



A geometric approach to inequalities for the Hilbert–Schmidt norm

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Abstract. We show that if X and Y are two non-zero Hilbert–Schmidt operators, then for any $\lambda \geq 0$,

$$\begin{aligned} \cos^2 \Theta_{X,Y} &\leq \frac{1}{1+\lambda} \sqrt{\cos \Theta_{|X^*|,|Y^*|}} \sqrt{\cos \Theta_{|X|,|Y|}} \frac{|\langle X, Y \rangle|}{\|X\|_2 \|Y\|_2} + \frac{\lambda}{1+\lambda} \cos \Theta_{|X^*|,|Y^*|} \cos \Theta_{|X|,|Y|} \\ &\leq \cos \Theta_{|X^*|,|Y^*|} \cos \Theta_{|X|,|Y|}. \end{aligned}$$

Here $\Theta_{A,B}$ denotes the angle between non-zero Hilbert–Schmidt operators A and B . This enables us to present some inequalities for the Hilbert–Schmidt norm. In particular, we prove that

$$\|X + Y\|_2 \leq \sqrt{\frac{\sqrt{2} + 1}{2}} \| |X| + |Y| \|_2.$$

1. Introduction and preliminaries

Let $\mathcal{B}(\mathcal{H})$ denote the C^* -algebra of all bounded linear operators acting on a Hilbert space $(\mathcal{H}, [\cdot, \cdot])$. For every $X \in \mathcal{B}(\mathcal{H})$, let $|X|$ denote the square root of X^*X , that is, $|X| = (X^*X)^{1/2}$. Let $X = U|X|$ be the polar decomposition of X , where U is some partial isometry. The polar decomposition satisfies

$$U^*X = |X|, U^*U|X| = |X|, U^*UX = X, X^* = |X|U^*, |X^*| = U|X|U^*. \quad (1)$$

Let $C_1(\mathcal{H})$ and $C_2(\mathcal{H})$ denote the trace-class and the Hilbert–Schmidt class in $\mathcal{B}(\mathcal{H})$, respectively. It is well known that $C_1(\mathcal{H})$ and $C_2(\mathcal{H})$ each form a two-sided $*$ -ideal in $\mathcal{B}(\mathcal{H})$ and $C_2(\mathcal{H})$ is itself a Hilbert space with the inner product

$$\langle X, Y \rangle = \sum_{i=1}^{\infty} [Xe_i, Ye_i] = \text{Tr}(Y^*X) \quad (2)$$

where $\{e_i\}_{i=1}^{\infty}$ is any orthonormal basis of \mathcal{H} and $\text{Tr}(\cdot)$ is the natural trace on $C_1(\mathcal{H})$. Three principal properties of the trace are that it is a linear functional and, for every X and Y , we have $\text{Tr}(X^*) = \overline{\text{Tr}(X)}$ and

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$\text{Tr}(XY) = \text{Tr}(YX)$. The Hilbert–Schmidt norm of $X \in C_2(\mathcal{H})$ is given by $\|X\|_2 = \sqrt{\langle X, X \rangle}$. One more fact that will be needed the sequel is that if $X \in C_2(\mathcal{H})$, then

$$\|X\|_2 = \|X^*\|_2 = \||X|\|_2 = \||X^*|\|_2. \tag{3}$$

The reader is referred to [8, 10] for further properties of the Hilbert–Schmidt class.

In Section 2, by using the angle $\Theta_{X,Y}$ between non-zero Hilbert–Schmidt operators X, Y , we present an extension of the Fuglede–Putnam theorem modulo the Hilbert–Schmidt class. We also prove that $|\langle X, Y \rangle|^2 \leq \langle |X^*|, |Y^*| \rangle \langle |X|, |Y| \rangle$ and then we apply it to obtain for any $\lambda \geq 0$,

$$\cos^2 \Theta_{X,Y} \leq \frac{1}{1 + \lambda} \sqrt{\cos \Theta_{|X^*|, |Y^*|}} \sqrt{\cos \Theta_{|X|, |Y|}} \frac{|\langle X, Y \rangle|}{\|X\|_2 \|Y\|_2} + \frac{\lambda}{1 + \lambda} \cos \Theta_{|X^*|, |Y^*|} \cos \Theta_{|X|, |Y|}.$$

In Section 3, by using the results in Section 2, we provide alternative proofs of some well-known inequalities for the Hilbert–Schmidt norm. For example we present a considerably briefer proof of an extension of the Araki–Yamagami inequality [1] obtained by Kittaneh [3]. In particular, we prove Lee’s conjecture [6, p. 584] on the sum of the square roots of operators. Some related inequalities and numerical examples are also presented.

