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PERFECT $\mathcal{P}$-SPLITTABILITY OF TOPOLOGICAL SPACES

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Abstract. We give several results about the inverse preservation of some topological properties under a special kind of perfect mappings.

1. Introduction

Let $\mathcal{M}$ be some class of continuous mappings and $\mathcal{P}$ some class of topological spaces. A space X is said to be **$(\mathcal{M},\mathcal{P})$-splittable** or **$\mathcal{M}$-splittable over $\mathcal{P}$** if for every $A\subset X$ there exist $Y\in\mathcal{P}$ and $f:X\to Y$ from $\mathcal{M}$ such that $f^{-1}f(A) = A$ (see [4], [5], [17]). In the case when $\mathcal{M}$ is the class of all continuous mappings we use the term " $\mathcal{P}$-splittable" or "splittable over $\mathcal{P}$" instead of "$(\mathcal{M},\mathcal{P})$-splittable"; when $\mathcal{M}$ is the class of all perfect mappings we speak about "perfectly $\mathcal{P}$-splittable" spaces. Clearly, if a space X can be mapped onto a space Y from a class $\mathcal{P}$ by a continuous one-to-one mapping, then X is splittable over $\mathcal{P}$; in this case we could say that X is absolutely splittable over $\mathcal{P}$. The paper [9] (see also [8]) contains many important and nice results about splittability (over the space R^{ω}). The usefulness of perfect $\mathcal{P}$-splittability is connected, in particular, with the fact that a compact space is splittable over some class of Hausdorff spaces if and only if it is perfectly splittable over that class.

There are several important classes of topological spaces which are not preserved in the preimage direction by perfect mappings (see [12], [14]). But the situation may be different for perfect mappings with additional conditions. For example, although metrizable spaces are not inversely preserved by perfect mappings they are an inverse invariant of open k-to-one mappings(each such mapping is perfect). In this article we give some results of the following

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type: if a space X is perfectly splittable over a class $\mathcal{P}$, then X is in $\mathcal{P}$. For example, if a space X is perfectly splittable over the class of metrizable spaces, then X is also metrizable. In this connection see also [6].

We use the usual terminology and notation as in [7] and [13]. We provide references (although not always the original source) where the definitions of undefined concepts can be found. All spaces are Hausdorff (unless stated otherwise) and all mappings are continuous and onto.

In the sequel we shall often use the following well-known result [13] :

LEMMA. If $f:X\to Y$ is a perfect mapping, then for any $B\subset Y$ the restriction $f_B:f^{-1}(B)\to B$ is perfect.

2. Convergence properties and splittability

In this section we answer two questions of A.V. Arhangel'skii (see Problems 5.7 and 5.8 in [5]): let a compact space X be splittable over the class of all (compact) Fréchet-Urysohn (resp. bisequential) spaces. Is X is a Fréchet-Urysohn (resp. bisequential) space ?

THEOREM 2.1. If a compact space X is splittable over the class of Fréchet-Urysohn spaces, then it is a strongly Fréchet-Urysohn space.

PROOF. Let us note that X is actually perfectly splittable over the class of compact strongly Fréchet-Urysohn spaces (see [1], [2] or [18] about these spaces). If A is any subset of X then there are a strongly Fréchet-Urysohn space Y and a perfect mapping $f:X\to Y$ such that $f^{-1}f(A) = A$. The set f(A) is strongly Fréchet-Urysohn, because this property is hereditary, so that f(A) is a k-space. Since the property "being a k-space" is a perfect inverse invariant, by Lemma we have that A is a k-space. So, every subspace of X is a k-space and thus, according to the well-known result of Arhangel'skii (see [7]), X is a Fréchet-Urysohn space. But every (countably) compact Fréchet-Urysohn space is strongly Fréchet-Urysohn [1], [2], and the theorem is proved.

The following theorem is, in some sense, a generalization of Theorem 2.1.

THEOREM 2.2. If a space X is perfectly splittable over the class of Fréchet-Urysohn spaces, then X is also a Fréchet-Urysohn space.

PROOF. First of all we shall prove that the tightness of X is countable: $t(X)\leq\omega$. According to a result of D. Rančin [20] for this it is enough to show that for every compact subset B in X one has $t(B)\leq\omega$. Since the class of Fréchet-Urysohn spaces is hereditary, B is splittable over the same class (see [5]). We have that B is splittable over the class of spaces of countable tightness. Thus the tightness of B is also countable, as was proved by Arhangel'skii in [5].

Let T be any countable subset of X. Fix a perfect mapping $f:X \to Y$ from X onto some Fréchet-Urysohn space Y such that $f^{-1}f(T) = T$. The set f(T) is a Fréchet-Urysohn space and consequently it is a k-space. According to Lemma we have that T is also a k-space. As every subset of T is countable we conclude: every subset of T is a k-space. Hence T (and every countable subset of X) is a Fréchet-Urysohn space. The spaces of countable tightness whose all countable subspaces are Fréchet-Urysohn are also Fréchet-Urysohn [2], [18], i.e. X is a Fréchet-Urysohn space. The theorem is proved.

Recall that a space X is said to be an $\aleph_0$-bisequential space if the following two conditions hold: (i) every countable subset of X is bisequential, (ii) $\aleph_0 \in Sp(X)$ (see [1], [2]). Here Sp(X) is the frequency spectrum of X.

THEOREM 2.3. If a space X is perfectly splittable over the class of bisequential spaces, then X is $\aleph_0$-bisequential.

PROOF. Let A be any subset of X. Let us take a bisequential space Y and a perfect mapping $f:X \to Y$ such that $f^{-1}f(A) = A$. The set f(A) is bisequential as a subspace of a bisequential space, and thus it is a bi-k-space (see [18]). Since the property "being a bi-k-space" is a perfect inverse invariant [18], by Lemma we have that A is a bi-k-space. So, every subspace of X is bi-k which implies that X is $\aleph_0$-bisequential [1].

