Filomat 37:30 (2023), 10445–10457
https://doi.org/10.2298/FIL2330445A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Approximation by Bernstein-Kantorovich type operators based on Beta function

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Abstract. With the idea taken from the King type operators which preserve some test functions, we introduce here some Durrmeyer variants of Bernstein operators based on Beta functions. Some direct approximation theorems are provided of this introduced sequence of operators. We also proved Voronovkaja type theorem. Furthermore, graphical and numerical examples are also given with the help of MATLAB.

1. Introduction

The integral modification of Bernstein operators, called Bernstein-Kantorovich operators, was defined by Kantorovich in [16] to obtain an approximation for Lebesgue integrable functions, Kantorovich operators expressed from those of Bernstein by replacing the sample values $\sigma(k/n)$ with $(m+1)\int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}}\sigma(t)\mathrm{d}t$ the mean values of σ in the interval $[\frac{k}{m},\frac{k+1}{m}]$; that is, for $y\in[0,1]$ and $m\in\mathbb{N}$:

$$\mathcal{K}_m(\sigma;y)=(m+1)\sum_{j=0}^{m}\mathcal{P}_{m,j}(y)\int_{\frac{j}{m+1}}^{\frac{j+1}{m+1}}\sigma(t)\mathrm{d}t,$$

where $\mathcal{P}_{m,j}(y)=\binom{m}{j}y^j(1-y)^{m-j}$. For more study on Kantorovich type operators, one can refer to [6, 12, 13, 15, 17, 19, 21]. In the approximation of functions by positive linear operators, Bernstein operators are the most studied and discussed operators. Many researchers have studied different variants of these operators. For some variations of Bernstein operators and interesting approximation results, see [1, 2, 14, 18]. In the recent past, research works have been carried out in such a direction to construct and modify the operators which fix some functions (see [18]). Szász type operators involving Charlier polynomials and associated approximation properties are given in [4], and different variants of Baskakov operators are discussed in [3, 5]. Recent studies on Gamma and Meyer-König and Zeller operators can be found in [8, 24]. Operators using q-calculus are also introduced and studied their approximation properties by many authors. Some of them can be seen in [9, 10, 20, 22].

2020 *Mathematics Subject Classification*. Primary 41A10; Secondary 41A25, 41A36.
Keywords. Keywords and phrases: Bernstein operators; Beta function; Modulus of continuity; Voronovskaya type theorem.
Received: 18 May 2022; Accepted: 23 May 2023
Communicated by Hemen Dutta
Research supported by King Khalid University.
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Very recently, Bhatt et al. [7] introduced a new sequence of Bernstein-type operators with the help of the beta function as follows:

For $y \in [0,1]$ and $\sigma \in C([0,1])$, Beta-Bernstein operator is defined as

$$\mathfrak{C}_m(\sigma; y) = \sum_{j=0}^{m} \varrho_{m,j}(y)\sigma\left(\frac{j}{m}\right), \tag{1}$$

where

$$\varrho_{m,j}(y) = \binom{m}{j}\frac{\beta(my + j + 1,\ 2m - j - my + 1)}{\beta(my + 1,\ m - my + 1)}. \tag{2}$$

Here $\beta(p, q)$ is the Beta function defined as

$$\beta(p, q) = \int_0^1 t^{p-1}(1 - t)^{q-1}\mathrm{d}t \quad (p, q > 0).$$

In [23], Özarslan and Duman studied various approximation properties of the following operators :

$$\mathcal{K}_{m,\alpha}(\sigma; y) = \sum_{j=0}^{m} \mathcal{P}_{m,j}(y) \int_0^1 \sigma\left(\frac{j + t^{\alpha}}{m + 1}\right)\mathrm{d}t.$$

By combining the ideas given in [7, 23], we construct a new sequence of operators as follows:

$$\mathfrak{C}^*_{m,\alpha}(\sigma; y) = \sum_{j=0}^{m} \varrho_{m,j}(y) \int_0^1 \sigma\left(\frac{j + t^{\alpha}}{m + 1}\right)\mathrm{d}t, \tag{3}$$

where $\varrho_{m,j}(y)$ is given by (2).

The rest of the paper is organized as follows. In Section 2, we recall the moments of operators defined in (1) we also establish the moments of our operators. Section 3 is devoted to proving some direct approximation theorems. In Section 4, we will also prove the Voronovkaja type theorem. In the last section, graphical and numerical examples are also given with the help of MATLAB.

2. Auxiliary results

In order to prove the moments of our operator, we recall the following lemma:

Lemma 2.1. *[7] For $x \in [0,1]$ and the operator given in (1), the following equalities hold true.*

1. $\mathfrak{C}_m(1; y) = 1;$
2. $\mathfrak{C}_m(t; y) = \dfrac{my + 1}{m + 2};$
3. $\mathfrak{C}_m(t^2; y) = \dfrac{(m - 1)(my + 1)(my + 2)}{m(m + 2)(m + 3)} + \dfrac{my + 1}{m(m + 2)};$
4. $\mathfrak{C}_m(t^3; y) = \dfrac{(m - 1)(m - 2)(my + 1)(my + 2)(my + 3)}{m^2(m + 2)(m + 3)(m + 4)} + \dfrac{3(m - 1)(my + 1)(my + 2)}{m^2(m + 2)(m + 3)} + \dfrac{my + 1}{m^2(m + 2)};$
5. $\mathfrak{C}_m(t^4; y) = \dfrac{(m - 1)(m - 2)(m - 3)(my + 1)(my + 2)(my + 3)(my + 4)}{m^3(m + 2)(m + 3)(m + 4)(m + 5)} + \dfrac{6(m - 1)(m - 2)(my + 1)(my + 2)(my + 3)}{m^3(m + 2)(m + 3)(m + 4)} + \dfrac{7(m - 1)(my + 1)(my + 2)}{m^3(m + 2)(m + 3)} + \dfrac{my + 1}{m^3(m + 2)}.$

