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Neutrosophic compactness via summability and sequential definitions of connectedness in neutrosophic spaces

Ahu Acikgoz[a], Huseyin Çakallı[b], Ferhat Esenbel[a]

[a]*Department of Mathematics, Balikesir University, 10145 Balikesir, Turkey*
[b]*Department of Mathematics, Maltepe University, 34857 İstanbul, Turkey*

Abstract. In this study, using the concept of the neutrosophic method introduced earlier, as done in different types of topological spaces, a new type of sequential compactness is introduced and investigated. After giving various definitions which constitute the cornerstones of our research, in the third section, a new perspective to the concept of connectedness is brought, which is among the most important characters of the topology world, based on the concept of the neutrosophic method. By giving examples of each new definition given in the second and third sections, a better understanding of the new concepts given is provided.

1. Introduction

Since the dawn of the mathematics, countless types of functions have been defined. However, as time passed, each function type has become insufficient. Various functions were needed to be found. Thereupon, a new type of function was introduced whose domain includes the set of convergent sequences and this new function was named as method. With the help of this new type of function, mathematicians were able to bring different perspectives to the concepts that form the cornerstones of the world of topology as in [1, 13, 14]. This concept was also used in studies on neutrosophic topological spaces and some other different topological spaces.

The concept of compactness has always been one of the fundamental and indispensable characters in topology. This concept forms the basis not only for general topology but also for some other non-standard topologies. Many research studies have been conducted on this concept, and some of them have shown deviations, as noted in [5–8, 11, 13]. Thus, these research studies have never reached a conclusion. However, as technology has advanced and industries have developed, people's needs have changed, rendering the basic concepts of these studies insufficient. In the face of developing technology and changing life conditions, scientists might use the sequential definition of compactness instead of the standard neighborhood definition of compactness for metric spaces to come up with stronger ideas. Undoubtedly, this shift has been a significant advantage for mathematicians, as many properties of set compactness can also be adapted to sequential compactness. Sequential compactness is widely recognized

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Email addresses: ahuacikgoz@gmail.com (Ahu Acikgoz), hcakalli@maltepe.edu.tr (Huseyin Çakallı), fesenbel@gmail.com (Ferhat Esenbel)

by mathematicians worldwide. Çakallı [13] introduced G-sequentially compactness, providing another aspect to the concept of sequential compactness.

The concept of connectedness, which is one of the indispensable members of the mathematical world, is a complementary part not only in mathematics, but also in some sub-branches of geography, population planning, and different fields of engineering such as robot manufacturing, which is an indispensable factor of the industry. As time progressed, many areas such as the industry and geography, where the concept of connectedness played important roles, became more complex. Therefore, almost all of the concepts in mathematics lost their importance in these areas. So, it was an inevitable necessity to present the concepts with more properties and to improve the existing concepts. Due to this requirement, scientists have spent very much time in the realm of mathematics in order to meet the demands of daily life and rapidly advancing technology, which is becoming increasingly complex. To illustrate this point, Çakallı [10], one of the scientists who was dissatisfied with these contributions, introduced new identities to the concept of connectedness in topology.

On the inadequacy of the classical set concept in general topology, Smarandache solved some of this requirement by introducing the concept of neutrosophic set in [16]. Using this new set concept, Salma and Albowi also contributed to overcoming problems by introducing the concept of neutrosophic topological space in [15]. This concept of neutrosophic set also paved the way for the definition of neutrosophic soft topological space, and these two topological spaces provided scientists with the opportunity to contribute to the world of mathematics like never-cultivated fields as in [2–4, 9].

In the present study, we introduce and investigate the concept of G-sequentially compactness in neutrosophic topological spaces. One of our main goals in this study is to bring a new perspective to the concept of compactness by introducing a new type of compactness and examine its basic properties. We also present a new type of connectedness in neutrosophic spaces using the concept of neutrosophic sequence and neutrosophic method previously defined in [1].

2. Preliminaries

In this section, we present the basic definitions related to neutrosophic set theory.

Definition 2.1. ([16]) A neutrosophic set A on the universe set X is defined as:

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\},$$

where $T, I, F : X \to]^-0, 1^+[$ and $^-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$.