2. Angle between two Hilbert–Schmidt operators

Let $X, Y \in C_2(\mathcal{H})$. Since $C_2(\mathcal{H})$ is a Hilbert space with the inner product (2), by the Cauchy–Schwarz inequality, we have

$$-\|X\|_2 \|Y\|_2 \leq -|\langle X, Y \rangle| \leq \text{Re} \langle X, Y \rangle \leq |\langle X, Y \rangle| \leq \|X\|_2 \|Y\|_2. \tag{4}$$

Therefore, when X and Y are non-zero operators, the inequality (4) implies

$$-1 \leq \frac{\text{Re} \langle X, Y \rangle}{\|X\|_2 \|Y\|_2} \leq 1.$$

This motivates defining the angle between X and Y as follows (see [12]).

Definition 2.1. For non-zero operators $X, Y \in C_2(\mathcal{H})$, the angle $\Theta_{X,Y}$ between X and Y is defined by

$$\cos \Theta_{X,Y} = \frac{\text{Re} \langle X, Y \rangle}{\|X\|_2 \|Y\|_2}; \quad 0 \leq \Theta_{X,Y} \leq \pi.$$

Example 2.2. Let us recall that by [8, p. 66] we have

$$\text{Tr}(a \otimes b) = [a, b] \quad \text{and} \quad \|a \otimes b\|_2 = \|a\| \|b\|,$$

for all $a, b \in \mathcal{H}$. Here, $a \otimes b$ denotes the rank one operator in $\mathcal{B}(\mathcal{H})$ defined by $(a \otimes b)c := [c, b]a$ for all $c \in \mathcal{H}$. Now, let $x, y, z \in \mathcal{H} \setminus \{0\}$. Put $X = x \otimes z$ and $Y = y \otimes z$. A simple calculation shows that $\langle X, Y \rangle = \|z\|^2 [x, y]$. Thus,

$$\cos \Theta_{X,Y} = \frac{\text{Re}[x, y]}{\|x\| \|y\|}.$$

Remark 2.3. Let $X, Y \in C_2(\mathcal{H}) \setminus \{0\}$.

- (i) One can see that $\cos \Theta_{\gamma X, \gamma Y} = \cos \Theta_{X,Y}$ for all $\gamma \in \mathbb{C} \setminus \{0\}$.
- (ii) For every $\alpha, \beta \in \mathbb{R} \setminus \{0\}$, it is easy to check that

$$\cos \Theta_{\alpha X, \beta Y} = \begin{cases} \cos \Theta_{X,Y} & \alpha\beta > 0 \\ -\cos \Theta_{X,Y} & \alpha\beta < 0. \end{cases}$$

(iii) By (2) and Definition 2.1, we have

$$\|X \pm Y\|_2^2 = \|X\|_2^2 + \|Y\|_2^2 \pm 2\|X\|_2\|Y\|_2 \cos \Theta_{X,Y}.$$

(iv) If X and Y are positive, then

$$\langle X, Y \rangle = \text{Tr}(Y^{1/2}Y^{1/2}X) = \text{Tr}(Y^{1/2}XY^{1/2}) = \text{Tr}((X^{1/2}Y^{1/2})^*(X^{1/2}Y^{1/2})).$$

Therefore, $\text{Re}\langle X, Y \rangle = \langle X, Y \rangle \geq 0$ and hence $0 \leq \Theta_{X,Y} \leq \frac{\pi}{2}$.

Let us recall the Fuglede–Putnam theorem modulo the Hilbert–Schmidt class due to Weiss [11, Theorem 1]: if X and Y are normal operators and Z is an operator on \mathcal{H} , then $\|XZ - ZY\|_2 = \|X^*Z - ZY^*\|_2$. In the following theorem, we present an extension of the Fuglede–Putnam theorem modulo the Hilbert–Schmidt class which provided by Kittaneh in [4]. Our proof is very different from that in [4, Theorem 5].

Theorem 2.4. *If X, Y and Z are operators on \mathcal{H} , then*

$$\|XZ - ZY\|_2^2 + \|X^*Z\|_2^2 + \|ZY^*\|_2^2 = \|XZ\|_2^2 + \|ZY\|_2^2 + \|X^*Z - ZY^*\|_2^2.$$

Proof. With no loss of generality we may assume that the operators XZ, ZY, X^*Z and ZY^* are Hilbert–Schmidt; otherwise, both sides of the last equation in the theorem are infinite. We may also assume that $XZ, ZY, X^*Z, ZY^* \neq 0$ otherwise the desired equality trivially holds. By (2) and Definition 2.1 we have

$$\begin{aligned} \|XZ\|_2\|ZY\|_2 \cos \Theta_{XZ,ZY} &= \text{Re}\langle XZ, ZY \rangle = \text{ReTr}(Y^*Z^*XZ) \\ &= \overline{\text{ReTr}(Z^*X^*ZY)} = \text{ReTr}(Z^*X^*ZY) \\ &= \text{ReTr}(YZ^*X^*Z) = \text{Re}\langle X^*Z, ZY^* \rangle \\ &= \|X^*Z\|_2\|ZY^*\|_2 \cos \Theta_{X^*Z,ZY^*}, \end{aligned}$$

and hence

$$\|XZ\|_2\|ZY\|_2 \cos \Theta_{XZ,ZY} = \|X^*Z\|_2\|ZY^*\|_2 \cos \Theta_{X^*Z,ZY^*}. \tag{5}$$

