Filomat 37:30 (2023), 10355–10368
https://doi.org/10.2298/FIL2330355P

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Study on distributional systems with economic applications

Liviu Popescu[a], Esmaeil Peyghan[b,*], Ramona Dimitrov[a], Gabriel Tica[a]

[a]*University of Craiova, Faculty of Economics and Business Administration, Dept. of Statistics and Economic Informatics, Al. I. Cuza st., no. 13, 200585 Craiova, Romania*
[b]*Arak University, Faculty of Science, Dept. of Mathematics, Arak, Iran*

Abstract. In the present work, two optimal control systems with economic applications are investigated, concerning the minimum cost of production and storage. The framework of distributional systems is used in the study of controllability, applying Lie geometric methods. In both cases of holonomic and nonholonomic distributions, Pontryagin's Maximum Principle leads to optimal solutions. In the case of a non-linear equation, a numerical solution is used. Finally, some numerical examples with graphical representations are given.

1. Introduction

It is well known that geometric Lie methods have been used since the last century in many research fields including optimal control theory. Distributional systems or driftless control affine systems are intensely studied, having applications in robotics, economics or sub-Riemannian geometry. In fact, the sub-Riemannian geometry can be regarded as the geometry of driftless control affine systems with quadratic cost and non-integrable (nonholonomic) distribution, which is bracket generating [3, 6]. Also, the case of integrable (holonomic) distribution provided very interesting applications and controllability information can be found in the structure of the Lie algebra formed by the vector fields that generate the distribution. In [4] the link between Lie theory and control is given. Control theory methods viewed from a geometrical point of view can be found, for example in [1, 7]. The framework of Lie algebroids is used in the investigation of distributional systems in the papers [9–12, 14, 16]. Applications of optimal control in economics have been studied in many scientific works, see for instance [2, 5, 17–19]. The framework of distributional systems in the study of optimization problems of production and storage costs can be found in [9]-[16]. One of the most used methods in solving an optimal control problem is Pontryagin's Maximum Principle. We say that a curve denoted by $c(t) = (x(t), u(t))$ is an optimal trajectory of the control system, if there is a lift of $x(t)$ to the dual space $(x(t), p(t))$ which satisfies the Hamilton-Jacobi-Bellman equations.

The main purpose of this paper is to study two optimal control systems with economic applications, concerning the minimum cost of production and storage. In the paper we will use geometric Lie methods

2020 *Mathematics Subject Classification*. 49K35; 91B02.
Keywords. Optimal control; Distributional systems; Controllability; Economic applications.
Received: 02 March 2023; Accepted: 25 June 2023
Communicated by Mića Stanković
The work of PhD students Ramona Dimitrov and Gabriel Tica was supported by the grant POCU380/6/13/123990, co-financed by the European Social Fund within the Sectorial Operational Human Capital 2014–2020.
* Corresponding author: Esmaeil Peyghan
Email addresses: liviu.popescu@edu.ucv.ro (Liviu Popescu), e-peyghan@araku.ac.ir (Esmaeil Peyghan), ramona.dimitrov@yahoo.com (Ramona Dimitrov), tica.gabriel@gmail.com (Gabriel Tica)

to investigate the controllability properties of systems in the both case of holonomic and nonholonomic distributions and Pontryagin's Maximum Principle to find the optimal solutions. We will see that if the distribution is integrable, the control system is not controllable and the distribution generates a foliation in three-dimensional space In the case of nonholonomic distribution, which is also strong bracket generating, the control system is mathematicaly controllable, but the positive conditions of trajectories, given by economical aspects lead to some final restriction. The work is organized in three sections as follows. In the second section some preliminaries about optimal control are presented, including distributional systems with holonomic and nonholonomic distribution. In the third section, which contains the novelty of the paper, two optimal control systems are investigated. The controllability problem is solved and optimal solutions are given. In the case of nonholonomic distribution, for a nonlinear equation a numerical solution is presented. At the end of each studied problem, several numerical examples with graphical representations are given.

2. Optimal Control

In what follows, we consider M to be an n-dimensional and smooth manifold and we will deal with control systems given by differential equations, which depend on certain parameters, in the form

$$\frac{dx^i}{dt} = f^i(x, u),$$

where $x \in M$ is the state of our system and $u \in U \subset \mathbb{R}^m$ is the controls. If we consider two points of M, given by x_0 and x_1, then an optimal control problem consists in determining trajectories of the control system that joins the points x_0 and x_1 and, at the same time, minimizes a functional cost

$$\min \int_0^T L(x(t), u(t))dt, \quad x(0) = x_0, \ x(T) = x_1,$$

where L is called the *Lagrangian* function. Control theory studies dynamic systems whose evolution over time can be influenced by some external variables. One of the most used tools for study optimal solutions for a control system is Pontryagin's Maxim Principle, which generates necessary, but not sufficient conditions for finding optimal solutions. For an optimal trajectory, given by a curve $c(t) = (x(t), u(t))$, this gives a lift on the dual space $(x(t), p(t))$ that satisfies the so-called Hamilton-Jacobi-Bellman equations. The Hamiltonian function has the form

$$H(x, p, u) = \langle p, f(x, u)\rangle - L(x, u), \quad p \in T^*M,$$

and the maximization condition given by

$$H(x(t), p(t), u(t)) = \max_s H(x(t), p(t), s),$$

implies $\frac{\partial H}{\partial u} = 0$, where the optimal trajectories satisfy the following equations

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}. \tag{1}$$

2.1. Distributional systems

A distributional system (driftless control affine system) has the following form [8]

$$\dot{x}(t) = \sum_{i=1}^{m} u_i(t) X_i(x(t)), \tag{2}$$

where $x = (x_1, ..., x_n)$ represent the local coordinates and $u(t) = (u_1(t), ..., u_m(t)) \in U \subset \mathbb{R}^m$, $m \leq n$, is called the control. Also, $X_1 ... X_m$ represent the smooth vector fields on the manifold M, which are called the input vector fields. Using [8] we have:

Definition 2.1. *A control system is called controllable if for every two points x_0 and x_1 on the manifold M, there exists a finite T and a control $u : [0, T] \to U$ such that for x satisfying $x(0) = x_0$ we have that $x(T) = x_1$.*

We can say that the control system is controllable, if for any two states x_0, x_1, there is a curve, which is the solution of the system of differential equations (2) and which, at the same time, connects x_0 to x_1. In other words, controllability is the ability to drive a system from any given initial state to any given final state in a finite time, using the available control functions. We recall that [8]

Definition 2.2. *A distribution denoted Δ on a smooth manifold M is an application that assigns to each point in M a subspace of the tangent space at that point $x \to \Delta(x) \subset T_xM$.*