Lemma 2.2. *For $m \in \mathbb{N}$, $\alpha > 0$ and $x \in [0,1]$, we have*

$$\mathfrak{C}^*_{m,\alpha}(e_i; y) = \frac{1}{(m+1)^i} \sum_{n=0}^{i} \frac{m^n}{\alpha(i-n)+1} \binom{i}{n} \mathfrak{C}_m(e_n; y),$$

where $\mathfrak{C}_m$ denotes the operators given by (1) *and $e_i(y) = y^i$, $\quad (i = 0, 1, 2, \dots)$.*

Proof. It follows from (3) that

$$\begin{aligned}
\mathfrak{C}^*_{m,\alpha}(e_i; y) &= \sum_{j=0}^{m} \varrho_{m,j}(y) \int_0^1 \left(\frac{j+t^\alpha}{m+1}\right)^i \mathrm{d}t \\
&= \frac{1}{(m+1)^i} \sum_{n=0}^{i} \binom{i}{n} \sum_{j=0}^{m} \varrho_{m,j}(y) j^n \int_0^1 t^{\alpha(i-n)} \mathrm{d}t \\
&= \frac{1}{(m+1)^i} \sum_{n=0}^{i} \frac{1}{\alpha(i-n)+1} \binom{i}{n} \sum_{j=0}^{m} \varrho_{m,j}(y) j^n \\
&= \frac{1}{(m+1)^i} \sum_{n=0}^{i} \frac{m^n}{\alpha(i-n)+1} \binom{i}{n} \mathfrak{C}_m(e_n; y)
\end{aligned}$$

which completes the proof. □

Lemma 2.3. *We have the moments for the defined operators* (3) *using Lemma 2.1 and Lemma 2.2 as follows:*

1. $\mathfrak{C}^*_{m,\alpha}(1; y) = 1$;
2. $\mathfrak{C}^*_{m,\alpha}(t; y) = \dfrac{m(my+1)}{(m+1)(m+2)} + \dfrac{1}{(m+1)(\alpha+1)}$;
3. $\mathfrak{C}^*_{m,\alpha}(t^2; y) = \dfrac{m(m-1)(my+1)(my+2)}{(m+1)^2(m+2)(m+3)} + \dfrac{m(my+1)}{(m+1)^2(m+2)}\left(1+\dfrac{2}{\alpha+1}\right) + \dfrac{1}{(m+1)^2(2\alpha+1)}$;
4. $\begin{aligned}[t]\mathfrak{C}^*_{m,\alpha}(t^3; y) &= \frac{m(m-1)(m-2)(my+1)(my+2)(my+3)}{(m+1)^3(m+2)(m+3)(m+4)} \\ &+ \frac{3m(m-1)(my+1)(my+2)}{(m+1)^3(m+2)(m+3)}\left(1+\frac{1}{\alpha+1}\right) \\ &+ \frac{m(my+1)}{(m+1)^3(m+2)}\left(1+\frac{3}{\alpha+1}+\frac{3}{2\alpha+1}\right) + \frac{1}{(m+1)^3(3\alpha+1)};\end{aligned}$
5. $\begin{aligned}[t]\mathfrak{C}^*_{m,\alpha}(t^4; y) &= \frac{m(m-1)(m-2)(m-3)(my+1)(my+2)(my+3)(my+4)}{(m+1)^4(m+2)(m+3)(m+4)(m+5)} \\ &+ \frac{2m(m-1)(m-2)(my+1)(my+2)(my+3)}{(m+1)^4(m+2)(m+3)(m+4)}\left(3+\frac{2}{\alpha+1}\right) \\ &+ \frac{m(m-1)(my+1)(my+2)}{(m+1)^4(m+2)(m+3)}\left(7+\frac{12}{\alpha+1}+\frac{6}{2\alpha+1}\right) \\ &+ \frac{m(my+1)}{(m+1)^4(m+2)}\left(1+\frac{4}{\alpha+1}+\frac{6}{2\alpha+1}+\frac{4}{3\alpha+1}\right) + \frac{1}{(m+1)^4(4\alpha+1)}.\end{aligned}$

3. Some direct approximation theorems

We start with the following lemma.

