x)\} \cap U_2\text{-}cl\{f(y)\})) = f^{-1}(V)$, where V is p-open in E (and similarly for B). Now $X = A \cup B$, $X \neq A$, $X \neq B$, which is a contradiction.

(i)$\Rightarrow$(iV) and (c) may be proved in a similar way.

COROLLARY. If $\mathcal{E}_B$ is the class of PT_1 spaces, then (iii)$\Rightarrow$(i) in Proposition 2.5. If $\mathcal{E}_B$ is the class of PT_2 spaces, then (iv)$\Rightarrow$(i) in Proposition 2.5.

DEFINITION 2.6. a) A set $G \subset X$ is called an $\mathcal{E}_B$-open neighbourhood of a subset B of X if $B \subset G$ and G is $\mathcal{E}_B$-open.

b) A set $W \subset X$ is said to be an $\mathcal{E}_B$-closed neighbourhood of a subset $B \subset X$ if there exist $E \in \mathcal{E}_B$, $f \in BC(X,E)$ and a p-open set $O \subset E$ such that $W = f^{-1}(U_k\text{-}clO)$ and $B \subset f^{-1}(O)$ for k=1,2.

c) A set $V \subset X$ is called a special $\mathcal{E}_B$-open neighbourhood of a subset $B \subset X$ if there exist $E \in \mathcal{E}_B$, $f \in BC(X,E)$ and a p-open neighbourhood O of $U_k\text{-}cl\{f(B)\}$ such that $V = f^{-1}(O)$ for k=1,2.

DEFINITION 2.7. Let X be a bitopological space and $x \in X$. Then the bitopological $\mathcal{E}_B$-quasicomponent of x is the set $\mathcal{E}_B\text{-}Q_x = \{y : f(x)=f(y) \text{ for each } E \in \mathcal{E}_B$ and each $f \in BC(X,E)\}$.

LEMMA 2.8. Let $\mathcal{E}_B$ be the class of PT_o spaces, X a bitopological space, $x \in X$ and

$$A_x = \{y : y \in X \text{ \& } y \text{ is in each } \mathcal{E}_B\text{-open neighbourhood of } x\},$$
$$B_x = \{y : y \in X \text{ \& } x \text{ is in each } \mathcal{E}_B\text{-open neighbourhood of } y\}.$$

Then $\mathcal{E}_B\text{-}Q_x = A_x \cap B_x$.

PROOF. Let $y \in A_x \cap B_x$ and $y \notin \mathcal{E}_B\text{-}Q_x$. Then there exist $E \in \mathcal{E}_B$ and $f \in BC(X,E)$ such that $f(x) \neq f(y)$. Since E is PT_o then there exists either (i) an U_1-open neighbourhood G of f(x) such that $f(y) \notin G$, or (ii) an U_2-open neighbourhood V of f(y) such that $f(x) \notin V$. Then:

(i) $f^{-1}(G)$ is an $\mathcal{E}_B$-open neighbourhood of x with $y \notin g^{-1}(G)$ which contradicts the fact $y \in A_x$;

(ii) $f^{-1}(V)$ is an $\mathcal{E}_B$-open neighbourhood of y and $x \notin f^{-1}(V)$ which contradicts the fact $y \in B_x$.

Now, let G be an $\mathcal{E}_B$-open neighbourhood of x and $y \in \mathcal{E}_B\text{-}Q_x$. Then there exist $E \in \mathcal{E}_B$, $f \in BC(X,E)$ and a p-open set $V \subset E$ such that $G = f^{-1}(V)$. Then $f(x)=f(y)$ and $f(x) \in V$. Therefore $y \in f^{-1}(V) = G$ and $y \in A_x$. Similarly, $y \in B_x$.

LEMMA 2.9. Let $\mathcal{E}_B$ be the class of PT_1 spaces, X a bitopological space, $x \in X$, A_x, B_x as in Lemma 2.8. Then the following hold:

(i) $\mathcal{E}_B\text{-}Q_x = A_x = B_x$.

(ii) $\mathcal{E}_B\text{-}Q_x = \{y \in X : X \text{ is } \mathcal{E}_B\text{-connected between } x \text{ and } y\}$.