We say that Δ is locally finitely generated if there is a family of vector fields $\{X_j\}_{j=\overline{1,m}}$ which generates Δ. The dimension of Δ is k if $\dim \Delta(x) = k$, for all points x in M. We recall the Lie bracket given by

$$[h, f](x) = \frac{\partial f}{\partial x}(x)h(x) - \frac{\partial h}{\partial x}(x)f(x),$$

A distribution Δ on M is said to be involutive if for $x \in M$ and $h(x), f(x) \in \Delta(x)$ it results $\left[h, f\right](x) \in \Delta(x)$. If the involutive distribution is also locally finitely generated by the vector fields $\{X_k\}_{k=\overline{1,m}}$ then it results

$$[X_r, X_s](x) = \sum_{i=1}^{m} L^i_{rs}(x)X_i(x),$$

and the Lie bracket can be written as a linear combination of the system vector fields. We recall that a foliation denoted $\{F_\alpha\}_{\alpha\in A}$ on the manifold M is a partition of $M = \bigcup_{\alpha\in A} F_\alpha$ of M into disjoint connected submanifolds F_α, which are called leaves.

Definition 2.3. *The distribution Δ, which has constant dimension on M, is named holonomic (or integrable) if there exists a foliation $\{F_\alpha\}_{\alpha\in A}$ on M whose tangent space is Δ, that is $T_xF = \Delta(x)$, where F is the leaf passing through x.*

Theorem 2.4. *(Frobenius) If the distribution Δ has constant dimension, then Δ is integrable if and only if it is involutive.*

Using [8] we have:

Definition 2.5. *The distribution Δ generated by vector fields $\{X_1, ..., X_m\}$ on M is called bracket generating if the iterated*

$$X_r, [X_r, X_s], [X_r, [X_s, X_k]], \cdots, \quad 1 \le r, s, k \le m,$$

span the tangent bundle TM of M in any point.

Now, with the help of the Lie brackets of the vector fields we have

$$\Delta \subset \Delta^2 \subset \cdots \subset \Delta^k \subset \cdots \subset TM,$$

where

$$\Delta^2 = \Delta + [\Delta, \Delta], ..., \Delta^{k+1} = \Delta^k + \left[\Delta, \Delta^k\right],$$

and

$$\left[\Delta, \Delta^k\right] = span\{[Y, Z] : Y \in \Delta,\ Z \in \Delta^k\}.$$

For any $k \ge 2$ for which $\Delta^k = TM$, then Δ is bracket generating distribution and k is named the step of the distribution. In this case the distribution Δ is called nonholonomic (non-integrable). This condition is also mentioned in the specialized literature with the name *strong Hörmander condition*, or *Lie algebra rank condition*. For $k = 2$ the distribution is called *strong bracket generating*.

Next, if we return to the case of distributional systems, where the vector fields X_k, $k = \overline{1, m}$, generate a distribution with constant rank Δ on the connected manifold M. We can characterize the controllability using the properties of Lie algebra generated by the vector fields.

Theorem 2.6. *(Chow-Rashevsky) If the distribution* $\Delta = span\{X_1, ..., X_m\}$ *is bracket generating (nonholonomic), then the driftless control affine system is controllable.*

In the case of the integrable distribution Δ with constant rank, the control system is not controllable and Δ determines a foliation on the manifold M having the property that any trajectory is contained in a single leaf of the foliation, and the restriction of Δ at each leaf of the foliation is bracket generating.

3. Application to optimization of production and storage costs

Suppose that an economic process includes the manufacture of three products P_1, P_2, P_3 in given final quantities and in a given period of time, such that the rate of production for third product depends on the production rates of the first two (which are controllable) by a fixed law. It is known that the unit production costs for the first two products have a linear increase with the level of production and the costs of production operations for the third product are sufficiently small and are not taken into account. We know the storage unit costs for each type of product denoted with $(\beta_1, \beta_2, \beta_3)$ and and we have no restrictions on production capacity, having the possibility to rent certain machines, but with the increase of production costs for P_1 and P_2. An optimal production plan is sought with the manufacture of the products in the specified quantities in the fixed period of time, so that the total cost of storage and production is minimal. We will have the notations: T is the fixed period of time required to manufacture the products; s_i, $i = \overline{1, 3}$ are the final quantities required; $x^i(t)$, are the manufactured quantities until time t; $p^i(t)$ are the production rates at time t; (c_1, c_2) are the production costs per unit for the first two products. We can also make the assumption that we have no initial quantity of each product type. We have the rate of production for third product given by the equation $\dot{x}^3 = (u^1 + u^2)(x^1 + x^2)$, where u^1, u^2 are control variables given by $\dot{x}^1 = u^1$, $\dot{x}^2 = u^2$. The unit production costs are given by $c_i = \alpha_i p^i$, where $\alpha_1, \alpha_2 > 0$, $i = 1, 2$. Considering the initial stock zero for all products, that is $x^i(0) = 0$ we have

$$x^i(t) = \int_0^t p^i(y)dy,$$

and the rate of change of stock level $\dot{x}^i(t)$ is the production rate and it results $\dot{x}^i(t) = p^i(t)$. Also, the total cost of production is given by $c_1p^1 + c_2p^2 = \alpha_1(p^1)^2 + \alpha_2(p^2)^2 = \alpha_1(\dot{x}^1)^2 + \alpha_2(\dot{x}^2)^2 = \alpha_1(u^1)^2 + \alpha_2(u^2)^2$. It follows that the total cost generated by production and storage has the form $\alpha_1(u^1)^2 + \alpha_2(u^2)^2 + \beta_1 x^1 + \beta_2 x^2 + \beta_3 x^3$. From the above, From the previous ones, an optimal control system results, in the form:

$$\begin{cases} \dot{x}^1 = u^1 \\ \dot{x}^2 = u^2 \\ \dot{x}^3 = (u^1 + u^2)(x^1 + x^2) \\ x^i(0) = 0,\ x^i(T) = s_i \\ u^1, u^2 \geq 0,\ \alpha_{1,}\alpha_2, \beta_1, \beta_2, \beta_3 > 0. \end{cases} \tag{3}$$

We are interested in finding an optimal production plan with minimum cost, given by

$$\min_{u(\cdot)} \int_0^T \left(\alpha_1(u^1(t))^2 + \alpha_2(u^2(t))^2 + \beta_1 x^1 + \beta_2 x^2 + \beta_3 x^3\right) dt.$$