So, by (5) and Remark 2.3(iii), we have

$$\begin{aligned} \|XZ - ZY\|_2^2 + \|X^*Z\|_2^2 + \|ZY^*\|_2^2 &= \|XZ\|_2^2 + \|ZY\|_2^2 - 2\|XZ\|_2\|ZY\|_2 \cos \Theta_{XZ,ZY} + \|X^*Z\|_2^2 + \|ZY^*\|_2^2 \\ &= \|XZ\|_2^2 + \|ZY\|_2^2 - 2\|X^*Z\|_2\|ZY^*\|_2 \cos \Theta_{X^*Z,ZY^*} + \|X^*Z\|_2^2 + \|ZY^*\|_2^2 \\ &= \|XZ\|_2^2 + \|ZY\|_2^2 + \|X^*Z - ZY^*\|_2^2. \end{aligned}$$

□

Remark 2.5. *Let X and Y be normal operators and let Z be an operator on \mathcal{H} . By (2) we have*

$$\begin{aligned} \|X^*Z\|_2^2 + \|ZY^*\|_2^2 &= \langle X^*Z, X^*Z \rangle + \langle ZY^*, ZY^* \rangle \\ &= \text{Tr}(Z^*XX^*Z) + \text{Tr}(YZ^*ZY^*) \\ &= \text{Tr}(Z^*XX^*Z) + \text{Tr}(Z^*ZY^*Y) \\ &= \text{Tr}(Z^*X^*XZ) + \text{Tr}(Z^*ZY^*Y^*) \\ &= \text{Tr}(Z^*X^*XZ) + \text{Tr}(Y^*Z^*ZY) \\ &= \langle XZ, XZ \rangle + \langle ZY, ZY \rangle = \|XZ\|_2^2 + \|ZY\|_2^2. \end{aligned}$$

Therefore if in Theorem 2.4, X and Y are assumed to be normal operators, then we retain the Fuglede–Putnam theorem modulo the Hilbert–Schmidt class.

We now state a interesting inequality for Hilbert–Schmidt operators.

Theorem 2.6. *If X and Y are Hilbert–Schmidt operators on \mathcal{H} , then*

$$|\langle X, Y \rangle|^2 \leq \langle |X^*|, |Y^*| \rangle \langle |X|, |Y| \rangle. \tag{6}$$

Moreover, the inequality in (6) becomes an equality if and only if $\zeta Y^* X$ is positive for some scalar ζ .

Proof. Let $X = U|X|$ and $Y = V|Y|$ be the polar decompositions of X and Y , respectively. By (1), (2) and the Cauchy–Schwarz inequality, we have

$$\begin{aligned} |\langle X, Y \rangle|^2 &= \left| \text{Tr}(|Y|V^*U|X|) \right|^2 \\ &= \left| \text{Tr}(|X|^{1/2}|Y|V^*U|X|^{1/2}) \right|^2 \\ &= \left| \langle |Y|^{1/2}V^*U|X|^{1/2}, |Y|^{1/2}|X|^{1/2} \rangle \right|^2 \\ &\leq \left\| |Y|^{1/2}V^*U|X|^{1/2} \right\|_2^2 \left\| |Y|^{1/2}|X|^{1/2} \right\|_2^2 \\ &= \text{Tr}(|X|^{1/2}U^*V|Y|V^*U|X|^{1/2}) \text{Tr}(|X|^{1/2}|Y||X|^{1/2}) \\ &= \text{Tr}(V|Y|V^*U|X|U^*) \text{Tr}(|Y||X|) \\ &= \text{Tr}(|Y^*||X^*|) \text{Tr}(|Y||X|) \\ &= \text{Tr}(|Y^*||X^*|) \text{Tr}(|Y^*||X|) = \langle |X^*|, |Y^*| \rangle \langle |X|, |Y| \rangle. \end{aligned}$$

The inequality in (6) becomes an equality if and only if

$$|Y|^{1/2}|X|^{1/2} = \zeta |Y|^{1/2}V^*U|X|^{1/2}$$

for some $\zeta \in \mathbb{C}$, and hence $|Y||X| = \zeta |Y|V^*U|X|$. So, by (1), we obtain $|Y||X| = \zeta Y^* X$. Therefore, the inequality in (6) becomes an equality if and only if $\zeta Y^* X$ is positive for some $\zeta \in \mathbb{C}$. \square

As a consequence of Theorem 2.6, we have the following result for trace-class operators.

Corollary 2.7. *If X is a trace-class operator on \mathcal{H} , then*

$$|\text{Tr} X| \leq \text{Tr} |X|. \tag{7}$$

Moreover, the inequality in (7) becomes an equality if and only if ζX is positive for some scalar $|\zeta| = 1$.