Lemma 3.1. *For each $m \in \mathbb{N}, \alpha > 0$ and $y \in [0,1]$, we have*

$$\mathfrak{C}^*_{m,\alpha}(\phi_y^2; y) \leq \frac{2}{m+1}(my(1-y) + A_\alpha),$$

where

$$\phi_y^2(t) = (t-y)^2 \ \ and \ \ A_\alpha = \frac{3\alpha^3 + 10\alpha^2 + 11\alpha + 3}{(1+\alpha)^2(2\alpha+1)}. \tag{4}$$

Proof. We have

$$\begin{aligned}
\mathfrak{C}^*_{m,\alpha}(\phi_y^2; y) &= \frac{-2m^3 + 11m^2 + 17m + 6}{(m+1)^2(m+2)(m+3)} y^2 + \frac{2m^3 - 8m^2 - 6m}{(m+1)^2(m+2)(m+3)} y \\
&\quad + \frac{2m - 2y(3m+2)}{(m+1)^2(m+2)} . \frac{1}{1+\alpha} + \frac{1}{(m+1)^2(2\alpha+1)} + \frac{3m^2+m}{(m+1)^2(m+2)(m+3)} \\
&= \frac{1}{(m+1)^2}\Bigg[3y^2 + \frac{-2m^3 + 8m^2 + 2m - 12}{(m+2)(m+3)} y^2 \\
&\quad + \frac{2m^3 - 8m^2 - 6m}{(m+1)^2(m+2)(m+3)} y + \frac{2m - 2y(3m+2)}{m+2} . \frac{1}{1+\alpha} + \frac{1}{2\alpha+1} + \frac{3m^2+m}{(m+2)(m+3)}\Bigg] \\
&\leq \frac{1}{(m+1)^2}\left[\frac{2m^3 - 8m^2 - 2m}{(m+2)(m+3)} y(1-y) + 3y^2 + \frac{2(1-y)}{1+\alpha} + \frac{1}{2\alpha+1} + \frac{3m^2+m}{(m+2)(m+3)}\right],
\end{aligned}$$

or

$$\frac{2m^3 - 8m^2 - 2m}{(m+2)(m+3)} \leq 2m \ \text{ and } \ \frac{3m^2+m}{(m+2)(m+3)} \leq 3,$$

we get

$$\mathfrak{C}^*_{m,\alpha}(\phi_y^2; y) \leq \frac{1}{(m+1)^2}(2my(1-y) + B_\alpha(y)),$$

where

$$B_\alpha(y) = 3y^2 + \frac{2(1-y)}{1+\alpha} + \frac{1}{2\alpha+1} + 3.$$

Now, since

$$\max_{y \in [0,1]} B_\alpha(y) = \frac{18\alpha^3 + 60\alpha^2 + 64\alpha + 17}{3(1+\alpha)^2(2\alpha+1)},$$

we deduce that

$$\mathfrak{C}^*_{m,\alpha}(\phi_y^2; y) \leq \frac{2}{m+1}\left[my(1-y) + \frac{3\alpha^3 + 10\alpha^2 + 11\alpha + 3}{(1+\alpha)^2(2\alpha+1)}\right].$$

□

Before getting the next result, we give the K-functional [25] defined as :

$$K_2(\sigma, \delta) = \inf\left\{\|\sigma - g\| + \delta\|g''\|, \ g \in C^2([0,1])\right\}, \quad \delta > 0,$$

and the second-order modulus of smoothness of $f \in C([0,1])$

$$w_2(\sigma, \delta) = \sup_{0<h<\delta} \sup_{y \pm h \in [0,1]} |\sigma(y-h) - 2\sigma(y) + \sigma(y+h)|, \quad \delta > 0.$$

We recall that (see [11]) there exists a positive constant C such that

$$K_2(\sigma, \delta^2) \leq C w_2(\sigma, \delta) \ \text{ for } \ \delta > 0. \tag{5}$$

Theorem 3.2. *Let σ be a continuous function on $[0,1]$, the sequence $\mathfrak{C}^*_{m,\alpha}(\sigma, y)$ converges uniformly to σ.*

Proof. Let $e_i(y) = y^i$, $i = 0,1,2$. In virtue of Bohman-Korovkin theorem it suffices to prove that $\mathfrak{C}^*_{m,\alpha}(e_i, y)$ converge to $e_i(y)$ uniformly in $[0,1]$.
If we take into consideration (1) – (3) of Lemma 2.3, the proof is completed. □

Theorem 3.3. *Let $\sigma \in C([0,1])$ and $y \in [0,1]$, there exists an absolute constant $C > 0$ such that*

$$\left|\mathfrak{C}^*_{m,\alpha}(\sigma, y) - \sigma(y)\right| \leq C w_2\left(\sigma, \sqrt{\rho_{m,\alpha}(y)}\right) + w(\sigma, \sigma_{m,\alpha}(y)), \text{ for all } m \in \mathbb{N} \text{ and } y \in [0,1],$$

where

$$\rho_{m,\alpha}(y) = \frac{1}{4(m+1)^2}\left[my(1-y) + A_\alpha + \max\left\{\left(1 - 3y + \frac{1}{1+\alpha}\right)^2, \left(-y + \frac{1}{1+\alpha}\right)^2\right\}\right],$$

$$\sigma_{m,\alpha}(y) = \frac{1}{m+1}\max\left\{\left|1 - 3y + \frac{1}{1+\alpha}\right|, \left|-y + \frac{1}{1+\alpha}\right|\right\},$$

and A_α is given by (4).