We are looking for the optimal trajectories that leave from the initial point $(0,0,0)$ and arrive at the final point (s_1, s_2, s_3). The control system (3) can be rewritten in the following form:

$$\begin{aligned} &\dot{x} = u_1 X_1 + u_2 X_2, \quad x = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \in R^3,\ X_1 = \begin{pmatrix} 1 \\ 0 \\ x^1 + x^2 \end{pmatrix},\ X_2 = \begin{pmatrix} 0 \\ 1 \\ x^1 + x^2 \end{pmatrix}, \\ &\min_{u(\cdot)} \int_0^T \mathcal{F}(u(t))dt,\ \mathcal{F}(u) = \alpha_1 (u^1(t))^2 + \alpha_2 (u^2(t))^2 + \beta_1 x^1 + \beta_2 x^2 + \beta_3 x^3. \end{aligned} \tag{4}$$

The distribution $\Delta = span\{X_1, X_2\}$ has constant dimension, $\dim \Delta = 2$, for all points $x \in R^3$. Also,in the canonical frame $\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right\}$ the vector fields are given by

$$X_1 = \frac{\partial}{\partial x^1} + (x^1 + x^2)\frac{\partial}{\partial x^3}, \quad X_2 = \frac{\partial}{\partial x^2} + (x^1 + x^2)\frac{\partial}{\partial x^3}.$$

The Lie bracket is

$$[X_1, X_2] = \left[\frac{\partial}{\partial x^1} + (x^1 + x^2)\frac{\partial}{\partial x^3}, \frac{\partial}{\partial x^2} + (x^1 + x^2)\frac{\partial}{\partial x^3}\right] = \frac{\partial}{\partial x^3} - \frac{\partial}{\partial x^3} = 0,$$

and it turns out that we are dealing with an involutive Δ distribution. Using Frobenius' theorem, we obtain that the distribution Δ is integrable and it generates a foliation in three dimensional space. As a consequence, it follows that two points can be joined by an optimal trajectory if and only if they belong to the same leaf of the foliation. In other words, the economic system is not controllable, which means that only a certain final amount of stock can be reached. Moreover, using the system (3) it results $\dot{x}^3 = (\dot{x}^1 + \dot{x}^2)(x^1 + x^2)$ which leads through integration to $x^3 = \frac{1}{2}\left(x^1 + x^2\right)^2 + c$, $c \in R$, which are surfaces in R^3, which give a foliation. Also, using the initial condition $x^i(0) = 0$, $i = \overline{1,3}$ it results $x^3 = \frac{1}{2}\left(x^1 + x^2\right)^2$. From the final conditions $x^i(T) = s_i$, $i = \overline{1,3}$ it results that the system is mathematically controllable if and only if $s_3 = \frac{1}{2}(s_1 + s_2)^2$.

To obtain the optimal solution, we apply Pontryagin's Maximum Principle. The Hamiltonian function on the dual space has the form

$$H = \sum_{i=1}^{3} p_i \dot{x}^i - \mathcal{F},$$

from which it results

$$H = p_1 u^1 + p_2 u^2 + p_3 (u^1 + u^2)(x^1 + x^2) - \alpha_1 (u^1)^2 - \alpha_2 (u^2)^2 - \beta_1 x^1 - \beta_2 x^2 - \beta_3 x^3,$$

where we have denoted by p_1, p_2, p_3 the momentum variables on cotangent space. The condition $\frac{\partial H}{\partial u} = 0$ yields the following equations

$\frac{\partial H}{\partial u^1} = 0 \Rightarrow p_1 + p_3(x^1 + x^2) - 2\alpha_1 u^1 = 0 \Rightarrow u^1 = \frac{p_1 + p_3(x^1 + x^2)}{2\alpha_1}$,

$\frac{\partial H}{\partial u^2} = 0 \Rightarrow p_2 + p_3(x^1 + x^2) - 2\alpha_2 u^2 = 0 \Rightarrow u^2 = \frac{p_2 + p_3(x^1 + x^2)}{2\alpha_2}$.

In what follows, we replace the expressions of u^1, u^2 in the Hamiltonian function and after some calculations it results:

$$H = \frac{(p_1 + p_3(x^1 + x^2))^2}{4\alpha_1} + \frac{(p_2 + p_3(x^1 + x^2))^2}{4\alpha_2} - \beta_1 x^1 - \beta_2 x^2 - \beta_3 x^3.$$

Using the equations (1) we obtain

$\dot{x}^1 = \frac{\partial H}{\partial p_1} = \frac{p_1 + p_3(x^1 + x^2)}{2\alpha_1}$,

$\dot{x}^2 = \frac{\partial H}{\partial p_2} = \frac{p_2+p_3(x^1+x^2)}{2\alpha_2}$,
$\dot{x}^3 = \frac{\partial H}{\partial p_3} = \left(\frac{p_1+p_3(x^1+x^2)}{2\alpha_1} + \frac{p_2+p_3(x^1+x^2)}{2\alpha_2}\right)\left(x^1 + x^2\right)$,
$\dot{p}_1 = -\frac{\partial H}{\partial x^1} = -\frac{(p_1+p_3(x^1+x^2))p_3}{2\alpha_1} - \frac{(p_2+p_3(x^1+x^2))p_3}{2\alpha_2} + \beta_1$,
$\dot{p}_2 = -\frac{\partial H}{\partial x^2} = -\frac{(p_1+p_3(x^1+x^2))p_3}{2\alpha_1} - \frac{(p_2+p_3(x^1+x^2))p_3}{2\alpha_2} + \beta_2$,
$\dot{p}_3 = -\frac{\partial H}{\partial x^3} = \beta_3$.

We we denote

$$\mu_1 = p_1 + p_3(x^1 + x^2),\ \mu_2 = p_2 + p_3(x^1 + x^2),$$

and it results

$$\dot{x}^1 = \frac{\mu_1}{2\alpha_1},\ \dot{x}^2 = \frac{\mu_2}{2\alpha_2}, \tag{5}$$

and by direct computation

$$\dot{\mu}_1 = \beta_1 + (x^1 + x^2)\beta_3,\ \dot{\mu}_2 = \beta_2 + (x^1 + x^2)\beta_3, \tag{6}$$

From (5) and (6) we obtain

$$\ddot{x}^1 = \frac{\dot{\mu}_1}{2\alpha_1} = \frac{\beta_1}{2\alpha_1} + (x^1 + x^2)\frac{\beta_3}{2\alpha_1}, \tag{7}$$

$$\ddot{x}^2 = \frac{\dot{\mu}_2}{2\alpha_2} = \frac{\beta_2}{2\alpha_2} + (x^1 + x^2)\frac{\beta_3}{2\alpha_2}, \tag{8}$$

which lead to

$$\ddot{x}^1 + \ddot{x}^2 = (x^1 + x^2)\left(\frac{\beta_3}{2\alpha_1} + \frac{\beta_3}{2\alpha_2}\right) + \frac{\beta_1}{2\alpha_1} + \frac{\beta_2}{2\alpha_2}.$$