Proof. Let $X = U|X|$ be the polar decomposition of X . Since $|X|^{1/2}U$ and $|X|^{1/2}$ are Hilbert–Schmidt operators on \mathcal{H} , by Theorem 2.6 and (1), the result follows. \square

Remark 2.8. *Let X and Y be Hilbert–Schmidt operators on \mathcal{H} and let $\lambda \in \mathbb{R}^{\geq 0}$. By Theorem 2.6, we have*

$$\begin{aligned} |\langle X, Y \rangle|^2 &\leq \sqrt{\langle |X^*|, |Y^*| \rangle} \sqrt{\langle |X|, |Y| \rangle} |\langle X, Y \rangle| \\ &\leq \sqrt{\langle |X^*|, |Y^*| \rangle} \sqrt{\langle |X|, |Y| \rangle} |\langle X, Y \rangle| + \lambda \left(\langle |X^*|, |Y^*| \rangle \langle |X|, |Y| \rangle - |\langle X, Y \rangle|^2 \right). \end{aligned}$$

This implies

$$|\langle X, Y \rangle|^2 \leq \frac{1}{1 + \lambda} \sqrt{\langle |X^*|, |Y^*| \rangle} \sqrt{\langle |X|, |Y| \rangle} |\langle X, Y \rangle| + \frac{\lambda}{1 + \lambda} \langle |X^*|, |Y^*| \rangle \langle |X|, |Y| \rangle. \tag{8}$$

Note that by Theorem 2.6 we have

$$\begin{aligned} & \frac{1}{1+\lambda} \sqrt{\langle |X^*|, |Y^*| \rangle} \sqrt{\langle |X|, |Y| \rangle} |\langle X, Y \rangle| + \frac{\lambda}{1+\lambda} \langle |X^*|, |Y^*| \rangle \langle |X|, |Y| \rangle \\ & \leq \frac{1}{1+\lambda} \langle |X^*|, |Y^*| \rangle \langle |X|, |Y| \rangle + \frac{\lambda}{1+\lambda} \langle |X^*|, |Y^*| \rangle \langle |X|, |Y| \rangle \\ & = \langle |X^*|, |Y^*| \rangle \langle |X|, |Y| \rangle, \end{aligned}$$

and hence the inequality (8) refines the inequality (6).

Here we present one of the main results of this paper. In fact, the following theorem enables us to provide alternative proofs of some well-known inequalities for the Hilbert–Schmidt norm.

Theorem 2.9. For non-zero Hilbert–Schmidt operators X and Y on \mathcal{H} the following properties hold.

- (i) $\cos^2 \Theta_{X,Y} \leq \cos \Theta_{|X^*|,|Y^*|} \cos \Theta_{|X|,|Y|}$.
- (ii) $|\cos \Theta_{X,Y}| \leq \min \{ \sqrt{\cos \Theta_{|X^*|,|Y^*|}}, \sqrt{\cos \Theta_{|X|,|Y|}} \}$.
- (iii) $\sin^2 \Theta_{|X^*|,|Y^*|} + \sin^2 \Theta_{|X|,|Y|} \leq 2 \sin^2 \Theta_{X,Y}$.

Proof. (i) By (3), (4), Definition 2.1 and Theorem 2.6 we have

$$\begin{aligned} \cos^2 \Theta_{X,Y} &= \left(\frac{\operatorname{Re}\langle X, Y \rangle}{\|X\|_2 \|Y\|_2} \right)^2 \\ &\leq \frac{|\langle X, Y \rangle|^2}{\|X\|_2^2 \|Y\|_2^2} \\ &\leq \frac{\langle |X^*|, |Y^*| \rangle \langle |X|, |Y| \rangle}{\|X\|_2 \|Y\|_2 \|X\|_2 \|Y\|_2} \\ &= \frac{\operatorname{Re}\langle |X^*|, |Y^*| \rangle}{\| |X^*| \|_2 \| |Y^*| \|_2} \frac{\operatorname{Re}\langle |X|, |Y| \rangle}{\| |X| \|_2 \| |Y| \|_2} \\ &= \cos \Theta_{|X^*|,|Y^*|} \cos \Theta_{|X|,|Y|}. \end{aligned} \tag{9}$$

(ii) The proof follows immediately from (i).

(iii) By the arithmetic-geometric mean inequality and (i) we have

$$\begin{aligned} \sin^2 \Theta_{|X^*|,|Y^*|} + \sin^2 \Theta_{|X|,|Y|} &= 2 - \left(\cos^2 \Theta_{|X^*|,|Y^*|} + \cos^2 \Theta_{|X|,|Y|} \right) \\ &\leq 2 - 2 \cos \Theta_{|X^*|,|Y^*|} \cos \Theta_{|X|,|Y|} \\ &\leq 2 - 2 \cos^2 \Theta_{X,Y} = 2 \sin^2 \Theta_{X,Y}. \end{aligned}$$