Proof. Consider the operator

$$\overline{\mathfrak{C}^*_{m,\alpha}}(\sigma, y) = \mathfrak{C}^*_{m,\alpha}(\sigma, y) + \sigma\left(\mathfrak{C}^*_{m,\alpha}(e_1, y)\right) + \sigma(y), \tag{6}$$

we can easily check that

$$\overline{\mathfrak{C}^*_{m,\alpha}}(e_0, y) = 1 \text{ and } \overline{\mathfrak{C}^*_{m,\alpha}}(\phi_y, y) = 0. \tag{7}$$

Applying Taylor formula on $h \in C^2([0,1])$, we get

$$h(z) = h(y) + (z-y)h'(y) + \int_y^z (z-s)h''(s)\mathrm{d}s \text{ for } z \in [0,1]$$

using (6) and (7), we have

$$\overline{\mathfrak{C}^*_{m,\alpha}}(h, y) - h(y) = \overline{\mathfrak{C}^*_{m,\alpha}}\left(\int_y^z \phi_s(z)h''(s)\mathrm{d}s,\ y\right).$$

From (6) we can see that

$$\overline{\mathfrak{C}^*_{m,\alpha}}(h, y) - h(y) = \mathfrak{C}^*_{m,\alpha}\left(\int_y^z \phi_s(z)h''(s)\mathrm{d}s,\ y\right) - \int_y^{\mathfrak{C}^*_{m,\alpha}(e_1,y)} \left(\mathfrak{C}^*_{m,\alpha}(e_1, y) - s\right)h''(s)\mathrm{d}s \tag{8}$$

On the one hand, since

$$\left|\int_y^z \phi_s(z)h''(s)\mathrm{d}s\right| \leq \frac{\|h''\|}{2}\phi_y^2(z),$$

we get

$$\left|\mathfrak{C}^*_{m,\alpha}\left(\int_y^z \phi_s(z)h''(s)\mathrm{d}s,\ y\right)\right| \leq \frac{\|h''\|}{2}\mathfrak{C}^*_{m,\alpha}\left(\phi_y^2, y\right). \tag{9}$$

On the other hand

$$\begin{aligned}
&\left|\int_y^{\mathfrak{C}^*_{m,\alpha}(e_1,y)} \left(\mathfrak{C}^*_{m,\alpha}(e_1, y) - s\right)h''(s)\mathrm{d}s\right| \\
&\qquad \leq \frac{\|h''\|}{2(n+1)^2}\left(\frac{(1-3y)m - 2y}{n+2} + \frac{1}{\alpha+1}\right)^2 \\
&\qquad \leq \frac{\|h''\|}{2(n+1)^2}\max\left\{\left(1 - 3y + \frac{1}{1+\alpha}\right)^2, \left(-y + \frac{1}{1+\alpha}\right)^2\right\}.
\end{aligned} \tag{10}$$

Combining (8)-(10) we deduce that

$$\left|\overline{\mathfrak{C}^*_{m,\alpha}}(h,y)-h(y)\right| \le \frac{\|h''\|}{2}\left[\mathfrak{C}^*_{m,\alpha}\left(\phi_y^2,y\right)+\frac{\|h''\|}{2(n+1)^2}\max\left\{\left(1-3y+\frac{1}{1+\alpha}\right)^2,\left(-y+\frac{1}{1+\alpha}\right)^2\right\}\right],$$

thus from Lemma 2.3, we have

$$\left|\overline{\mathfrak{C}^*_{m,\alpha}}(h,x)-h(y)\right| \le \frac{\|h''\|}{2(n+1)^2}\left(A_\alpha+my(1-y)+\max\left\{\left(1-3y+\frac{1}{1+\alpha}\right)^2,\left(-y+\frac{1}{1+\alpha}\right)^2\right\}\right).$$

Now, let $\sigma\in C([0,1])$, $y\in[0,1]$ and $h\in C^2([0,1])$, we get

$$\begin{aligned}\left|\mathfrak{C}^*_{m,\alpha}(\sigma,y)-\sigma(y)\right| &\le \left|\overline{\mathfrak{C}^*_{m,\alpha}}(\sigma-h,y)-(\sigma-h)(y)\right|+\left|\overline{\mathfrak{C}^*_{m,\alpha}}(h,y)-h(y)\right|\\ &\quad+\left|\sigma\left(\frac{(1-3y)m-2y}{(m+1)(m+2)}+\frac{1}{(m+1)(\alpha+1)}\right)-\sigma(y)\right|.\end{aligned}$$

Since $\left|\overline{\mathfrak{C}^*_{m,\alpha}}(\sigma,y)\right|\le 3\|\sigma\|$, and

$$\left|\frac{(1-3y)m-2y}{(m+1)(m+2)}+\frac{1}{(m+1)(\alpha+1)}\right| \le \frac{1}{m+1}\max\left\{\left|1-3y+\frac{1}{1+\alpha}\right|,\left|-y+\frac{1}{1+\alpha}\right|\right\},$$

we conclude that

$$\begin{aligned}\left|\mathfrak{C}^*_{m,\alpha}(\sigma,y)-\sigma(y)\right| &\le 4(\|\sigma-h\|+\rho_{m,\alpha}(y)\|h''\|)\\ &\quad+w\left(\sigma,\frac{1}{m+1}\max\left\{\left|1-3y+\frac{1}{1+\alpha}\right|,\left|-y+\frac{1}{1+\alpha}\right|\right\}\right),\end{aligned}$$

which implies that

$$\begin{aligned}\left|\mathfrak{C}^*_{m,\alpha}(\sigma,y)-\sigma(y)\right| &\le 4\left(\|\sigma-h\|+\rho_{m,\alpha}(y)\|h''\|\right)+w(\sigma,\sigma_{m,\alpha}(y))\\ &\quad+4K_2(\sigma,\rho_{m,\alpha}(y))+w(\sigma,\sigma_{m,\alpha}(y)).\end{aligned}$$

Using (5), we get

$$\left|\mathfrak{C}_{m,\alpha}(\sigma,y)-\sigma(y)\right| \le Cw_2\left(\sigma,\sqrt{\rho_{m,\alpha}(y)}\right)+w(\sigma,\sigma_{m,\alpha}(y)).$$