Denoting $y = x^1 + x^2$ we obtain a nonhomogeneous differential equation

$$\ddot{y} = \frac{(\alpha_1 + \alpha_2)\beta_3}{2\alpha_1\alpha_2} y + \frac{\beta_1\alpha_2 + \beta_2\alpha_1}{2\alpha_1\alpha_2}. \tag{9}$$

Considering the homogeneous equation

$$\ddot{y} - \frac{(\alpha_1 + \alpha_2)\beta_3}{2\alpha_1\alpha_2} y = 0, \tag{10}$$

with characteristic equation $\lambda^2 - \frac{(\alpha_1+\alpha_2)\beta_3}{2\alpha_1\alpha_2} = 0$ with the solutions $\lambda_{1,2} = \pm\sqrt{\frac{(\alpha_1+\alpha_2)\beta_3}{2\alpha_1\alpha_2}}$, which yield the solution

$$y(t) = c_1 e^{\sqrt{\frac{(\alpha_1+\alpha_2)\beta_3}{2\alpha_1\alpha_2}}t} + c_2 e^{-\sqrt{\frac{(\alpha_1+\alpha_2)\beta_3}{2\alpha_1\alpha_2}}t}.$$

The solution of nonhomogeneous equation (9) has the form

$$y(t) = c_1 e^{\sqrt{\frac{(\alpha_1+\alpha_2)\beta_3}{2\alpha_1\alpha_2}}t} + c_2 e^{-\sqrt{\frac{(\alpha_1+\alpha_2)\beta_3}{2\alpha_1\alpha_2}}t} - \frac{\beta_1\alpha_2 + \beta_2\alpha_1}{(\alpha_1 + \alpha_2)\beta_3}. \tag{11}$$

Now, replace this result in the equation (7) and it results

$$\ddot{x}^1 = \frac{\beta_1}{2\alpha_1} + \frac{\beta_3}{2\alpha_1}\left(c_1 e^{\sqrt{\frac{(\alpha_1+\alpha_2)\beta_3}{2\alpha_1\alpha_2}}t} + c_2 e^{-\sqrt{\frac{(\alpha_1+\alpha_2)\beta_3}{2\alpha_1\alpha_2}}t} - \frac{\beta_1\alpha_2 + \beta_2\alpha_1}{(\alpha_1 + \alpha_2)\beta_3}\right),$$

which is equivalent with

$$\ddot{x}^1 = \frac{\beta_3}{2\alpha_1}\left(c_1 e^{\sqrt{\frac{(\alpha_1+\alpha_2)\beta_3}{2\alpha_1\alpha_2}}t} + c_2 e^{-\sqrt{\frac{(\alpha_1+\alpha_2)\beta_3}{2\alpha_1\alpha_2}}t}\right) + \frac{\beta_1-\beta_2}{2(\alpha_1+\alpha_2)}.$$

By twice integration, we obtain

$$x^1(t) = \frac{\alpha_2}{\alpha_1+\alpha_2}\left(c_1 e^{\sqrt{\frac{(\alpha_1+\alpha_2)\beta_3}{2\alpha_1\alpha_2}}t} + c_2 e^{-\sqrt{\frac{(\alpha_1+\alpha_2)\beta_3}{2\alpha_1\alpha_2}}t}\right) + \frac{\beta_1-\beta_2}{4(\alpha_1+\alpha_2)}t^2 + c_3 t + c_4. \tag{12}$$

In the same way, in accordance with (8) and (9) it results

$$x^2(t) = \frac{\alpha_1}{\alpha_1+\alpha_2}\left(c_1 e^{\sqrt{\frac{(\alpha_1+\alpha_2)\beta_3}{2\alpha_1\alpha_2}}t} + c_2 e^{-\sqrt{\frac{(\alpha_1+\alpha_2)\beta_3}{2\alpha_1\alpha_2}}t}\right) - \frac{\beta_1-\beta_2}{4(\alpha_1+\alpha_2)}t^2 - c_3 t - c_4 - \frac{\beta_1\alpha_2+\beta_2\alpha_1}{(\alpha_1+\alpha_2)\beta_3}. \tag{13}$$

and

$$x^3(t) = \frac{(x^1(t)+x^2(t))^2}{2}. \tag{14}$$

The conditions $x^i(0) = 0$, $x^i(T) = s_i$, $i = 1, 2$ generate the linear system

$$\begin{cases} \alpha_2(c_1+c_2) + (\alpha_1+\alpha_2)c_4 = 0 \\ \alpha_1(c_1+c_2) - (\alpha_1+\alpha_2)c_4 = \frac{\beta_1\alpha_2+\beta_2\alpha_1}{\beta_3} \\ \alpha_2\left(c_1e^{\alpha} + c_2e^{-\alpha}\right) + (\alpha_1+\alpha_2)Tc_3 + (\alpha_1+\alpha_2)c_4 = (\alpha_1+\alpha_2)s_1 - \frac{(\beta_1-\beta_2)T^2}{4} \\ \alpha_1\left(c_1e^{\alpha} + c_2e^{-\alpha}\right) - (\alpha_1+\alpha_2)Tc_3 - (\alpha_1+\alpha_2)c_4 = (\alpha_1+\alpha_2)s_2 + \frac{(\beta_1-\beta_2)T^2}{4} + \frac{\beta_1\alpha_2+\beta_2\alpha_1}{\beta_3} \end{cases}$$

where we have denoted $\alpha = \sqrt{\frac{(\alpha_1+\alpha_2)\beta_3}{2\alpha_1\alpha_2}}T$.

By straightforward computation, we obtain the solution

$$c_1 = \frac{1}{e^{\alpha}-e^{-\alpha}}\left(s_1 + s_2 + \left(1-e^{-\alpha}\right)\frac{\beta_1\alpha_2+\beta_2\alpha_1}{(\alpha_1+\alpha_2)\beta_3}\right), \tag{15}$$

$$c_2 = \frac{1}{e^{-\alpha}-e^{\alpha}}\left(s_1 + s_2 + \left(1-e^{\alpha}\right)\frac{\beta_1\alpha_2+\beta_2\alpha_1}{(\alpha_1+\alpha_2)\beta_3}\right), \tag{16}$$

