Zbornik radova Filozofskog fakulteta u Nišu
Serija Matematika 3(1989), 13-18

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# MONOTONE NORMALITY IN TOPOLOGICAL GROUPS

(Received 31.01.1989.)

**Abstract.** We give an example of a non-metrizable topological group which is stratifiable. We also show that certain topological groups are monotonically normal and pose some questions in this area.

Since first countable topological groups are metrizable, there has been little work on generalized metric spaces in relation to topological groups. We gave an example [H2] of a non-stratifiable topological group that is countable and hence a paracompact $\sigma$-space; and in this note we give an example of a non-metrizable topological group that is stratifiable and we show that certain topological groups are all monotonically normal. We also pose some questions in this area which, we believe, may be interesting.

Note: In this paper we consider only Hausdorff spaces ("$T_2$-spaces").

DEFINITION 1. [HLZ] A Hausdorff topological space X is monotonically normal iff one of the following equivalent conditions A,B and C holds:

(A) there is a function G that assigns, to each ordered pair (H,K) of non-intersecting closed sets, an open neighborhood G(H,K) of H such that:(i) G(H,K) $\cap$ G(K,H) = $\emptyset$ (ii) whenever $H^*$ and $K^*$ are non-intersecting closed sets and $H^*$ contains H and K contains $K^*$, then $G(H^*,K^*)$ contains G(H,K);

(B) same as Condition A except that "ordered pair (H,K) of non-intersecting closed sets" is replaced by "ordered pair (H,K) of mutually separated sets"(sets H and K such that H $\cap$[clo(K)]=K $\cap$ [clo(H)]=$\emptyset$) [likewise for $(H^*,K^*)$];

(C) there is a function g that assigns, to each point p and to each closed subset M in X $\setminus$ {p}, an open neighborhood g(p,M) of p such that: (i) g(p,M) $\cap$ M=$\emptyset$ (ii) whenever $M^*$ is a closed subset of M, then g(p,M) is a subset of $g(p,M^*)$, (iii) whenever p$\neq$q, g(p,{q}) $\cap$ g(q,{p}) = $\emptyset$.

AMS Subject Classification (1980): 54D15, 54E20, 22

THEOREM 1. Every topological group X in which the identity element of the group has a nested local neighborhood base is monotonically normal. (Hence , every topological group that is an LOB space [Da] is monotonically normal.)

PROOF. Denote the group operation by + and the identity by 0, and pick a well-ordered strictly decreasing local base $\{U_\alpha\}_{\alpha<K}$, of 0. Then, for each closed subset M of $X \setminus \{0\}$, let $\beta = \min\{\delta : (U_\delta \cap [\bigcup\{x+U_\delta : x \text{ in } M\}] = \emptyset\}$ and let $g(0,M) = U_\beta$. [Note that if, for every $\delta$, there is an $x_\delta$ in M with $U_\delta \cap (x_\delta + U_\delta) \neq \emptyset$, so there exist $u_\delta$ and $v_\delta$ in $U_\delta$ with $u_\delta = x_\delta + v_\delta$ (so that $x_\delta = u_\delta - v_\delta$), then, by the continuity of + and of −, the nets $\{u_\delta\}$ and $\{v_\delta\}$ and hence $\{x_\delta\}$ all converge to 0, contrary to 0 not being in the closure of M.] For any p and any closed subset M of $X \setminus \{p\}$, let $g(p,M) = p + g(0,-p+M)$. This function g satisfies (C) in Definition 1; hence X is monotonically normal.

There is a simple example, due to Dieudonne, of such a topological group.

EXAMPLE 1. (Dieudonne [Di], [HR]). A non-metrizable topological group satisfying the hypothesis of Theorem 1, and hence monotonically normal.

**Description:** Let $\Omega$ denote the set of all countable ordinals, and let G be the set of all real-valued functions defined on $\Omega$ . Then G, with the usual addition of functions, is an abelian group. Furthermore, the linear order $<$ defined by "$f < g$ whenever there exists an $\alpha$ in $\Omega$ such that (i) $f(\alpha) < g(\alpha)$ and (ii) for all $\beta < \alpha$, $f(\beta) = g(\beta)$" provides G with an order topology with respect to which G is a topological group. (For further details see 4.19 of [HR]; note, for example, that every countable set in G is closed and the intersection of every countable family of open sets is open).