PROOF. (i) is trivial so that we shall prove (ii). Let X be $\mathcal{E}_B$-connected between x and y. Then there are $E \in \mathcal{E}_B$ and $f \in BC(X,E)$ such that $U_i\text{-cl}\{f(x)\} \cap U_j\text{-cl}\{f(y)\} \neq \emptyset$, for $i \neq j$, $i,j=1,2$. As E is PT_1 we have $U_k\text{-cl}\{f(x)\} = \{f(x)\}$ for $k=1,2$ and every point x in X. Therefore $f(x) = f(y)$, i.e. $y \in \mathcal{E}_B\text{-}Q_x$. The converse is true for every class $\mathcal{E}_B$.

LEMMA 2.10. Let $\mathcal{E}_B$ be the class of PT_2 spaces, X a bitopological space and $x \in X$. Then $\mathcal{E}_B\text{-}Q_x$ is the set of all points y in X such that x and y cannot be separated by $\mathcal{E}_B$-open sets.

PROOF. Suppose that x and y can be separated by $\mathcal{E}_B$-open sets. Then there exist $E, E_1 \in \mathcal{E}_B$, $f \in BC(X,E)$, $g \in BC(X,E_1)$ and p-open sets $O \subset E$, $O_1 \subset E_1$ such that $x \in f^{-1}(O)$, $y \in g^{-1}(O_1)$ and $f^{-1}(O) \cap g^{-1}(O_1) = \emptyset$. Then $f(x) \in O$, $f(y) \notin O$ and $f(y) \in O_1$, $f(x) \notin O_1$. Thus $f(x) \neq f(y)$, i.e. $y \notin \mathcal{E}_B\text{-}Q_x$.

Let $y \in X$ be a point such that x and y cannot be separated by $\mathcal{E}_B$-open sets.

Let $E \in \mathcal{E}_B$ and $f \in BC(X,E)$ be such that $f(x) \neq f(y)$. Since E is a PT_2 space, there exist disjoint neighbourhoods $G \in U_1$ and $V \in U_2$ of $f(x)$ and $f(y)$, respectively. We have $x \in f^{-1}(G) \in T_1$, $y \in f^{-1}(V) \in T_2$ and $f^{-1}(G) \cap f^{-1}(V) = \emptyset$, i.e. $f^{-1}(G)$ and $f^{-1}(V)$ are disjoint $\mathcal{E}_B$-open sets which is a contradiction.

DEFINITION 2.11. The bitopological $\mathcal{E}_B$-component of a set $A \subset X$ is its maximal $\mathcal{E}_B$-connected subset. $\mathcal{E}_B$-K_x denotes the $\mathcal{E}_B$-component of a point $x \in X$.

PROPOSITION 2.12. Let A be an $\mathcal{E}_B$-connected subset of a bitopological space X. Then T_k-clA is $\mathcal{E}_B$-connected if and only if $\mathcal{E}_B$ is the class of PT_1 spaces.

PROOF. Let $A \subset X$ be $\mathcal{E}_B$-connected and let $f: T_k\text{-cl}A \to E$ be a p-continuous function, for k=1,2 and $E \in \mathcal{E}_B$. Then $f(A) = \{a\} \subset E$ and the set $B = f^{-1}(a)$ is T_k-closed in T_k-clA. Hence B is T_k-closed in X and $A \subset B \subset T_k\text{-cl}A$. Therefore T_k-clA= B and f is constant. This means that T_k-clA is $\mathcal{E}_B$-connected.

Let $E \in \mathcal{E}_B$, $x \in X$. Consider the set $\mathcal{E}_B$-K_x. This set is equal to $\{x\}$. Hence $\{x\} = T_k\text{-cl}\{x\}$, k=1,2, because $T_k\text{-cl}\{x\}$ is $\mathcal{E}_B$-connected and $\mathcal{E}_B$-K_x is maximal $\mathcal{E}_B$-connected. Therefore E is a PT_1 space.

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$\mathcal{E}_B$-POVEZANOST BITOPOLOŠKIH PROSTORA

U radu se definiše povezanost bitopoloških prostora u smislu Preuss-a i proučavaju osobine takvih prostora u zavisnosti od aksioma separacije.

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