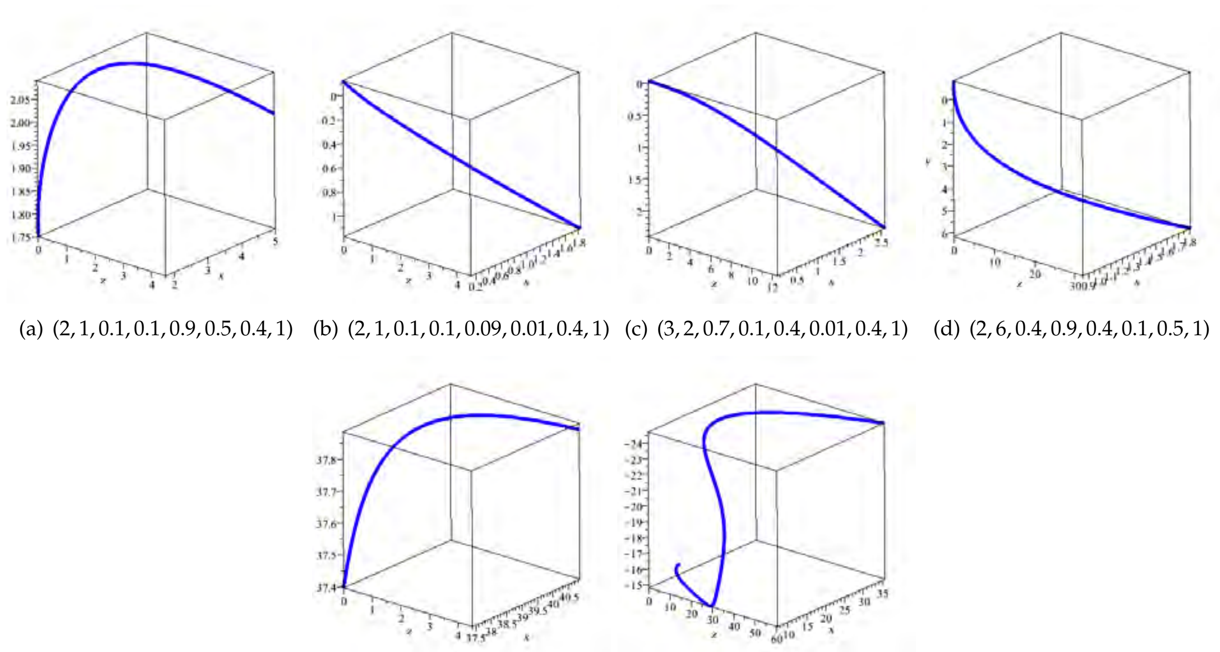
$$c_3 = \frac{\alpha_1 s_1 - \alpha_2 s_2}{T} - \frac{(\beta_1-\beta_2)T}{4}, \tag{17}$$

$$c_4 = -\frac{\alpha_2(\beta_1\alpha_2+\beta_2\alpha_1)}{\beta_3(\alpha_1+\alpha_2)^2}. \tag{18}$$

The solution is optimal because the Hamilton function is convex. Now, we can discuss on answers on an interval $[0, T]$. Indeed, computing c_1, c_2, c_3, c_4 is an easy process. Using them, we can compute the 3D spatial curve $(x_1(t), x_2(t), x_3(t))$ for $0 \le t \le T$. Through the fact that the range of this curve should not have any negative coordinates for practical purposes, we only accept non-negative answers for a parameter list $(s_1, s_2, \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, T)$. The economic condition of positivity of the solutions in the numerical examples must also be taken into account, the inequalities $x^i(t) > 0$ cannot be solved by classical methods. Below, is a computing paradigm to achieve the corresponding curve $(x_1(t), x_2(t), x_3(t))$ for such a parameters list.

Algorithm 1 Calculate spatial curve $(x^1(t), x^2(t), x^3(t))$ in the interval $[0, T]$

Require: $(s_1, s_2, \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, T)$
  Compute c_1, c_2, c_3, c_4 via equations (15), (16), (17), (18)
  Apply (c_1, c_2, c_3, c_4) into $x^i(t)$s in the interval $[0, T]$
  if $x^i(t)$ have no negative value in the interval $[0, T]$ **then**
    Plot the spatial curve $(x^1(t), x^2(t), x^3(t))$ in the interval $[0, T]$
  else
    print The parameters list $(s_1, s_2, \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, T)$ does not lead to a practical answer.
  end if



(a) $(2, 1, 0.1, 0.1, 0.9, 0.5, 0.4, 1)$ (b) $(2, 1, 0.1, 0.1, 0.09, 0.01, 0.4, 1)$ (c) $(3, 2, 0.7, 0.1, 0.4, 0.01, 0.4, 1)$ (d) $(2, 6, 0.4, 0.9, 0.4, 0.1, 0.5, 1)$

(e) $(2, 1, 0.9, 0.4, 0.4, 0.07, 0.01, 2)$ (f) $(8, 3, 0.01, 0.5, 0.8, 0.15, 0.1, 6)$

Figure 1: Some sample outputs for the spatial curve $(x^1(t), x^2(t), x^3(t))$ in the interval $[0, T]$ from the list of parameters $(s_1, s_2, \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, T)$. Figures a, b, c, d, and e are acceptable practical answers. But, the output f meets the space in some negative coordinates and so is not acceptable.

3.1. *The case of no storage cost*

In this case we consider that we have no storage cost for any product and the production rate for the third product has the form $\dot{x}^3 = u^2 x^1$. The optimal control system becomes

$$\begin{cases} \dot{x}^1 = u^1 \\ \dot{x}^2 = u^2 \\ \dot{x}^3 = u^2 x^1 \\ x^i(0) = 0,\ x^i(T) = s_i \\ u^1, u^2 \geq 0,\ \alpha_1, \alpha_2 > 0. \end{cases} \tag{19}$$

It is necessary to find the production plan with the minimum cost, given by

$$\min_{u(\cdot)} \int_0^T \left(\alpha_1 (u^1(t))^2 + \alpha_2 (u^2(t))^2 \right) dt,$$

in which the optimal trajectories start from the initial point $(0,0,0)$ and arrive at the final point (s_1,s_2,s_3). The control system (19) can be rewritten in the form of a distributional system

$$\dot{x} = u_1X_1 + u_2X_2, \quad x = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \in R^3,\ X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ X_2 = \begin{pmatrix} 0 \\ 1 \\ x^1 \end{pmatrix},$$
$$\min_{u(\cdot)} \int_0^T \mathcal{F}(u(t))dt,\ \mathcal{F}(u) = \alpha_1(u^1(t))^2 + \alpha_2(u^2(t))^2.$$

The distribution $\Delta = span\{X_1,X_2\}$ has constant dimension, $\dim\Delta = 2$, for all points $x \in R^3$. Also, in the canonical frame $\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right\}$ we get

$$X_1 = \frac{\partial}{\partial x^1}, \quad X_2 = \frac{\partial}{\partial x^2} + x^1\frac{\partial}{\partial x^3}.$$

The Lie bracket is given by

$$[X_1,X_2] = \left[\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2} + x^1\frac{\partial}{\partial x^3}\right] = \frac{\partial}{\partial x^3}.$$

We will denote $X_3 = [X_1,X_2] = \frac{\partial}{\partial x^3}$ and it follows that the distribution $\Delta = span\{X_1,X_2\}$ is nonholonomic with constant rank 2. Moreover, $[X_1,X_3] = [X_2,X_3] = 0$ and it results that Δ is strong bracket generating, because $\{X_1,X_2,[X_1,X_2]\}$ span the entire space R^3. Indeed,

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x^1 & 1 \end{vmatrix} = 1,$$

and $\{X_1,X_2,X_3\}$ is a base in R^3. By using the Chow-Rashevsky theorem and the fact that the distribution $\Delta = span\{X_1,X_2\}$ is strong bracket generating, we obtain that the system (19) is controllable by mathematically point of view.