Scientifically, membership functions, indeterminacy functions and non-membership functions of a neutrosophic set take value from real standart or nonstandart subsets of $]^-0, 1^+[$. However, these subsets are sometimes inconvenient to be used in real life applications such as economical and engineering problems. On account of this fact, we consider the neutrosophic sets, whose membership functions, indeterminacy functions and non-membership functions take values from subsets of $[0, 1]$.

Definition 2.2. ([12]) Let X be a nonempty set. If r, t, s are real standard or non standard subsets of $]^-0, 1^+[$ then the neutrosophic set $x_{r,t,s}$ is called a neutrosophic point in X given by

$$x_{r,t,s}\left(x_p\right) = \begin{cases} (r, t, s), & \text{if } x = x_p \\ (0, 0, 1), & \text{if } x \neq x_p \end{cases}$$

For $x_p \in$X, it is called the support of $x_{r,t,s}$, where r denotes the degree of membership value, t denotes the degree of indeterminacy and s is the degree of non-membership value of $x_{r,t,s}$.

It is clear that every neutrosophic set is the union of its neutrosophic points.

Definition 2.3. ([15]) Let A be a neutrosophic set over the universe set X. The complement of A is denoted by A^c and is defined by:

$$A^c = \left\{\langle x,\ F_{\widetilde{F(e)}}(x), 1\ -\ I_{\widetilde{F(e)}}(x),\ T_{\widetilde{F(e)}}(x)\rangle : x \in X\right\}$$

It is obvious that $[A^c]^c = A$.

Definition 2.4. ([15]) Let A and B be two neutrosophic sets over the universe set X. A is said to be a neutrosophic subset of B if $T_A(x) \leq T_B(x), I_A(x) \leq I_B(x), F_A(x) \geq F_B(x)$, for every $x \in X$. It is denoted by $A \subseteq B$. A is said to be neutrosophic equal to B if $A \subseteq B$ and $B \subseteq A$. It is denoted by $A = B$.

Definition 2.5. ([15]) Let F_1 and F_2 be two neutrosophic sets over the universe set X. Then, their union is denoted by $F_1 \cup F_2 = F_3$ and is defined by

$$F_3 = \{\langle x,\ T_{F_3}(x),\ I_{F_3}(x),\ F_{F_3}(x)\rangle : x \in X\},$$

where

$$T_{F_3}(x) = max\{T_{F_1}(x), T_{F_2}(x)\},$$

$$I_{F_3}(x) = max\{I_{F_1}(x), I_{F_2}(x)\},$$

$$F_{F_3}(x) = min\{F_{F_1}(x), F_{F_2}(x)\}.$$

Definition 2.6. ([15]) Let F_1 and F_2 be two neutrosophic sets over the universe set X. Then their intersection is denoted by $F_1 \cap F_2 = F_4$ is defined by:

$$F_4 = \{\langle x, T_{F_4}(x), I_{F_4}(x), F_{F_4}(x)\rangle : x \in X\},$$

where

$$T_{F_4(x)} = \min\{T_{F_1(x)}, T_{F_2}(x)\},$$

$$I_{F_4(x)} = \min\{I_{F_1(x)}, I_{F_2}(x)\},$$

$$F_{F_4(x)} = \max\{F_{F_1(x)}, F_{F_2}(x)\}.$$

Definition 2.7. ([15]) A neutrosophic set F over the universe set X is said to be a null neutrosophic set if $T_F(x) = 0, I_F(x) = 0, F_F(x) = 1$, for every $x \in X$. It is denoted by 0_X.

Definition 2.8. ([15]) A neutrosophic set F over the universe set X is said to be an absolute neutrosophic set if $T_F(x) = 1, I_F(x) = 1, F_F(x) = 0$, for every $x \in X$. It is denoted by 1_X.

Clearly $0_X^c = 1_X$ and $1_X^c = 0_X$.

Definition 2.9. ([15]) Let $NS(X)$ be the family of all neutrosophic sets over the universe set X and $\tau \subset NS(X)$. Then, τ is said to be a neutrosophic topology on X, if:
1. 0_X and 1_X belong to τ,
2. The union of any number of neutrosophic sets in τ belongs to τ ,
3. The intersection of a finite number of neutrosophic sets in τ belongs to τ.