□

Remark 2.10. Let X and Y be non-zero Hilbert–Schmidt operators on \mathcal{H} . For any $\lambda \geq 0$, by (3) and (8), we have

$$\begin{aligned} \left(\frac{\operatorname{Re}\langle X, Y \rangle}{\|X\|_2 \|Y\|_2} \right)^2 &\leq \frac{|\langle X, Y \rangle|^2}{\|X\|_2^2 \|Y\|_2^2} \\ &\leq \frac{\frac{1}{1+\lambda} \sqrt{\langle |X^*|, |Y^*| \rangle} \sqrt{\langle |X|, |Y| \rangle} |\langle X, Y \rangle| + \frac{\lambda}{1+\lambda} \langle |X^*|, |Y^*| \rangle \langle |X|, |Y| \rangle}{\|X\|_2^2 \|Y\|_2^2} \\ &= \frac{1}{1+\lambda} \sqrt{\frac{\operatorname{Re}\langle |X^*|, |Y^*| \rangle}{\| |X^*| \|_2 \| |Y^*| \|_2}} \sqrt{\frac{\operatorname{Re}\langle |X|, |Y| \rangle}{\| |X| \|_2 \| |Y| \|_2}} \frac{|\langle X, Y \rangle|}{\|X\|_2 \|Y\|_2} \\ &\quad + \frac{\lambda}{1+\lambda} \frac{\operatorname{Re}\langle |X^*|, |Y^*| \rangle}{\| |X^*| \|_2 \| |Y^*| \|_2} \frac{\operatorname{Re}\langle |X|, |Y| \rangle}{\| |X| \|_2 \| |Y| \|_2}. \end{aligned}$$

So, by Definition 2.1 it follows that

$$\cos^2 \Theta_{X,Y} \leq \frac{1}{1+\lambda} \sqrt{\cos \Theta_{|X^*||Y^*|}} \sqrt{\cos \Theta_{|X||Y|}} \frac{|\langle X, Y \rangle|}{\|X\|_2 \|Y\|_2} + \frac{\lambda}{1+\lambda} \cos \Theta_{|X^*||Y^*|} \cos \Theta_{|X||Y|}. \tag{10}$$

Note that by (9) we have

$$\begin{aligned} & \frac{1}{1+\lambda} \sqrt{\cos \Theta_{|X^*||Y^*|}} \sqrt{\cos \Theta_{|X||Y|}} \frac{|\langle X, Y \rangle|}{\|X\|_2 \|Y\|_2} + \frac{\lambda}{1+\lambda} \cos \Theta_{|X^*||Y^*|} \cos \Theta_{|X||Y|} \\ & \leq \frac{1}{1+\lambda} \cos \Theta_{|X^*||Y^*|} \cos \Theta_{|X||Y|} + \frac{\lambda}{1+\lambda} \cos \Theta_{|X^*||Y^*|} \cos \Theta_{|X||Y|} \\ & = \cos \Theta_{|X^*||Y^*|} \cos \Theta_{|X||Y|}. \end{aligned}$$

Therefore, the inequality (10) is a refinement of the inequality in Theorem 2.9(i).

3. Inequalities for the Hilbert–Schmidt norm

In this section, by using Theorem 2.9, we provide alternative proof of some well-known inequalities for the Hilbert–Schmidt norm. First we present a considerably briefer proof of an extension of the Araki–Yamagami inequality [1] obtained by Kittaneh [3, Theorem 2].

Theorem 3.1. *If X and Y are Hilbert–Schmidt operators on \mathcal{H} , then*

$$\| |X^*| - |Y^*| \|_2^2 + \| |X| - |Y| \|_2^2 \leq 2\|X - Y\|_2^2.$$

Proof. Since the desired inequality trivially holds when $X = 0$ or $Y = 0$, we may assume $X, Y \neq 0$. By (3), Remark 2.3(iii), Theorem 2.9(i) and the arithmetic-geometric mean inequality we have

$$\begin{aligned} \| |X^*| - |Y^*| \|_2^2 + \| |X| - |Y| \|_2^2 &= \| |X^*| \|_2^2 + \| |Y^*| \|_2^2 - 2\| |X^*| \|_2 \| |Y^*| \|_2 \cos \Theta_{|X^*||Y^*|} \\ &\quad + \| |X| \|_2^2 + \| |Y| \|_2^2 - 2\| |X| \|_2 \| |Y| \|_2 \cos \Theta_{|X||Y|} \\ &= 2\|X\|_2^2 + 2\|Y\|_2^2 - 2\|X\|_2 \|Y\|_2 (\cos \Theta_{|X^*||Y^*|} + \cos \Theta_{|X||Y|}) \\ &\leq 2\|X\|_2^2 + 2\|Y\|_2^2 - 4\|X\|_2 \|Y\|_2 \sqrt{\cos \Theta_{|X^*||Y^*|} \cos \Theta_{|X||Y|}} \\ &\leq 2\|X\|_2^2 + 2\|Y\|_2^2 - 4\|X\|_2 \|Y\|_2 |\cos \Theta_{X,Y}| \\ &\leq 2\|X\|_2^2 + 2\|Y\|_2^2 - 4\|X\|_2 \|Y\|_2 \cos \Theta_{X,Y} = 2\|X - Y\|_2^2. \end{aligned}$$

□

As an immediate consequence of Theorem 3.1, we get the Araki–Yamagami inequality [1, Theorem 1].