□

We denote by $AC([0,1])$ the space of all absolutely continuous functions in $[0,1]$.
Let $\phi(y)=\sqrt{y(1-y)}$, and $\delta>0$. The second-order modified K-functional for $\sigma\in C([0,1])$ is defined by

$$K^2_\phi(\sigma,\delta)=\inf\left\{\|\sigma-g\|+\delta\|\phi^2g''\|+\delta^2\|g''\|,\ g\in\ W^2(\phi)\right\},$$

where

$$W^2(\phi)=\left\{g\in C([0,1]):g'\in AC([0,1]),\phi^2g''\in C([0,1])\right\}.$$

The second-order Ditzian-Totik modulus is given by

$$w^2_\phi(\sigma,\delta)=\sup_{0<h<\delta}\ \sup_{y\pm h\phi\in[0,1]}|\sigma(y-h\phi)-2\sigma(y)+\sigma(y+h\phi)|.$$

It is well-known (see [11]) that, for any $\delta>0$,

$$K^2_\phi(\sigma,\delta^2)\le Cw^2_\phi(\sigma,\delta),$$

holds for some positive constant C.
Finally, we consider the first-order Ditzian-Totik modulus, which is defined by

$$\overrightarrow{w}_\psi(\sigma,\delta)=\sup_{0<h<\delta}\ \sup_{y+h\psi\in[0,1]}|\sigma(y+h\psi)-\sigma(y)|.$$

Theorem 3.4. *Let $m \in \mathbb{N}$ and $\alpha > 0$. Then, for every $\sigma \in C([0,1])$ and $y \in [0,1]$, there exists a positive constant C such that*

$$\left|\mathfrak{C}^*_{m,\alpha}(\sigma, y) - \sigma(y)\right| \le C w^2_{\phi}\left(\sigma, \frac{1}{\sqrt{m+1}}\right) + \overrightarrow{w}_{\psi_\alpha}\left(\sigma, \frac{1}{1+m}\right),$$

where $\psi_\alpha = (1+\alpha)y + 2$.

Proof. As in the proof of Theorem 3.2, for a given $h \in w^2_{\phi}$, we obtain that

$$\overline{\mathfrak{C}^*_{m,\alpha}}(h, y) - h(y) \le \left|\mathfrak{C}^*_{m,\alpha}\left(\int_y^z \phi_s(z) h''(s)\mathrm{d}s,\ y\right)\right| + \left|\int_y^{\mathfrak{C}^*_{m,\alpha}(e_1,y)} \left(\mathfrak{C}^*_{m,\alpha}(e_1, y) - s\right) h''(s)\mathrm{d}s\right|, \tag{11}$$

and we set

$$\delta_m(y) = y(1-y) + \frac{1}{m+1}.$$

Using changing variable $s = \tau y + (1-\tau)u, \tau \in [0,1]$ and the fact that $\tau\delta_m(y) \le \delta_m(s)$ by the concavity of δ_m, we get

$$\begin{aligned}
\left|\int_y^u \phi_s(u) h''(s)\mathrm{d}s\right| &= \left|\int_0^1 \tau(y-u)^2 h''(s)\mathrm{d}\tau\right| \\
&\le \left|\int_0^1 \frac{\phi^2_y(u)}{\delta_m(y)} \delta_m(s) h''(s)\mathrm{d}\tau\right| \\
&\le \frac{\|\delta_m h''\|}{\delta_m(y)} \phi^2_y(u).
\end{aligned}$$

Thus, we get from (11) that

$$\overline{\mathfrak{C}^*_{m,\alpha}}(h, y) - h(y) \le \frac{\|\delta_m h''\|}{\delta_m(y)} \mathfrak{C}^*_{m,\alpha}\left(\phi^2_y,\ y\right) + \frac{\|\delta_m h''\|}{\delta_m(y)} \left(\frac{m(my+1)}{(m+1)(m+2)} + \frac{1}{(m+1)(\alpha+1)} - y\right)^2.$$

Therefore from Lemma 3.1, we have

$$\begin{aligned}
&\left|\overline{\mathfrak{C}^*_{m,\alpha}}(h, y) - h(y)\right| \\
&\le \frac{\|\delta_m h''\|}{(m+1)^2\delta_m(y)} \left(A_\alpha + my(1-y) + \max\left\{\left(1 - 3y + \frac{1}{1+\alpha}\right)^2, \left(-y + \frac{1}{1+\alpha}\right)^2\right\}\right) \\
&\le \frac{\|\delta_m h''\|}{(m+1)\delta_m(y)} \left(y(1-y) + \frac{A_\alpha}{m+1} + \frac{1}{m+1} \max\left\{\left(1 - 3y + \frac{1}{1+\alpha}\right)^2, \left(-y + \frac{1}{1+\alpha}\right)^2\right\}\right) \\
&\le \frac{\|\delta_m h''\|}{(m+1)\delta_m(y)} \left(y(1-y) + \frac{3}{m+1} + \frac{4}{m+1}\right) \\
&\le \frac{7\|\delta_m h''\|}{(m+1)} \\
&\le \frac{7}{(m+1)} \left(\|\phi^2 h''\| + \frac{1}{m+1}\|h''\|\right).
\end{aligned}$$