Then, (X, τ) is said to be a neutrosophic topological space over X. Each member of τ is said to be a neutrosophic open set [15].

Definition 2.10. ([15]) Let (X, τ) be a neutrosophic topological space over X and F be a neutrosophic set over X. Then F is said to be a neutrosophic closed set iff its complement is a neutrosophic open set.

Definition 2.11. ([1]) A neutrosophic point $x_{r,t,s}$ is said to be *neutrosophic quasi-coincident* (*neutrosophic q-coincident*, for short) with F, denoted by $x_{r,t,s}\ q\ F$ if and only if $x_{r,t,s} \not\subseteq F^c$. If $x_{r,t,s}$ is not neutrosophic quasi-coincident with F, we denote by $x_{r,t,s}\ \tilde{q}\ F$.

Definition 2.12. ([1]) A neutrosophic set F in a neutrosophic topological space (X, τ) is said to be a *neutrosophic q-neighborhood* of a neutrosophic point $x_{r,t,s}$ if and only if there exists a neutrosophic open set G such that $x_{r,t,s}\ q\ G \subset F$.

Definition 2.13. ([1]) A neutrosophic set G is said to be *neutrosophic quasi-coincident* (*neutrosophic q-coincident*, for short) with F, denoted by $G\ q\ F$ if and only if $G \not\subseteq F^c$. If G is not neutrosophic quasi-coincident with F, we denote by $G\ \tilde{q}\ F$.

Definition 2.14. ([1]) A neutrosophic point $x_{r,t,s}$ is said to be a *neurosophic cluster point* of a neutrosophic set F if and only if every neutrosophic open q-neighborhood G of $x_{r,t,s}$ is q-coincident with F. The union of all neutrosophic cluster points of F is called the *neutrosophic closure* of F and denoted by $\overline{F}$.

Definition 2.15. ([1]) A neutrosophic point $x_{r,t,s}$ is said to be *a neurosophic boundary point* of a neutrosophic set F if and only if every neutrosophic open q-neighborhood G of $x_{r,t,s}$ is q-coincident with F and F^c. The union of all neutrosophic boundary points of F is called the *neutrosophic limit* of F and denoted by F^b.

Definition 2.16. ([1]) A *neutrosophic sequence* in a neutrosophic topological space (X, τ) is a function $S : N \to (X, \tau)$, where N is the set of naturel numbers. We write $\left\{x_{n_{r_n,t_n,s_n}}\right\}_{n\in N}$ to denote the sequence of neutrosophic points in (X, τ) indexed by N.

Definition 2.17. ([1]) A *neutrosophic subsequence* of a neutrosophic sequence $S : N \to (X, \tau)$ is a composition $S \circ P$, where $P : N \to N$ is an increasing cofinal function. That is,

a) $P(n_1) \leq P(n_2)$, whenever $n_1 \leq n_2$ (P is increasing),

b) For each $n_1 \in N$, there exists a naturel number $n_2 \in N$ such that $n_1 \leq P(n_2)$ (P is cofinal in N).

For $k \in N$, the neutrosophic point $(S \circ P)(k)$ will often be written $x_{n_{r_n,t_n,s_n}}$.

Definition 2.18. ([1]) Let $\left\{x_{n_{r_n,t_n,s_n}}\right\}_{n\in N}$ be a neutrosophic sequence in a neutrosphic topological space (X, τ). Then, $\left\{x_{n_{r_n,t_n,s_n}}\right\}_{n\in N}$ converges to a neutrosophic point $x^e_{\alpha,\beta,\gamma}$ in (X, τ) (written $x_{n_{r_n,t_n,s_n}} \to x_{r,t,s}$) provided that, for each neutrosophic q-neighbourhood $\left(\widetilde{U}, E\right)$ of $x^e_{\alpha,\beta,\gamma}$ there exists $n_0 \in N$ such that $n \geq n_0$ implies $x^e_{\alpha,\beta,\gamma} q \left(\widetilde{U}, E\right)$.