To obtain the optimal solution, we apply Pontryagin's Maximum Principle. The Hamiltonian function on the cotangent space has the following form:

$$H = \sum_{i=1}^{3} p_i\dot{x}^i - \mathcal{F},$$

and it results

$$H = p_1u^1 + p_2u^2 + p_3u^2x^1 - \alpha_1(u^1)^2 - \alpha_2(u^2)^2,$$

where p_1,p_2,p_3 represent the momentum variables on the cotangent space. The condition $\frac{\partial H}{\partial u} = 0$ yields the following equations

$\frac{\partial H}{\partial u^1} = 0 \Rightarrow p_1 - 2\alpha_1u^1 = 0 \Rightarrow u^1 = \frac{p_1}{2\alpha_1}$,

$\frac{\partial H}{\partial u^2} = 0 \Rightarrow p_2 + p_3x^1 - 2\alpha_2u^2 = 0 \Rightarrow u^2 = \frac{p_2+p_3x^1}{2\alpha_2}$.

In what follows, we introduce the expressions of u^1, u^2 into the Hamiltonian function and by direct calculation we obtain

$$H = \frac{(p_1)^2}{4\alpha_1} + \frac{(p_2 + p_3x^1)^2}{4\alpha_2}.$$

Using the equations (1) we obtain

$\dot{x}^1 = \frac{\partial H}{\partial p_1} = \frac{p_1}{2\alpha_1}$,

$\dot{x}^2 = \frac{\partial H}{\partial p_2} = \frac{p_2+p_3x^1}{2\alpha_2}$,

$\dot{x}^3 = \frac{\partial H}{\partial p_3} = \frac{(p_2+p_3x^1)x^1}{2\alpha_2}$,

$\dot{p}_1 = -\frac{\partial H}{\partial x^1} = -\frac{(p_2+p_3x^1)p_3}{2\alpha_2}$,

$\dot{p}_2 = -\frac{\partial H}{\partial x^2} = 0 \Rightarrow p_2 = a = const.$

$\dot{p}_3 = -\frac{\partial H}{\partial x^3} = 0 \Rightarrow p_3 = b = const.$

Also, we get

$$\ddot{x}^1 = \frac{\dot{p}_1}{2\alpha_1} = -\frac{(p_2 + p_3x^1)p_3}{4\alpha_1\alpha_2},$$

which leads to a nonhomogeneus second order differential equation

$$\ddot{x}^1 = -\frac{b^2}{4\alpha_1\alpha_2}x^1 - \frac{ab}{4\alpha_1\alpha_2}. \tag{20}$$

Considering the homogeneous second order differential equation

$$\ddot{x}^1 = -\frac{b^2}{4\alpha_1\alpha_2}x^1,$$

with characteristic equation

$$\lambda^2 + \frac{b^2}{4\alpha_1\alpha_2} = 0,$$

with the complex solutions $\lambda_{1,2} = \pm\frac{bi}{2\sqrt{\alpha_1\alpha_2}} = 0$, it results the general solution of the homogeneous equation

$$x^1(t) = a_1 \cos\frac{bt}{2\sqrt{\alpha_1\alpha_2}} + c\sin\frac{bt}{2\sqrt{\alpha_1\alpha_2}} + d.$$

Replace into the equation (20) and after some direct computation, it results

$$-\frac{b^2d}{4\alpha_1\alpha_2} = \frac{ab}{4\alpha_1\alpha_2},$$

and $d = -a/b$, which yields

$$x^1(t) = a_1 \cos\frac{bt}{2\sqrt{\alpha_1\alpha_2}} + c\sin\frac{bt}{2\sqrt{\alpha_1\alpha_2}} - \frac{a}{b}.$$

Using the initial condition $x^1(0) = 0$, we get $a_1 = a/b$ and

$$x^1(t) = -\frac{a}{b}\left(1 - \cos\frac{bt}{2\sqrt{\alpha_1\alpha_2}}\right) + c\sin\frac{bt}{2\sqrt{\alpha_1\alpha_2}}. \tag{21}$$

Next, from the differential equation

$$\dot{x}^2(t) = \frac{a}{2\alpha_2} + \frac{b}{2\alpha_2}x^1(t),$$

it results

$$\dot{x}^2(t) = \frac{a}{2\alpha_2}\cos\frac{bt}{2\sqrt{\alpha_1\alpha_2}} + \frac{bc}{2\alpha_2}\sin\frac{bt}{2\sqrt{\alpha_1\alpha_2}},$$

which leads to

$$x^2(t) = \frac{a\sqrt{\alpha_1}}{b\sqrt{\alpha_2}}\sin\frac{bt}{2\sqrt{\alpha_1\alpha_2}} - \frac{c\sqrt{\alpha_1}}{\sqrt{\alpha_2}}\cos\frac{bt}{2\sqrt{\alpha_1\alpha_2}} + d_1.$$

Using the initial condition $x^2(0) = 0$, we get $d_1 = \frac{c\sqrt{\alpha_1}}{\sqrt{\alpha_2}}$ and

$$x^2(t) = \frac{c\sqrt{\alpha_1}}{\sqrt{\alpha_2}}\left(1 - \cos\frac{bt}{2\sqrt{\alpha_1\alpha_2}}\right) + \frac{a\sqrt{\alpha_1}}{b\sqrt{\alpha_2}}\sin\frac{bt}{2\sqrt{\alpha_1\alpha_2}}. \tag{22}$$