Corollary 3.2. *If X and Y are Hilbert–Schmidt operators on \mathcal{H} , then*

$$\| |X| - |Y| \|_2 \leq \sqrt{2}\|X - Y\|_2.$$

Remark 3.3. *In [1], Araki and Yamagami remarked that $\sqrt{2}$ is the best possible coefficient for a general X and Y . Now let X and Y be normal operators. Since X and Y are normal operators, the spectral theorem (see [8]) implies that $|X^*| = |X|$, $|Y^*| = |Y|$ and hence, by Theorem 3.1 we obtain*

$$\| |X| - |Y| \|_2 \leq \|X - Y\|_2.$$

Therefore, if X and Y are restricted to be normal, then the best coefficient in the Araki–Yamagami inequality is 1 instead of $\sqrt{2}$.

The following result is a special case of [6, Theorem 2.1].

Theorem 3.4. *If X and Y are Hilbert–Schmidt operators on \mathcal{H} , then*

$$\|X + Y\|_2^2 \leq \| |X^*| + |Y^*| \|_2 \| |X| + |Y| \|_2.$$

Proof. We may assume that $X, Y \neq 0$ otherwise the desired inequality trivially holds. By (3), Remark 2.3(iii), Theorem 2.9(i) and the arithmetic-geometric mean inequality we have

$$\begin{aligned} \| |X^*| + |Y^*| \|_2^2 \| |X| + |Y| \|_2^2 &= (\| |X^*| \|_2^2 + \| |Y^*| \|_2^2 + 2\| |X^*| \|_2 \| |Y^*| \|_2 \cos \Theta_{|X^*|, |Y^*|}) \\ &\quad \times (\| |X| \|_2^2 + \| |Y| \|_2^2 + 2\| |X| \|_2 \| |Y| \|_2 \cos \Theta_{|X|, |Y|}) \\ &= (\|X\|_2^2 + \|Y\|_2^2 + 2\|X\|_2 \|Y\|_2 \cos \Theta_{|X^*|, |Y^*|}) \\ &\quad \times (\|X\|_2^2 + \|Y\|_2^2 + 2\|X\|_2 \|Y\|_2 \cos \Theta_{|X|, |Y|}) \\ &= (\|X\|_2^2 + \|Y\|_2^2)^2 + 4\|X\|_2^2 \|Y\|_2^2 \cos \Theta_{|X^*|, |Y^*|} \cos \Theta_{|X|, |Y|} \\ &\quad + 2\|X\|_2 \|Y\|_2 (\|X\|_2^2 + \|Y\|_2^2) (\cos \Theta_{|X^*|, |Y^*|} + \cos \Theta_{|X|, |Y|}) \\ &\geq (\|X\|_2^2 + \|Y\|_2^2)^2 + 4\|X\|_2^2 \|Y\|_2^2 \cos \Theta_{|X^*|, |Y^*|} \cos \Theta_{|X|, |Y|} \\ &\quad + 4\|X\|_2 \|Y\|_2 (\|X\|_2^2 + \|Y\|_2^2) \sqrt{\cos \Theta_{|X^*|, |Y^*|} \cos \Theta_{|X|, |Y|}} \\ &\geq (\|X\|_2^2 + \|Y\|_2^2)^2 + 4\|X\|_2^2 \|Y\|_2^2 \cos^2 \Theta_{X, Y} \\ &\quad + 4\|X\|_2 \|Y\|_2 (\|X\|_2^2 + \|Y\|_2^2) |\cos \Theta_{X, Y}| \\ &\geq (\|X\|_2^2 + \|Y\|_2^2)^2 + 4\|X\|_2^2 \|Y\|_2^2 \cos^2 \Theta_{X, Y} \\ &\quad + 4\|X\|_2 \|Y\|_2 (\|X\|_2^2 + \|Y\|_2^2) \cos \Theta_{X, Y} \\ &= (\|X\|_2^2 + \|Y\|_2^2 + 2\|X\|_2 \|Y\|_2 \cos \Theta_{X, Y})^2 = \|X + Y\|_2^4. \end{aligned}$$

□

Next, we provide alternative proof of an inequality for the Hilbert–Schmidt norm due to Kittaneh [5, Theorem 2.1].