We will use boldface letters $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$ for neutrosophic sequences $\mathbf{x} = \left\{x^{e_n}_{n_{r_n,t_n,s_n}}\right\}_{n\in N}$, $\mathbf{y} = \left\{y^{e_n}_{n_{r_n,t_n,s_n}}\right\}_{n\in N}$, $\mathbf{z} = \left\{z^{e_n}_{n_{r_n,t_n,s_n}}\right\}_{n\in N}, \ldots$ of neutrosophic points in (X, τ). s(X) and c(X) denote the set of all neutrosophic sequences in (X, τ) and the set of all convergent neutrosophic sequences in (X, τ), respectively.

Definition 2.19. ([1]) Let (X, Δ) is a neutrosophic group in a neutrosophic topological space (X, τ), where Δ is a binary operation defined on (X, τ) such that the following conditions hold:

a) Closure: For all neutrosophic points $x1_{r_1,t_1,s_1}$, $x2_{r_2,t_2,s_2}$ in (X, τ), $x1_{r_1,t_1,s_1}\ \Delta\ x2_{r_2,t_2,s_2}$ is a uniquely defined neutrosophic point in (X, τ),

b) Associativity: For all neutrosophic points $x1_{r_1,t_1,s_1}, x2_{r_2,t_2,s_2}, x3_{r_3,t_3,s_3}$ in (X, τ), we have $x1_{r_1,t_1,s_1}\ \Delta \left(x2_{r_2,t_2,s_2}\ \Delta\ x3_{r_3,t_3,s_3}\right)$.

c) Identity: There exists an identity neutrosophic point $e_{\alpha,\beta,\gamma}$ in (X, τ) such that $x1_{r_1,t_1,s_1}\ \Delta\ e_{\alpha,\beta,\gamma} = e_{\alpha,\beta,\gamma}\ \Delta\ x1_{r_1,t_1,s_1} = x1_{r_1,t_1,s_1}$ for any neutrosophic point $x1_{r_1,t_1,s_1}$ in (X, τ),

d) Inverses: For any neutrosophic point $x1_{r_1,t_1,s_1}$ in (X, τ), there exists an inverse neutrosophic point $\left(x1_{r_1,t_1,s_1}\right)^{-1}$ in (X, τ) such that $x1_{r_1,t_1,s_1}\ \Delta \left(x1_{r_1,t_1,s_1}\right)^{-1} = e_{\alpha,\beta,\gamma}$ and $\left(x1_{r_1,t_1,s_1}\right)^{-1} \Delta x1_{r_1,t_1,s_1} = e_{\alpha,\beta,\gamma}$.

Definition 2.20. ([1]) Let $(s(X),^*)$ be a group of neutrosophic sequences and (X, Δ) be a neutrosophic group in a neutrosophic topological space (X, τ). A *neutrosophic method* is a function G defined on a subgroup $(c_G(X),^*)$ of $(s(X),^*)$ into (X, τ) such that $G(x^* y) = G(x)\Delta G(y)$ for all neutrosophic sequences x, y in (X, τ).

Definition 2.21. ([1]) A neutrosophic sequence $\mathbf{x} = \left\{x_{n_{r_n,t_n,s_n}}\right\}_{n\in N}$ is said to be G-convergent to $x_{r,t,s}$, if $x \in c_G(X)$ and $G(x) = x_{r,t,s}$.

Definition 2.22. ([1]) A neutrosophic method G is called *neutrosophic regular*, if every convergent neutrosophic sequence $\mathbf{x} = \{x_{n_{r_n,t_n,s_n}}\}_{n\in N}$ is G-convergent with $G(x) = x_{r,t,s}$, where $\mathbf{x}$ converges to $x_{r,t,s}$.

Definition 2.23. ([1]) A neutrosophic point $u_{r,t,s}$ is in the neutrosophic G-sequential derived set of A (or called a *neutrosophic G-sequential accumulation point* of A) if there exists a neutrosophic sequence $\mathbf{x} = \left\{x_{n_{r_n,t_n,s_n}}\right\}_{n\in N}$ of neutrosophic points, where $x_{n_{r_n,t_n,s_n}} q(u_{r,t,s})^c$ for all $n \in N$, in A such that $G(\mathbf{x}) = u_{r,t,s}$. We denote neutrosophic G-sequential derived set (the set of all neutrosophic G-sequential accumulation points) of a neutrosophic set A by $(A')^G$.