Finally, using the equation

$$\dot{x}^3 = \frac{a}{2\alpha_2}x^1 + \frac{b}{2\alpha_2}\left(x^1\right)^2,$$

we obtain

$$\begin{aligned}\dot{x}^3(t) &= \frac{a^2}{2\alpha_2 b}\cos^2\frac{bt}{2\sqrt{\alpha_1\alpha_2}} + \frac{bc^2}{2\alpha_2}\sin^2\frac{bt}{2\sqrt{\alpha_1\alpha_2}} + \frac{ac}{2\alpha_2}\sin\frac{bt}{\sqrt{\alpha_1\alpha_2}}\\ &\quad -\frac{a^2}{2\alpha_2 b}\cos\frac{bt}{2\sqrt{\alpha_1\alpha_2}} - \frac{ac}{2\alpha_2}\sin\frac{bt}{2\sqrt{\alpha_1\alpha_2}},\end{aligned}$$

which, together with initial condition $x^3(0) = 0$, lead to

$$\begin{aligned}x^3(t) &= \frac{a^2 + b^2c^2}{4b\alpha_2}t + \frac{a^2 - b^2c^2}{4b^2}\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_2}}\sin\frac{bt}{\sqrt{\alpha_1\alpha_2}} - \frac{ac}{2b}\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_2}}\left(1 + \cos\frac{bt}{\sqrt{\alpha_1\alpha_2}}\right)\\ &\quad -\frac{a^2}{b^2}\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_2}}\sin\frac{bt}{2\sqrt{\alpha_1\alpha_2}} + \frac{ac}{b}\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_2}}\cos\frac{bt}{2\sqrt{\alpha_1\alpha_2}}.\end{aligned} \tag{23}$$

We denote $\varphi = \frac{bt}{2\sqrt{\alpha_1\alpha_2}}$ and from the final conditions $x^i(T) = s_i$, $i = \overline{1,3}$ it result the following nonlinear system

$$\begin{cases} c\sin\varphi - \frac{a}{b}(1 - \cos\varphi) = s_1\\ \frac{c\sqrt{\alpha_1}}{\sqrt{\alpha_2}}(1 - \cos\varphi) + \frac{a\sqrt{\alpha_1}}{b\sqrt{\alpha_2}}\sin\varphi = s_2\\ \frac{a^2+b^2c^2}{4b\alpha_2}T + \frac{a^2-b^2c^2}{4b^2}\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_2}}\sin 2\varphi - \frac{ac}{2b}\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_2}}(1 + \cos 2\varphi) - \frac{a^2}{b^2}\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_2}}\sin\varphi + \frac{ac}{b}\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_2}}\cos\varphi = s_3 \end{cases}$$

If we multiply the first equation by $\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_2}}\sin\varphi$ and the second equation by $(1 - \cos\varphi)$, then we add them and it results

$$c = \frac{s_1}{2}ctg\frac{\varphi}{2} + \frac{s_2}{2}\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_2}}. \tag{24}$$

Also, from the first equation we obtain

$$a = b\left[\left(\frac{s_1}{2}ctg\frac{\varphi}{2} + \frac{s_2}{2}\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_2}}\right)ctg\frac{\varphi}{2} - \frac{s_1}{2\sin^2\frac{\varphi}{2}}\right]. \tag{25}$$

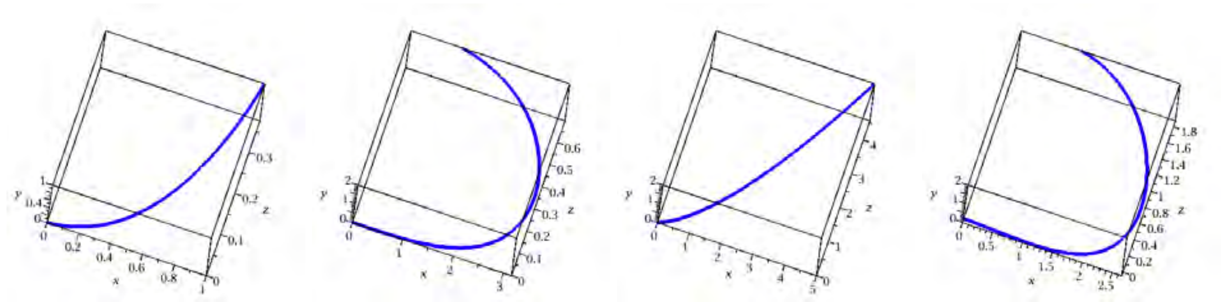
We replace the values of c and a from (24) and (25) in the last equation of the system and we get

$$\begin{aligned} s_3 \;=\;& \frac{b}{4\alpha_2}\left\{\left[\left(\frac{s_1}{2}ctg\frac{\varphi}{2}+\frac{s_2}{2}\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_2}}\right)ctg\frac{\varphi}{2}-\frac{s_1}{2\sin^2\frac{\varphi}{2}}\right]^2+\left(\frac{s_1}{2}ctg\frac{\varphi}{2}+\frac{s_2}{2}\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_2}}\right)^2\right\}T \\ &+\frac{\sqrt{\alpha_1}}{4\sqrt{\alpha_2}}\sin 2\varphi\left\{\left[\left(\frac{s_1}{2}ctg\frac{\varphi}{2}+\frac{s_2}{2}\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_2}}\right)ctg\frac{\varphi}{2}-\frac{s_1}{2\sin^2\frac{\varphi}{2}}\right]^2-\left(\frac{s_1}{2}ctg\frac{\varphi}{2}+\frac{s_2}{2}\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_2}}\right)^2\right\} \\ &-\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_2}}\cos^2\frac{\varphi}{2}\left(\frac{s_1}{2}ctg\frac{\varphi}{2}+\frac{s_2}{2}\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_2}}\right)\left[\left(\frac{s_1}{2}ctg\frac{\varphi}{2}+\frac{s_2}{2}\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_2}}\right)ctg\frac{\varphi}{2}-\frac{s_1}{2\sin^2\frac{\varphi}{2}}\right] \\ &-\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_2}}\sin\varphi\left[\left(\frac{s_1}{2}ctg\frac{\varphi}{2}+\frac{s_2}{2}\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_2}}\right)ctg\frac{\varphi}{2}-\frac{s_1}{2\sin^2\frac{\varphi}{2}}\right]^2 \\ &+\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_2}}\cos\varphi\left(\frac{s_1}{2}ctg\frac{\varphi}{2}+\frac{s_2}{2}\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_2}}\right)\left[\left(\frac{s_1}{2}ctg\frac{\varphi}{2}+\frac{s_2}{2}\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_2}}\right)ctg\frac{\varphi}{2}-\frac{s_1}{2\sin^2\frac{\varphi}{2}}\right]. \end{aligned} \tag{26}$$