Now, we remark that

$$\begin{aligned}
\left|\mathfrak{C}^*_{m,\alpha}(\sigma, y) - \sigma(y)\right| \le\ & \left|\overline{\mathfrak{C}^*_{m,\alpha}}(\sigma - h, y) - (\sigma - h)(y)\right| + \left|\overline{\mathfrak{C}^*_{m,\alpha}}(h, y) - h(y)\right| \\
& + \left|\sigma\left(\frac{m(my+1)}{(m+1)(m+2)} + \frac{1}{(m+1)(\alpha+1)}\right) - \sigma(y)\right|,
\end{aligned}$$

which implies that

$$\begin{aligned}
\left|\mathfrak{C}^*_{m,\alpha}(\sigma, y) - \sigma(y)\right| \le\ & 7\left(\|\sigma - h\| + \tfrac{1}{m+1}\|\phi^2 h''\| + \tfrac{1}{(m+1)^2}\|h''\|\right) \\
& + \left|\sigma\left(\frac{m(my+1)}{(m+1)(m+2)} + \frac{1}{(m+1)(\alpha+1)}\right) - \sigma(y)\right|.
\end{aligned}$$

Then

$$\left|\mathfrak{C}^*_{m,\alpha}(\sigma,y)-\sigma(y)\right| \le 7K_\phi^2\left(\sigma,\frac{1}{m+1}\right)+\left|\sigma\left(\frac{m(my+1)}{(m+1)(m+2)}+\frac{1}{(m+1)(\alpha+1)}\right)-\sigma(y)\right|.$$

The last term of the previous inequality can be estimated as follows

$$\begin{aligned}
&\left|\sigma\left(\frac{m(my+1)}{(m+1)(m+2)}+\frac{1}{(m+1)(\alpha+1)}\right)-\sigma(y)\right| \\
&\quad= \left|\sigma\left(y+\psi_\alpha(y)\frac{\frac{m(my+1)}{(m+1)(m+2)}+\frac{1}{(m+1)(\alpha+1)}-y}{\psi_\alpha(y)}\right)-\sigma(y)\right| \\
&\quad\le \sup_{t\in I_\alpha(y)}\left|\sigma\left(t+\psi_\alpha(t)\frac{\frac{m(1-3y)-2y}{m+2}+\frac{1}{\alpha+1}}{(m+1)\psi_\alpha(y)}\right)-\sigma(y)\right| \\
&\quad\le \overrightarrow{w}_{\psi_\alpha}\left(\sigma,\frac{\left|\frac{m(1-3y)-2y}{m+2}+\frac{1}{\alpha+1}\right|}{(m+1)\psi_\alpha(y)}\right),
\end{aligned}$$

or we have

$$\left|\frac{m(1-3y)-2y}{m+2}+\frac{1}{\alpha+1}\right| \le \max\left\{\left|1-3y+\frac{1}{1+\alpha}\right|, \left|-y+\frac{1}{1+\alpha}\right|\right\} \le 2 \le \psi_\alpha(y).$$

We deduce that

$$\left|\sigma\left(\frac{m(my+1)}{(m+1)(m+2)}+\frac{1}{(m+1)(\alpha+1)}\right)-\sigma(y)\right| \le \overrightarrow{w}_{\psi_\alpha}\left(\sigma,\frac{1}{m+1}\right),$$

where

$$A_\alpha(y) := \left\{t+\psi_\alpha(t)\frac{\frac{m(1-3y)-2y}{m+2}+\frac{1}{\alpha+1}}{(m+1)\psi_\alpha(x)} : t\in[0,1]\right\}.$$

□

4. Voronovskaya type theorem

In this last section, we deal with the Voronovskaya-type asymptotic theorem for $\mathfrak{C}^*_{m,\alpha}$. We start with the following lemma

Lemma 4.1. *For every $y \in [0;1]$, we have*

(1) $\lim_{m\to\infty} m\mathfrak{C}^*_{m,\alpha}(t-y,y) = 1-3y+\frac{1}{\alpha+1}$;

(2) $\lim_{m\to\infty} m\mathfrak{C}^*_{m,\alpha}\left((t-y)^2,y\right) = 2y(1-y)$;

(3) $\lim_{m\to\infty} m^2\mathfrak{C}^*_{m,\alpha}\left((t-y)^4,y\right)$ *exists.*

Proof. (1) From (2) of lemma 2.3 we can easily deduce this assertion.

(2) It suffices to use the relation (4)

(3) By the linearity of $\mathfrak{C}^*_{m,\alpha}(.,y)$, we can write

$$m^2\mathfrak{C}^*_{m,\alpha}((t-y)^4,y) = m^2\mathfrak{C}^*_{m,\alpha}(t^4,y) - 4ym^2\mathfrak{C}^*_{m,\alpha}(t^3,y) + 6m^2y^2\mathfrak{C}^*_{m,\alpha}(t^2,y) - 4m^2y^3\mathfrak{C}^*_{m,\alpha}(t,y) + m^2y^4.$$

On the one hand, it is easy to get the following equalities, where the function $\varepsilon(y,n)$, different from line to line, and satisfies $\lim\limits_{n\to+\infty}\varepsilon(y,n)$ exists.