Definition 2.24. ([1]) Let A be a neutrosophic set and $x_{r,t,s}$ be a neutrosophic point in (X, τ). Then, $x_{r,t,s}$ is in the *neutrosophic G-sequential closure* of A (or neutrosophic G-hull of A), if there is a neutrosophic sequence $\mathbf{x} = \left\{x_{n_{r_n,t_n,s_n}}\right\}_{n\in N}$ of neutrosophic points in A such that $G(\mathbf{x}) = x_{r,t,s}$. We denote neutrosophic G-sequential closure of a neutrosophic set A by $\overline{A}^G$. We say that a neutrosophic set is *neutrosophically G-sequentially closed* if it contains all the neutrosophic points in its neutrosophic G-closure. And, a neutrosophic set is said to be *neutrosophically G-sequentially open* if its complement is neutrosophically G-sequentially closed.

3. Neutrosophic compactness via summability

In the following, we introduce definitions. Then, main results are presented at the end of this section.

Definition 3.1. A $\sigma \subset \tau$ is said to be a *neutrosophic open base* of τ if for every $F \in \tau$ there is a subfamily $\{F_i\}_{i\in\Delta} \subset \sigma$ such that $F = \bigcup_{i\in\Delta} F_i$.

If the neutrosophic topology τ has a countable neutrosophic open basis σ, then (X, τ) is said to be a *neutrosophic second countable* space.

Example 3.2. Let (X, τ) be any neutrosophic topological space, where $X = \{a, b, c\}$ and $\tau = \{0_X, 1_X, a_{1,1,0}, b_{1,1,0} \cup c_{1,1,0}\}$. Consider a subfamily σ defined as $\sigma = \{0_X, a_{1,1,0}, b_{1,1,0} \cup c_{1,1,0}\}$. Then, σ is a countable neutrosophic open basis for τ and (X, τ) is a neutrosophic second countable space.

Definition 3.3. A collection of neutrosophic open sets $\sigma(x_{r,t,s})$ is said to form a *neutrosophic local base* at the neutrosophic point $x_{r,t,s}$ if for any $F \in \tau$ with $x_{r,t,s} \in F$ there is a neutrosophic set $G \in \sigma(x_{r,t,s})$ such that $x_{r,t,s} q G \subset F$.

If every neutrosophic point $x_{r,t,s}$ in the neutrosophic topological space (X, τ) has a countable neutrosophic local basis $\sigma(x_{r,t,s})$, then (X, τ) is said to be a *neutrosophic first countable space*.

Example 3.4. Let (X, τ) be any neutrosophic topological space, where $X = \{a, b, c\}$ and $\tau = \{0_X, 1_X, a_{1,1,0} \cup b_{1,1,0}, c_{1,1,0}\}$. Consider a neutrosophic point $a_{1,1,0}$ and subfamily $\sigma(a_{1,1,0})$ defined as $\sigma(a_{1,1,0}) = \{a_{1,1,0} \cup b_{1,1,0}, c_{1,1,0}\}$. Then $\sigma(a_{1,1,0})$ is a neutrosophic local base at the neutrosophic point $a_{1,1,0}$. It is clear that $\sigma(a_{1,1,0})$ is a neutrosophic local base at every neutrosophic point in the neutrosophic topological space (X, τ). So, (X, τ) is a neutrosophic first countable space.

Definition 3.5. A neutrosophic topological space (X, τ) is said to be *neutrosophically Hausdorff* if for any two neutrosophic points $x_{r,t,s}$ and $y_{m,n,p}$ with different supports, there exist $F, G \in \tau$ such that $x_{r,t,s} \in F$, $y_{m,n,p} \in G$ and $F\tilde{q}G$.

Example 3.6. Let (X, τ) be any neutrosophic topological space, where $X = \{a, b\}$ and $\tau = \{0_X, 1_X, a_{1,1,0}, b_{1,1,0}\}$. Then (X, τ) is neutrosophically Hausdorff.

Definition 3.7. A neutrosophic topological space (X, τ) is said to be *strongly neutrosophically Hausdorff* if for any two neutrosophic points $x_{r,t,s}$ and $x_{m,n,p}$ with same supports, there exist $F, G \in \tau$ such that $x_{r,t,s} \in F$, $x_{m,n,p} \in G$ and $F\tilde{q}G$ for all points x of X.