Theorem 3.5. *If X and Y are Hilbert–Schmidt operators on \mathcal{H} , then*

$$\| |X| - |Y| \|_2^2 \leq \|X + Y\|_2 \|X - Y\|_2.$$

Proof. Since the desired inequality trivially holds when $X = 0$ or $Y = 0$, we may assume $X, Y \neq 0$. By the arithmetic-geometric mean inequality we have

$$\|X\|_2 \|Y\|_2 (1 + \cos \Theta_{|X|, |Y|}) \leq 2\|X\|_2 \|Y\|_2 \leq \|X\|_2^2 + \|Y\|_2^2.$$

Then, since $\cos \Theta_{|X|, |Y|} \geq 0$, we obtain

$$\|X\|_2 \|Y\|_2 (\cos \Theta_{|X|, |Y|} + \cos^2 \Theta_{|X|, |Y|}) \leq \cos \Theta_{|X|, |Y|} (\|X\|_2^2 + \|Y\|_2^2),$$

and so

$$\|X\|_2 \|Y\|_2 \cos \Theta_{|X|, |Y|} \leq \cos \Theta_{|X|, |Y|} (\|X\|_2^2 + \|Y\|_2^2) - \|X\|_2 \|Y\|_2 \cos^2 \Theta_{|X|, |Y|}. \tag{11}$$

Therefore by (3), Remark 2.3(iii), Theorem 2.9(ii) and (11) we have

$$\begin{aligned} \|X + Y\|_2^2 \|X - Y\|_2^2 &= (\|X\|_2^2 + \|Y\|_2^2)^2 - 4\|X\|_2^2 \|Y\|_2^2 \cos^2 \Theta_{X,Y} \\ &\geq (\|X\|_2^2 + \|Y\|_2^2)^2 - 4\|X\|_2^2 \|Y\|_2^2 \cos \Theta_{|X|,|Y|} \\ &\geq (\|X\|_2^2 + \|Y\|_2^2)^2 - 4\|X\|_2 \|Y\|_2 (\|X\|_2^2 + \|Y\|_2^2) \cos \Theta_{|X|,|Y|} + 4\|X\|_2^2 \|Y\|_2^2 \cos^2 \Theta_{|X|,|Y|} \\ &= (\|X\|_2^2 + \|Y\|_2^2 - 2\|X\|_2 \|Y\|_2 \cos \Theta_{|X|,|Y|})^2 \\ &= (\| |X| \|_2^2 + \| |Y| \|_2^2 - 2\| |X| \|_2 \| |Y| \|_2 \cos \Theta_{|X|,|Y|})^2 = \| |X| - |Y| \|_2^4. \end{aligned}$$

□

The following result may be stated as well.

Theorem 3.6. *If X and Y are Hilbert–Schmidt operators on \mathcal{H} , then*

$$\| |X^*| + |Y^*| \|_2 \leq \sqrt{2} \| |X| + |Y| \|_2.$$

Proof. We may assume that $X, Y \neq 0$ otherwise the desired inequality trivially holds. Since $-2 \leq \cos \Theta_{|X^*|,|Y^*|} - 2 \cos \Theta_{|X|,|Y|} \leq 1$, by the arithmetic-geometric mean inequality we get

$$2\|X\|_2 \|Y\|_2 (\cos \Theta_{|X^*|,|Y^*|} - 2 \cos \Theta_{|X|,|Y|}) \leq 2\|X\|_2 \|Y\|_2 \leq \|X\|_2^2 + \|Y\|_2^2.$$

Hence

$$2\|X\|_2 \|Y\|_2 \cos \Theta_{|X^*|,|Y^*|} \leq \|X\|_2^2 + \|Y\|_2^2 + 4\|X\|_2 \|Y\|_2 \cos \Theta_{|X|,|Y|}. \tag{12}$$

So, by (3), Remark 2.3(iii) and (12) we have

$$\begin{aligned} \| |X^*| + |Y^*| \|_2^2 &= \| |X^*| \|_2^2 + \| |Y^*| \|_2^2 + 2\| |X^*| \|_2 \| |Y^*| \|_2 \cos \Theta_{|X^*|,|Y^*|} \\ &= \|X\|_2^2 + \|Y\|_2^2 + 2\|X\|_2 \|Y\|_2 \cos \Theta_{|X^*|,|Y^*|} \\ &\leq 2\|X\|_2^2 + 2\|Y\|_2^2 + 4\|X\|_2 \|Y\|_2 \cos \Theta_{|X|,|Y|} \\ &= 2(\| |X| \|_2^2 + \| |Y| \|_2^2 + 2\| |X| \|_2 \| |Y| \|_2 \cos \Theta_{|X|,|Y|}) = 2\| |X| + |Y| \|_2^2. \end{aligned}$$

□

In [6, p. 584], Lee conjectured that for arbitrary n -by- n matrices A and B , the inequality

$$\|A + B\|_2 \leq \sqrt{\frac{\sqrt{2} + 1}{2}} \| |A| + |B| \|_2$$

holds. Very recently, a proof of Lee’s conjecture has been presented in [7] as our work was in progress. We end this section by a proof of Lee’s conjecture for operators.