- $$m^2\mathfrak{C}^*_{m,\alpha}(t^4,y) = \frac{m^{10}-6m^9}{(m+1)^4(m+2)...(m+5)}y^4 + \frac{10m^9}{(m+1)^4(m+2)...(m+5)}y^3 + \frac{2m^8}{(m+1)^4(n+2)(m+3)(m+4)}\left(3+\frac{2}{1+\alpha}\right)y^3 + \varepsilon(y,m);$$
- $$-4ym^2\mathfrak{C}^*_{m,\alpha}(t^3,y) = \frac{-4m^{10}-12m^9}{(m+1)^4(m+2)...(m+5)}y^4 + -\frac{24m^9}{(m+1)^4(m+2)...(m+5)}y^3 + \frac{-12m^8}{(m+1)^4(n+2)(m+3)(m+4)}\left(1+\frac{1}{1+\alpha}\right)y^3 + \varepsilon(y,m);$$
- $$6m^2y^2\mathfrak{C}^*_{m,\alpha}(t^2,y) = \frac{6m^{10}+60m^9}{(m+1)^4(n+2)...(m+5)}y^4 + \frac{18m^9}{(m+1)^4(m+2)...(m+5)}y^3 + \frac{6m^8}{(m+1)^4(n+2)(m+3)(m+4)}\left(1+\frac{2}{1+\alpha}\right)y^3 + \varepsilon(y,m);$$
- $$-4m^2y^3\mathfrak{C}^*_{m,\alpha}(t,x) = \frac{-4m^{10}-60m^9}{(m+1)^4(m+2)...(m+5)}y^4 + \frac{-4m^9}{(m+1)^4(m+2)...(m+5)}y^3 - \frac{4m^8}{(m+1)^4(m+2)(m+3)(m+4)}\frac{1}{1+\alpha}y^3 + \varepsilon(y,m);$$
- $$m^2y^4 = \frac{m^2(m+1)^4(m+2)...(m+5)}{(m+1)^4(m+2)...(m+5)}y^4 = \frac{m^{10}+18m^9}{(m+1)^4(n+2)(m+3)(m+4)(m+5)}y^4 + \varepsilon(y,m).$$

On the other hand, by straightforward calculations, we get

$$m^2\mathfrak{C}^*_{m,\alpha}((t-y)^4,y) = 0 + \varepsilon(y,m),$$

which gives that

$$\lim_{m\to+\infty} m^2\mathfrak{C}^*_{m,\alpha}((t-y)^4,y) \text{ exists.}$$

□

We will prove the following result:

Theorem 4.2. *For $\sigma \in C^2([0,1])$ and $\sigma \in [0,1]$, we have*

$$\lim_{m\to\infty} m\left(\mathfrak{C}^*_{m,\alpha}(\sigma,y) - \sigma(y)\right) = \left(1 - 3y + \frac{1}{\alpha+1}\right)\sigma'(y) + y(1-y)\sigma''(y).$$

Proof. Let $\sigma \in C^2([0,1])$ and $y \in [0,1]$. By Taylor's formula, we write

$$\sigma(u) - \sigma(y) = (u-y)\sigma'(y) + \frac{(u-y)^2}{2}\sigma''(y) + \varepsilon_y(u)(u-y)^2,$$

where $\lim\limits_{u\to y}\varepsilon_y(u) = 0$. Applying $\mathfrak{C}^*_{m,\alpha}(.;y)$ to both sides of the above equality, we get

$$m\left(\mathfrak{C}^*_{m,\alpha}(\sigma,y) - \sigma(y)\right) = m\sigma'(y)\mathfrak{C}^*_{m,\alpha}(u-y;y) - \tfrac{m}{2}\sigma''(y)\mathfrak{C}^*_{m,\alpha}\left((u-y)^2;y\right) + m\mathfrak{C}^*_{m,\alpha}\left(\varepsilon_y(u)(u-y)^2;y\right).$$

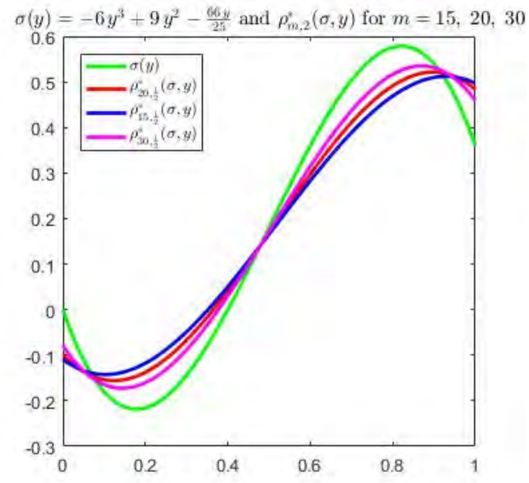

Figure 1: Approximation to $\sigma(x)$ by $\mathfrak{C}^*_{m,\alpha}(\sigma)$ for $\alpha = 2, \sigma(x) = -6x^3 + 9x^2 - \frac{66}{25}x$ and $m = 15, 20, 30$.