Example 3.8. Let (X, τ) be any neutrosophic topological space, where $X = \{a, b\}$ and $\tau = \{0_X, 1_X\} \cup \{\{\langle a, 1-k, r, k\rangle, \langle b, 1-k, r, k\rangle\} \mid k, r \in [0, 1]\}$. Then, (X, τ) is strongly neutrosophically Hausdorff.

Definition 3.9. A neutrosophic set F in a neutrosophic topological space (X, τ) is said to be *neutrosophically sequentially closed* if every neutrosophic convergent sequence of neutrosophic points in F converges to a neutrosophic point in F.

Definition 3.10. Let $\mathbf{x} = \left\{x_{k_{r_k,t_k,s_k}}\right\}_{k\in N}$ be a neutrosophic sequence in a neutrosphic topological space (X, τ). A neutrosophic sequence $\mathbf{y} = \left\{y_{i_{m_i,n_i,p_i}}\right\}_{i\in N}$ of neutrosophic points in (X, τ) is said to be a *generalized neutrosophic subsequence* of $\mathbf{x}$ if $y_{i_{m_i,n_i,p_i}} \subset x_{k_{i r_{k_i}, t_{k_i}, s_{k_i}}}$ for some $k_1 < k_2 < k_3 < ...$) for any $i \in N$.
Any generalized neutrosophic subsequence of a convergent neutrosophic sequence is convergent to the same neutrosophic point.

Example 3.11. Let $\mathbf{x} = \{k_{1,1,0}\}_{k\in N}$ be a neutrosophic sequence in a neutrosphic topological space (N, τ), where N is the set of naturel numbers. Then, $\mathbf{y} = \{k_{0,5,0,5,0,5}\}_{k\in N}$ is a generalized neutrosophic subsequence of $\mathbf{x}$.

Definition 3.12. A neutrosophic set F is said to *exhibit generalized neutrosophic sequential compactness* if any neutrosophic sequence $\mathbf{x} = \left\{x_{n_{r_n,t_n,s_n}}\right\}_{n\in N}$ in F has a generalized neutrosophic subsequence $\mathbf{y} = \left\{y_{i_{m_i,n_i,p_i}}\right\}_{i\in N}$ which is neutrosophically convergent in F.
In this study, we will mean generalized neutrosophic sequential compactness by neutrosophic sequential compactness.

Definition 3.13. A neutrosophic set F is said to be *neutrosophically G-sequentially compact* if any neutrosophic sequence $x = \left\{x_{n_{r_n,t_n,s_n}}\right\}_{n\in N}$ in F has a generalized neutrosophic subsequence $\mathbf{y} = \left\{y_{i_{m_i,n_i,p_i}}\right\}_{i\in N}$ with $G(\mathbf{y}) \in F$.
For regular methods any neutrosophically sequentially compact subset of X is also neutrosophically G-sequentially compact and the converse is not always true.

Definition 3.14. A neutrosophic set F in a neutrosophic topological space (X, τ) is said to be *neutrosophically G-sequentially Frechet compact* if any neutrosophic subset K of F with $T_A(x) > 0$ or $I_A(x) > 0$ or $F_A(x) < 1$ for infinitely many points $x \in X$ has at least one neutrosophic G-sequential accumulation point in F.

Definition 3.15. A neutrosophic method G is called *neutrosophic subsequential* if, for any neutrosophic G-sequence $\mathbf{x}$ such that $G(\mathbf{x}) = x_{r,t,s}$, there exists a generalized neutrosophic subsequence of $\mathbf{x}$ that converges to $x_{r,t,s}$.

Firstly, we note that if F is a fuzzy subset in a neutrosophic topological space (X, τ) with $T_A(x) > 0$ or $I_A(x) > 0$ or $F_A(x) < 1$ for only finite number of points of X, then it immediately follows that F is neutrosophically G-sequentially compact. The union of two neutrosophically G-sequentially compact subsets in (X, τ) is neutrosophically G-sequentially compact and the intersection of any neutrosophically G-sequentially compact subsets in (X, τ) is neutrosophically G-sequentially compact.