COROLLARY 2.4. If a separable compact space X is splittable over the class of bisequential space, then X is bisequential.

PROOF. This follows from Theorem 2.3 (because X is perfectly splittable over the class of bisequential spaces) and the fact that every separable $\aleph_0$-bisequential space is bisequential [2].

3. Generalized metric spaces and splittability

The definitions of Moore spaces, σ-spaces, Σ-spaces and strong Σ-spaces can be found in [15] or [16].

We start with the following nice result:

THEOREM 3.1. If a space X is perfectly splittable over the class of metrizable spaces, then X is also metrizable.

PROOF. The space X is perfect. Indeed, if F is a closed subspace of X and Y a metrizable space, $f:X \to Y$ a perfect mapping with $f^{-1}f(F) = F$, then f(F) is closed in Y so that it is a G_δ-set. Thus $F = f^{-1}f(F)$ is a G_δ-set in X.

Let A be any subset of X. Fix a metrizable space Y and a perfect mapping $f: X \to Y$ such that $f^{-1}f(A) = A$. The set f(A) is metrizable. Using Lemma and taking into account the well-known result of Arhangel'skii-Morita (see [7], [13]) one concludes that A is a paracompact p-space. So, every subset of X is a paracompact p-space, i.e. X is an F_{pp}-space in terminology of [3] (see also [10]). X is a metrizable space by the following result of Balogh-Pytkeev [10], [19] : an F_{pp}-space is metrizable if and only if it is perfect. The theorem is proved.

THEOREM 3.2. If a space X is perfectly splittable over the class of all σ-spaces, then X is a σ-space.

PROOF. For any $A \subset X$ fix a σ-space Y and a perfect mapping $f: X \to Y$ such that $f^{-1}f(A) = A$. The set f(A) is a σ-space and consequently it is a strong Σ-space. The perfectness of $f|A$ witnesses that A is also a strong Σ-space because this property is a perfect inverse invariant [12]. Moreover X is a perfect space since every σ-space is perfect [15], [16] and all mappings are closed. It remains to apply the following result of Z. Balogh [11]: a perfect space X is a σ-space if and only if X is a hereditarily strong Σ-space.

THEOREM 3.3. If a completely regular space X is perfectly splittable over the class of Moore spaces, then X is also a Moore space.

PROOF. Xis perfect (this can be proved as in Theorems 3.1 and 3.2). Let A be any subset of X. There are a Moore space Y and a perfect mapping $f: X \to Y$ such that $f^{-1}f(A) = A$. The set f(A) is a Moore space. Thus Lemma and fact that the perfect inverse images (with completely regular domains) of a Moore space are subparacompact p-spaces (see [12])guarantee that X is a hereditarily p-space and (hereditarily) θ-refinable (because subparacompacness implies θ-refinability). To end the proof we have to apply the following result of Pytkeev [19]: a hereditarily p-space X is developable if and only if it is perfect and θ-refinable. So, X is a Moore space and the theorem is proved.

THEOREM 3.4. If a space X is perfectly splittable over the class of spaces of weight (netweight) $\leq \tau$, τ is a cardinal, then X has weight (netweight) $\leq \tau$.

PROOF. Let A be a subset of X. Let Y be a space of weight (netweight) $\leq \tau$ and $f: X \to Y$ a perfect mapping such that $f^{-1}f(A) = A$. The set f(A) has weight (netweight) $\leq \tau$ and the mapping $f|A$ is perfect by Lemma. Therefore for every subset of X there exists a perfect mapping onto a space of weight (netweight) $\leq \tau$. Then the theorem of Arhangel'skii-Pytkeev [3], [19]: if every subspace of a space X admits a perfect mapping onto a space of weight (netweight) $\leq \tau$, then

X has weight (netweight) $\leq \tau$, says that $w(X) \leq \tau$ ($nw(X) \leq \tau$). The theorem is proved.

The last theorem is connected with one idea of Arhangel'skii [5] developed in [17]. Let φ be a topological cardinal function, $\mathcal{M}$ a class of continuous mappings and τ an infinite cardinal. For a space X we define (the $\mathcal{M}$-splittable version of φ)

$$\varphi_{\mathcal{M},s}(X) = \min\{\tau : X \text{ is } \mathcal{M}\text{-splittable over the class of all spaces } Y \text{ with } \varphi(Y) \leq \tau\}.$$

When $\mathcal{M}$ is the class of perfect mappings we write $\varphi_{p,s}$ instead of $\varphi_{\mathcal{M},s}$. So Theorem 3.4 and the simple fact that $\varphi_{\mathcal{M},s} \leq \varphi$ for every cardinal function φ give us

COROLLARY 3.5. If X is a Hausdorff space, then

(i) [17] $w_{p,s}(X) = w(X)$;

(ii) $nw_{p,s}(X) = nw(X)$.

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SAVRŠENA $\mathcal{P}$-RACEPLJENOST TOPOLOŠKIH PROSTORA

Ako je $\mathcal{P}$ klasa topoloških prostora onda za prostor X kažemo da je savršeno racepljen nad $\mathcal{P}$ ako za svaki $A \subset X$ postoji savršeno preslikavanje iz X na neki prostor iz $\mathcal{P}$ tako da je $f^{-1}f(A) = A$. U radu se izučava kada prostor X savršeno racepljen nad nekim klasama prostora i sam ima osobine prostora iz $\mathcal{P}$. Na primer, ako je X savršeno racepljen nad klasom metrizabilnih prostora, onda je i on sam metrizabilan. Slične rezultate dobijamo i za još neke klase prostora.

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