Note: All countable powers of G (above) also satisfy the hypothesis of Theorem 1 and so are monotonically normal, but of course G is not stratifiable (compare this with the theorem from [HLZ] that, if $X^2$ is monotonically normal and some countable subset of X has a limit point, then X is stratifiable).

QUESTION 1. If G is a monotonically normal topological group, must $G^2$ be monotonically normal ?

DEFINITION 2. A $T_2$-space X is semi-stratifiable if there is a sequence $g_1, g_2, g_3, \ldots$ such that (i) for every x in X, $g_n(x)$ is an open neighborhood of x and (ii) if y belongs to $g_n(x_n)$ for all n, $\{x_n\}$ converges to y. For X to be stratifiable , replace (ii) by (ii'): if, for every n, y is in the closure of $\bigcup\{g_n(x_n) : n=1,2,\ldots\}$, then y is in the closure of $\{x_n : n=1,2,\ldots\}$ [H4]. A $\sigma$-space is a $T_3$-space with a $\sigma$-discrete network.

Note: Every $\sigma$-space is semi-stratifiable. Also a first countable semi-stratifiable space is semi-metrizable and conversely [H3]. Theorem 2 could be taken as a definition of stratifiable spaces (see [B1], [HLZ]).

**THEOREM 2.** (Heath, Lutzer, Zenor |HLZ|) A Hausdorff space X is stratifiable iff X is monotonically normal and semi-stratifiable.

Note: An $M_1$-space (a $T_3$-space with a $\sigma$-closure preserving basis) is stratifiable. The converse is not known, but Gruenhage has shown that all $\sigma$-metrizable stratifiable spaces are $M_1$-spaces [G]. A proof of Theorem 3 is to be found in the proof of non-stratifiability in [H1] or [H2]. The group operation on $2^c$ is vector addition modulo 2 (in each component); the topology on $2^c$ is the product topology.

**THEOREM 3** (Heath [H1],[H2]). If a topological subgroup G of the topological group $2^c$ is dense in $2^c$, then G is not stratifiable.

**EXAMPLE 2** (Heath [H1],[H2]). A countable $T_3$-space X which is a topological group but is not stratifiable. [Necessarily X is a $\sigma$-space; hence X is semi-stratifiable; and X is paracompact, too.]

**Description:** Let X be a countable dense subspace of the topological group $2^c$, and let G be the minimal subgroup of $2^c$ containing X ("close up X under addition"; each element of $2^c$ is its own additive inverse; note G is countable).

Also note the following theorem of van Douwen.

**THEOREM 4.** (van Douwen [vD1,pp.70-76]) There is a dense-in-itself, countable $T_3$-space, in which every nowhere dense set is closed (a "Nodec" space); and every such space is non-stratifiable.

We get a simple example of a stratifiable non-metrizable topological group as follows: let Y be the space obtained by identifying the axes in $Q^2$ to a point P, replace each point of $Y \setminus \{P\}$ with a copy of Y and repeat the process countably many times (on each new copy of Y). Example 3 is the same set algebraically described, with a more convenient topology (this version of the example was simultaneously obtained by Nyikos, but it is in fact a special case of an old theorem of van Douwen [vD2]).

**EXAMPLE 3.** A stratifiable, non-metrizable, countable topological group.

**Description:** Let H be the space of all rational sequences with at most finitely many non-zero terms [$s=s_1,s_2,s_3,\dots$ with each $s_i$ rational and such that, for some n, if $i \geqslant n$, $s_i=0$] with the box product topology [a basic neighborhood of s is any set of the form $\{t \text{ in } H: |s_i - t_i| < \delta_i,\ i=1,2,\dots\}$ with each $\delta_i > 0]$. For each natural number n and each s in H, let $g_n(s)=\{t \text{ in } H: |t_i - s_i| < \delta_i(s)\}$ where, for $s_k=0$, $\delta_k(s)=2^{-n}$ and, for $s_k \neq 0$, $\delta_k = 2^{-n}\min\{1,|s_k|\}$. Suppose that y is in H and y is not in the closure of a subset $\{x_n: n=1,2,3,\dots\}$ of H, so that there is a basic neighborhood, $U=\{t \text{ in } H: |y_k - t_k| < \delta_k,\ k=1,2,\dots\}$ of y that contains no $x_j$. Now if $y_k=0$ for all $k \geq n$, and, if $\delta_k > 2^{-m}$ for all $k < n$, then, for $V = \{t \text{ in } H: |y_k - t_k| < \delta_k/2,\ k=1,2,\dots\}$, $V \cap (\cup\{g_{m+n}(x_j): j=1,2,\dots\}) = \emptyset$. It follows by Definition 2 that H is stratifiable. It is easily seen that H is a non-metrizable topological group (vector space in fact).