Theorem 3.16. *Let G be a neutrosophic regular subsequential method in a neutrosophic topological space (X, τ). Then, a neutrosophic subset F in (X, τ) is neutrosophically G-sequentially compact if and only if it is neutrosophically sequentially compact.*

Proof. Let F be a neutrosophically G-sequentially compact subset of X and $\mathbf{x}$ be any neutrosophic sequence in F. As F is G-neutrosophically sequentially compact, there exists a generalized neutrosophic subsequence $\mathbf{y}$ of the sequence $\mathbf{x}$ such that $G(\mathbf{y}) \in F$. As G is a neutrosophic subsequential method, there is a generalized neutrosophic subsequence $\mathbf{z}$ of $\mathbf{y}$ that converges to $G(\mathbf{y})$. Hence F is neutrosophically sequentially compact. Now, take a neutrosophically sequentially compact subset F of X and let $\mathbf{x}$ be any neutrosophic sequence in F. As F is neutrosophically sequentially compact, there exists a generalized neutrosophic subsequence $\mathbf{y}$ of $\mathbf{x}$ that converges to a neutrosophic point $x_{r,t,s} \in F$. As G is neutrosophically regular, $G(\mathbf{y}) = x_{r,t,s}$. This completes the proof of the theorem. □

Theorem 3.17. *Any neutrosophically G-sequentially closed subset of a neutrosophically G-sequentially compact subset in a neutrosophic topological space* (X, τ) *is neutrosophically G-sequentially compact.*

Proof. Let F be any neutrosophically G-sequentially compact subset of X and K be a neutrosophically G-sequentially closed subset of F.Take any sequence $\mathbf{x}$ of neutrosophic points in K. Then, $\mathbf{x}$ is a sequence of neutrosophic points in F. Since F is neutrosophically G-sequentially compact, there is a generalized neutrosophic subsequence $\mathbf{y}$ of the sequence $\mathbf{x}$ such that $G(\mathbf{y}) \in F$. Now, the generalized neutrosophic subsequence $\mathbf{y}$ is also a sequence of neutrosophic points in K. Since K is neutrosophically G-sequentially closed, $G(\mathbf{y}) \in K$. Thus, $\mathbf{x}$ has a generalized neutrosophic subsequence $\mathbf{y}$ with $G(\mathbf{y}) \in K$. So, K is neutrosophically G-sequentially compact. □

Theorem 3.18. *Let G be a neutrosophic subsequential method,* (X, τ) *be neutrosophically Hausdorff and A be a neutrosophically G-sequentially compact subset A of X. Let* $B = \overline{A}^G$. *For any* $\alpha \in X$, *if* $T_B(\alpha) > 0$ *or* $I_B(\alpha) > 0$ *or* $F_B(\alpha) < 1$ *then* $T_A(\alpha) > 0$ *or* $I_A(\alpha) > 0$ *or* $F_A(\alpha) < 1$.

Proof. Let A be any neutrosophically G-sequentially compact subset of X and B be neutrosophic subset of X such that $B = \overline{A}^G$. Let $\alpha \in X$ be such that $T_B(\alpha) > 0$ or $I_B(\alpha) > 0$ or $F_B(\alpha) < 1$. Suppose that $T_A(\alpha) = 0$ and $I_A(\alpha) = 0$ and $F_A(\alpha) = 1$. Write $T_B(\alpha) = s$ and $I_B(\alpha) = t$ and $F_B(\alpha) = s$. Then, $\alpha_{r,t,s} \in B$. Then, there exists a neutrosophic sequence $\mathbf{x}$ of neutrosophic points in A such that $G(\mathbf{x}) = \alpha_{r,t,s}$. Since G is a neutrosophic subsequential method, there is a generalized neutrosophic subsequence $\mathbf{y}$ of the neutrosophic sequence $\mathbf{x}$ that converges to $\alpha_{r,t,s}$. From G-neutrosophic sequential compactness of A, there is a generalized neutrosophic subsequence $\mathbf{z}$ of the neutrosophic subsequence $\mathbf{y}$ such that $G(\mathbf{z}) \in A$. Let the support of $G(\mathbf{z})$ be β. Then, $T_A(\beta) > 0$ or $I_A(\beta) > 0$ or $F_A(\beta) < 1$. Again there is a generalized neutrosophic subsequence $\mathbf{w}$ of $\mathbf{z}$ that converges to $G(\mathbf{z})$. Since $\mathbf{w}$ is also a generalized neutrosophic subsequence of $\mathbf{y}$, then it is also neutrosophically convergent to $\alpha_{r,t,s}$. Since $T_A(\alpha) = 0$ and $I_A(\alpha) = 0$ and $F_A(\alpha) = 1$ and $G(\mathbf{z}) \in A$, $\beta \neq \alpha$. Since (X, τ) is neutrosophically Hausdorff, $\mathbf{w}$ cannot be neutrosophically convergent to both $\alpha_{r,t,s}$ and $G(\mathbf{z})$. This contradiction shows that $T_A(\alpha) > 0$ or $I_A(\alpha) > 0$ or $F_A(\alpha) < 1$. This completes the proof of the theorem. □