Theorem 3.7. *If X and Y are Hilbert–Schmidt operators on \mathcal{H} , then*

$$\|X + Y\|_2 \leq \sqrt{\frac{\sqrt{2} + 1}{2}} \| |X| + |Y| \|_2.$$

Proof. Since the desired inequality trivially holds when $X = 0$ or $Y = 0$, we may assume $X, Y \neq 0$. By (3), Remark 2.3(iii), Theorem 2.9(ii) we have

$$\begin{aligned} \frac{\sqrt{2}+1}{2} \| |X| + |Y| \|_2^2 &= \frac{\sqrt{2}+1}{2} (\| |X| \|_2^2 + \| |Y| \|_2^2 + 2\| |X| \|_2 \| |Y| \|_2 \cos \Theta_{|X||Y|}) \\ &= \frac{\sqrt{2}+1}{2} (\|X\|_2^2 + \|Y\|_2^2 + 2\|X\|_2 \|Y\|_2 \cos \Theta_{|X||Y|}) \\ &\geq \frac{\sqrt{2}+1}{2} (\|X\|_2^2 + \|Y\|_2^2 + 2\|X\|_2 \|Y\|_2 \cos^2 \Theta_{X,Y}) \\ &= \|X\|_2^2 + \|Y\|_2^2 + 2\|X\|_2 \|Y\|_2 \cos \Theta_{X,Y} + \frac{\sqrt{2}-1}{2} (\|X\|_2^2 + \|Y\|_2^2) \\ &\quad + \|X\|_2 \|Y\|_2 ((\sqrt{2}+1) \cos^2 \Theta_{X,Y} - 2 \cos \Theta_{X,Y}) \\ &= \|X+Y\|_2^2 + \frac{\sqrt{2}-1}{2} (\|X\|_2 - \|Y\|_2)^2 \\ &\quad + \|X\|_2 \|Y\|_2 ((\sqrt{2}-1) + (\sqrt{2}+1) \cos^2 \Theta_{X,Y} - 2 \cos \Theta_{X,Y}) \\ &= \|X+Y\|_2^2 + \frac{\sqrt{2}-1}{2} (\|X\|_2 - \|Y\|_2)^2 \\ &\quad + \|X\|_2 \|Y\|_2 (\sqrt{\sqrt{2}-1} - \sqrt{\sqrt{2}+1} \cos \Theta_{X,Y})^2 \\ &\geq \|X+Y\|_2^2. \end{aligned}$$

□

Remark 3.8. Suppose $M_2(\mathbb{C})$ is the algebra of all complex 2×2 matrices. Let $\text{Det}(A)$ denote the determinant of $A \in M_2(\mathbb{C})$. Recall (see e.g. [2, p. 460]) that for $A \in M_2(\mathbb{C})$ we have

$$|A| = \frac{1}{\sqrt{\text{Tr}(A^*A) + 2\sqrt{\text{Det}(A^*A)}}} (\sqrt{\text{Det}(A^*A)}I + A^*A).$$

Now, let $B = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 0 \\ 1 - \sqrt{2} & \sqrt{\sqrt{8}-2} \end{bmatrix}$. Then simple computations show that $|B| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $|B^*| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $|C| = |C^*| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $|D| = \begin{bmatrix} 3 - \sqrt{8} & -\sqrt{\sqrt{200}-14} \\ -\sqrt{\sqrt{200}-14} & \sqrt{8}-2 \end{bmatrix}$. Therefore,

$$\| |B^*| + |C^*| \|_2 = \left\| \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \right\|_2 = \sqrt{\text{Tr} \left(\begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \right)} = 2$$

and

$$\sqrt{2} \| |B| + |C| \|_2 = \sqrt{2} \left\| \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\|_2 = \sqrt{2} \sqrt{\text{Tr} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)} = 2.$$

Hence the inequality in Theorem 3.6 is sharp. In addition, we have

$$\|B+D\|_2 = \left\| \begin{bmatrix} 0 & 0 \\ -\sqrt{2} & \sqrt{\sqrt{8}-2} \end{bmatrix} \right\|_2 = \sqrt{\text{Tr} \left(\begin{bmatrix} 2 & -\sqrt{\sqrt{32}-4} \\ -\sqrt{\sqrt{32}-4} & \sqrt{8}-2 \end{bmatrix} \right)} = \sqrt[4]{8}$$

and

$$\begin{aligned} \sqrt{\frac{\sqrt{2}+1}{2}} \| |B| + |D| \|_2 &= \sqrt{\frac{\sqrt{2}+1}{2}} \left\| \begin{bmatrix} 4 - \sqrt{8} & -\sqrt{\sqrt{200}-14} \\ -\sqrt{\sqrt{200}-14} & \sqrt{8}-2 \end{bmatrix} \right\|_2 \\ &= \sqrt{\frac{\sqrt{2}+1}{2}} \sqrt{\operatorname{Tr} \left(\begin{bmatrix} 10 - \sqrt{72} & -\sqrt{\sqrt{3200}-56} \\ -\sqrt{\sqrt{3200}-56} & \sqrt{8}-2 \end{bmatrix} \right)} \\ &= \sqrt{\frac{\sqrt{2}+1}{2}} \sqrt{2\sqrt{8}-4} = \sqrt[4]{8}. \end{aligned}$$

So, the inequality in Theorem 3.7 is also sharp.

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