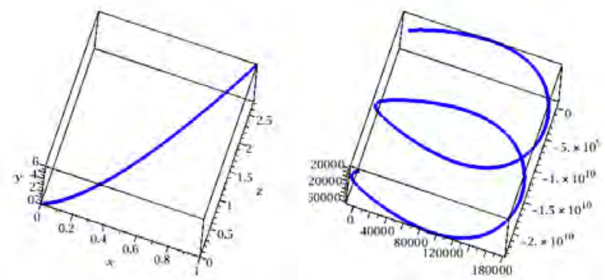
For given s_1, s_2, s_3, T, α_1, α_2 we can find c from eq. (24) and b from the last trigonometric equation (26) (numerical solution). After that we can find a from (25) and $x^1(t)$, $x_2(t)$, $x^3(t)$ with graphic representation for $0 \le t \le T$. The solution is optimal because the Hamilton function is convex. To this end, we create a computer numerical solution framework and try several different states of the parameters. An important condition that must be met operationally is the non-negativity of $x^i(t)$s for every $i = 1, 2, 3$ and so, despite being mathematically meaningful, it is practically meaningless. In fact, this does not necessarily happen for all parameters and we also show an example of the choice of parameters for which some amounts of $x^i(t)$s are negative and therefore considered as unacceptable answers. This shows that the choice of parameters is not arbitrary and therefore, the problem space is a search space so we have to be careful about the meaning of the answers. By fixing the list of parameters $(s_1, s_2, s_3, \alpha_1, \alpha_2, T)$, the Newton method can be used and so b can be obtained as the real root of the equation (26). Therefore, a can be obtained from the equation (25). Also c is obtained directly from the equation (24). Now by applying the tripet (a, b, c) in $x^i(t)$s, we can plot the values of spatial curve $(x^1(t), x^2(t), x^3(t))$ in the range $0 \le t \le T$. The following is a summary of this process.

Algorithm 2 Calculate spatial curve $(x^1(t), x^2(t), x^3(t))$ in the interval $[0, T]$ in the case of non storage cost

Require: $(s_1, s_2, s_3, \alpha_1, \alpha_2, T)$
  Compute c via equation (24)
  Use Newton method and compute b as the root of equation (26)
  Use b and equation (25) and find a
  Apply (a, b, c) into $x^i(t)$s in the interval $[0, T]$
  if $x^i(t)$s have no negative value in the interval $[0, T]$ **then**
    Plot the spatial curve $(x^1(t), x^2(t), x^3(t))$ in the interval $[0, T]$
  else
    print The parameters list $(s_1, s_2, s_3, \alpha_1, \alpha_2, T)$ does not lead to a practical answer.
  end if



(a) $(1, 1, 2, 0.5, 0.5, 1)$ (b) $(1, 2, 5, 0.1, 0.9, 2)$ (c) $(5, 2, 1, 0.01, 0.3, 1)$ (d) $(1, 2, 4, 0.2, 0.9, 2)$



(e) $(1, 6, 3, 0.9, 0.1, 1)$ (f) $(1, 1, 10, 0.2, 0.9, 10)$

Figure 2: Some sample outputs for the spatial curve $(x^1(t), x^2(t), x^3(t))$ in the interval $[0, T]$ from the list of parameters $(s_1, s_2, s_3, \alpha_1, \alpha_2, T)$. Figures a, b, c, d, and e are acceptable practical answers. But, the output f meets the space in some negative coordinates and so is not acceptable.

4. Conclusions

In this work, we solved two problems of optimizing production and storage costs, using the framework of optimal control. The controllability problems were studied using the Frobenius theorem for the first case of the integrable distribution and the Chow-Rashevski theorem for the second non-integrable case, but strong bracket generating distribution. In the first case, it was found that the economic system is not controllable, in the sense that we can only reach certain final stock quantities. In the second case, the system is controllable from a mathematical point of view, but in both cases the economic condition of positivity of the solutions must also be taken into account. The optimal solutions were found using Pontryagin's Maxim Principle. In the end, several scenarios were studied, considering different values of the parameters in the respective systems, considering those with positive solutions as acceptable.

References

[1] A. Agrachev, Y.L. Sachkov, *Control theory from the geometric viewpoint*. Encyclopedia of Mathematical Sciences, 87, Control Theory and Optimization, II. Springer-Verlag, Berlin, 2004.

[2] K. J. Arrow, *Applications of control theory of economic growth*. Mathematics of Decision Sciences, 2, AMS, 1968.

[3] A. Bellaiche, J. Rister (editors), *Sub-Riemannian geometry*, Birkhauser, 1996, 144.

[4] R. Brocket, *Lie algebra and Lie groups in control theory*, Geometrical Methods in Control Theory, Dordrecht, (1973), 43–82.

[5] M. Caputo, *Foundations of Dynamic Economic Analysis: Optimal Control Theory and Applications*, Cambridge University Press, 2005.

[6] D. Hrimiuc, L. Popescu, *Geodesics of sub-Finslerian Geometry*, In: Differential Geometry and its Applications, Proc. Conf. Prague 2004, Charles Univ. (2005), 59–67.

[7] V. Jurdjevic, *Geometric Control Theory*, Cambridge University Press, 1997.

[8] S. M. LaValle, *Planning Algorithms*, Cambridge University Press, 2006.

[9] E. Peyghan, L. Popescu, *Study on geometric structures on Lie algebroids with optimal control applications*, Journal of Nonlinear Mathematical Physics, **27(4)** 2020, 550–580.

[10] L. Popescu, *The geometry of Lie algebroids and applications to optimal control*, Annals Univ. Al. I. Cuza, Iasi, series I, Math., LI (2005), 155–170.

[11] L. Popescu, *Lie algebroids framework for distributional systems*, Annals Univ. Al. I. Cuza, Iasi, series I, Mathematics, **55 (2)** (2009) 257–274.

[12] L. Popescu, *Applications of driftless control affine systems to a problem of inventory and production*, Studies in Informatics and Control, **28(1)** (2019), 25–34.

[13] L. Popescu, *Applications of optimal control to production planning*, Information Technology and Control, **49(1)** (2020), 89–99.

[14] L. Popescu, D. Militaru, O.Mituca, *Optimal control applications in the study of production management*, International Journal of Computers, Communications & Control, **15(2)** (2020), 3859.

[15] L. Popescu, R. Dimitrov, *Applications of maximum principle to optimization of production and storage costs*, Archives of Control Sciences, **31(4)** (2021), 865–881.

[16] L. Popescu, D. Militaru, G. Tica, *Lie geometric methods in the study of driftless control affine systems with hononomic distribution and economic applications*, Mathematics, **10(4)** (2022), 545.

[17] A. Seierstad, K. Sydsater,*Optimal Control Theory with Economic Applications*, North-Holland, Amsterdam, NL, 1987.

[18] S. P. Sethi, G. L. Thompson, *Optimal Control Theory: Applications to Management Science and Economics*, 2nd ed., Springer, New York, NY, 2000.

[19] T. A. Weber, *Optimal control theory with applications in economics*, MIT Press, 2011.