Note: If the vector space H is generalized to higher cardinals, the resulting space may be monotonically normal but likely not stratifiable.

For any spaces $X_\alpha$ and any choice of points $p_\alpha$ in $X_\alpha$ for each $\alpha$ in I, $\Sigma_p = \{x \text{ in } (\prod\{X_\alpha : \alpha \text{ in } I\}): x_\alpha \neq p_\alpha$ for at most finitely many $\alpha$ in I$\}$ is the **direct sum** of the $X_\alpha$. Theorem 5 (Borges) is a generalization of van Douwen's theorem of which Example 5 is a special case.

**THEOREM 5** (Borges [B2]). Any direct sum of stratifiable spaces with the box product topology is stratifiable.

Theorem 6 extends Theorem 1, Example 3 and some cases of Theorem 5.

**THEOREM 6.** If G is an additive topological group in which the identity, 0, has a nested local neighborhood base, and H is the space of all sequences in G with at most finitely many non-zero terms, then H with the box product topology is monotonically normal.

**PROOF.** Pick a well-ordered, decreasing, local neighborhood base, $\{u_\alpha\}_{\alpha<K}$, for 0 in G with the properties: (i) K is minimal (ii) $u_\alpha = -u_\alpha$ for each $\alpha$, (iii) $u_\alpha$ contains $u_{\alpha+1} + u_{\alpha+1}$ for each $\alpha$. Note that, if K is countable so that G is metric, then, by Theorem 5, H is stratifiable hence monotonically normal. Assume K is uncountable. For each $\alpha < K$, let $g_\alpha(0) = u_\alpha$, and, for x in $G \setminus \{0\}$, let $f(x) = \min\{\beta: (x+u_\beta) \cap u_\beta = \emptyset\}$ and let $g_\alpha(x) = x + u_{f(x)+\alpha}$. For $x=(x_1,x_2,\dots)$ in H and $\alpha < K$, let $g_\alpha(x) = \prod\{g(x_i): i=1,2,\dots\}$. Now by precisely the same process as in Example 3, for any closed subset M of $H \setminus \{0^\omega\}$ there is some $\beta$ such that $0^\omega$ is not in the closure of $g_\beta(M) = \cup\{g_\beta(y): y \text{ in } M\}$ so that, (because K is uncountable) there is some $\delta$ such that $g_\delta(0^\omega) \cap g_\beta(M) = \emptyset$, so that, for some minimal $\alpha$, $g_\alpha(0^\omega) \cap g_\alpha(M) = \emptyset$; finally, let $g(0^\omega, M) = g_\alpha(0^\omega)$. As in the proof of Theorem 1, for any x in H and any closed set M in $H \setminus \{x\}$, let $g(x,M) = x + g(0^\omega, -x+M)$ and complete the proof as in Theorem 1.

Note that an alternate approach would be to show that the spaces H of

Theorem 6, and X of Theorem 1, are linearly stratifiable (see [V]). In fact X is k-metrizable.

**QUESTION 2.** What is a necessary and sufficient condition for a topological group to be stratifiable? (or to be an $M_1$-space ?)

One might still be able to answer the less ambitious question:

**QUESTION 3.** What is a sufficient condition for a topological group to be stratifiable ? (- or to be an $M_1$-space ?)

**QUESTION 4.** Is some subgroup of $2^c$ a stratifiable non-metrizable topological subgroup? (- with respect to the same group structure, or even with respect to a different group structure? - a countable one? - uncountable?)

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## MONOTONA NORMALNOST U TOPOLOŠKIM GRUPAMA

Dobro je poznato da je svaka topološka grupa koja zadovoljava prvu aksiomu prebrojivosti metrizabilna. Zato je od interesa ispitivanje generalisanih metričkih prostora u topološkim grupama. U ovom radu ispituje se kada je topološka grupa monotono normalna. Na primer, dokazano je da je takva svaka topološka grupa koja ima linearno uređenu lokalnu bazu neutralnog elementa. Dat je primer kružne (stratifiable) topološke grupe koja nije metrizabilna. Postavljeno je i nekoliko interesantnih pitanja.

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