Using the Cauchy Schwartz inequality, we have

$$m\mathfrak{C}^*_{m,\alpha}\left(\varepsilon_y(u)(u-y)^2; y\right) \leq \sqrt{\mathfrak{C}^*_{m,\alpha}\left(\varepsilon_y^2(u); y\right)}.\sqrt{m^2\mathfrak{C}^*_{m,\alpha}\left((u-y)^4; y\right)}. \tag{12}$$

From (3) of Lemma 4.1, $\lim_{m\to\infty} m^2\mathfrak{C}^*_{m,\alpha}\left((u-y)^4; y\right)$ exists and non-negative and by using uniform convergence of the operators $\mathfrak{C}^*_{m,\alpha}$, we have

$$\lim_{m\to\infty} \mathfrak{C}^*_{m,\alpha}\left(\varepsilon_y(u); y\right) = \varepsilon_y(y) = 0 \text{ uniformly for } y \in [0;1].$$

Hence, from (12) , we get

$$\lim_{m\to\infty} m\mathfrak{C}^*_{m,\alpha}\left(\varepsilon_y(u)(u-y)^2; y\right) = 0,$$

this and (12) we deduce that

$$\lim_{m\to\infty} m\left(\mathfrak{C}^*_{m,\alpha}(\sigma, y) - \sigma(y)\right) = \left(1 - 3y + \frac{1}{\alpha+1}\right)\sigma'(y) + y(1-y)\sigma''(y).$$

□

5. Graphical simulations

In this section, we show the approximation of some continuous functions by the operator $\mathfrak{C}^*_{m,\alpha}(\sigma)$ graphically using MATLAB.

We first consider the function $\sigma(y) = -6y^3 + 9y^2 - \frac{66}{25}y$, and take $\alpha = 2$. On the one hand, we show, in figure 1 and figure 2 respectively, the approximation to this function by the operators $\mathfrak{C}^*_{m,\alpha}(\sigma, x)$ and the graphical of Error(y) defined as $|\mathfrak{C}^*_{m,2}(\sigma, y) - \sigma(y)|$, for the values $m = 15, 20, 30$, respectively.
On the other hand, in the table below, we see that when the values of α increase, the maximum error for $m = 20, 30, 40$ increases too. Secondly, in figure 3 we state that the operator $\mathfrak{C}^*_{m,\frac{1}{2}}$ preserves the convexity of the function $\sigma(y) = 1 - \cos(\pi y - \frac{\pi}{2})$.

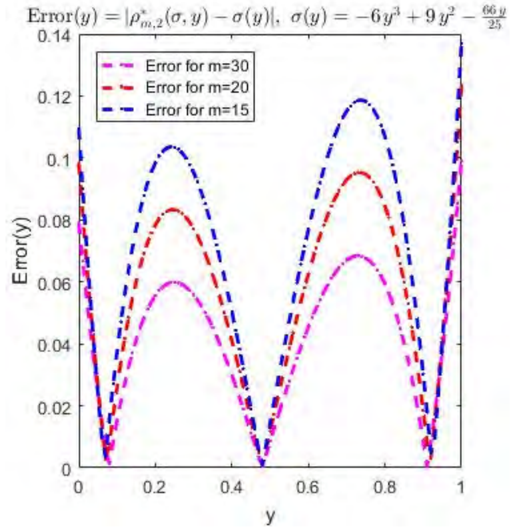

Figure 2: Absolute Error while Approximating to $\sigma(x)$ by $\mathfrak{C}^{*}_{m,\alpha}(\sigma)$ for $\alpha = 2, \sigma(x) = -6x^3 + 9x^2 - \frac{66}{25}x$ and $m = 15, 20, 30$.

α	Error($m = 20$)	Error($m = 30$)	Error($m = 40$)
1.00000	0.06392	0.04687	0.03703
1.50000	0.06417	0.04704	0.03716
2.00000	0.06456	0.04732	0.03737
2.50000	0.06495	0.04759	0.03759
3.00000	0.06530	0.04784	0.03778
3.50000	0.06561	0.04806	0.03795
4.00000	0.06587	0.04825	0.03810
4.50000	0.06610	0.04841	0.03823
5.00000	0.06631	0.04856	0.03834
5.50000	0.06648	0.04869	0.03844
6.00000	0.06664	0.04880	0.03853

Table : Error of approximation

Conclusion

Many researchers have studied different variants of Bernstein-Kantorovich operators, but in this study, we took the Bernstein basis based on Beta functions which were constructed in [7]. We study different kinds of approximation properties associated with these operators, e.g. we prove direct approximation results and approximation in weighted spaces as well. A Voronovskaya-type formula is also established. Finally, we provide a couple of numerical and graphical experiments to show the approximation properties of the newly defined operator. In the future, one can study the shape-preserving properties of these operators. Also, these bases can be used in computer-aided geometric design and approximate numerical solutions of differential and integral equations as well.

Data Availability

We do not have any data supporting our results.

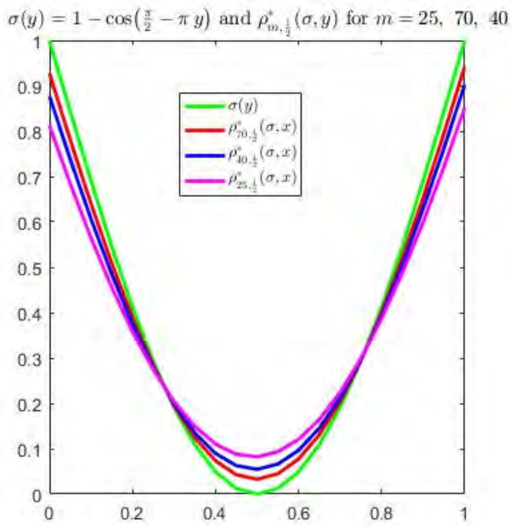

Figure 3: $\mathfrak{C}^*_{m,\alpha}(\sigma)$ preserves the convexity of $\sigma(y) = 1 - cos(\pi y - \frac{\pi}{2})$ on $[0, 1]$, where $\alpha = \frac{1}{2}, m = 25, 40, 70$.

Conflict of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Acknowledgement

The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through a research groups program under Grant number R.G.P. 1/151/43.

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