Corollary 3.19. *Let G be a neutrosophically regular subsequential method and* (X, τ) *be strongly neutrosophically Hausdorff. Then, any neutrosophically G-sequentially compact subset of X is neutrosophically G-sequentially closed.*

Corollary 3.20. *Let G be a neutrosophically regular subsequential method. Then, any neutrosophically G-sequentially compact subset of X is neutrosophically sequentially closed.*

Lemma 3.21. *Let G be a neutrosophically regular method. Then, G is a neutrosophically subsequential method if and only if* $\overline{A} = \overline{A}^G$ *for every neutrosophic subset A of X.*

Theorem 3.22. *Let G be a neutrosophically regular subsequential method and X be infinite and neutrosophically first countable. Then a neutrosophic subset A of X is G-neutrosophically sequentially compact if and only if it isneutrosophically G-sequentially Frechet compact.*

Proof. Let A be any neutrosophically G-sequentially compact subset of X and B a neutrosophic subset of A with $T_B(x) > 0$ or $I_B(x) > 0$ or $F_B(x) < 1$ at an infinite number of points of X. We can choose a sequence x of neutrosophic points of B with distinct supports. G-neutrosophic sequential compactness of A implies that the sequence x has a generalized neutrosophic subsequence y with $G(y) \in A$. Then, $G(y)$ is G-neutrosophic sequential accumulation point of B. Thus, A is neutrosophically G-sequentially Frechet compact.

Now suppose that A is any neutrosophically G-sequentially Frechet compact subset of X. Let $\mathbf{x}$ be any sequence of neutrosophic points in A. If infinitely many supports of neutrosophic points in $\mathbf{x}$ are equal, then clearly $\mathbf{x}$ has a generalized neutrosophic subsequence which is neutrosophically G-sequentially convergent in A. Otherwise assume that no point of X appears as a support of any neutrosophic point in $\mathbf{x}$ more than finite number of times. We now consider a subsequence $\mathbf{y}$ of $\mathbf{x}$ with distinct supports. Let us construct the

sequence of neutrosophic subsets (C_n) of X in the following way:
For any x of X,

$$T_{C_i}(x) = \begin{cases} m_k, & \text{if } x = y_k \textit{ and } k \geq i, \textit{ where } y_{k_{m_k,n_k,p_k}} \textit{ is a neutrosophic point in } \mathbf{y} \\ 0, & \textit{otherwise} \end{cases}$$

$$I_{C_i}(x) = \begin{cases} n_k, & \text{if } x = y_k \textit{ and } k \geq i, \textit{ where } y_{k_{m_k,n_k,p_k}} \textit{ is a neutrosophic point in } \mathbf{y} \\ 0, & \textit{otherwise} \end{cases}$$

$$F_{C_i}(x) = \begin{cases} p_k, & \text{if } x = y_k \textit{ and } k \geq i, \textit{ where } y_{k_{m_k,n_k,p_k}} \textit{ is a neutrosophic point in } \mathbf{y} \\ 1, & \textit{otherwise} \end{cases}$$

Clearly, $C_1 \supset C_2 \supset C_3 \ldots$ and $T_{C_n}(x) > 0$ or $F_{C_n}(x) < 1$, at an infinite number of points of X for each $n \in 1, 2, 3, \ldots$. So each C_n has a neutrosophically G-sequentially accumulation point. Then we must have $\bigcap_{n=1}^{\infty} \overline{C_n}^G \neq 0_X$. Let $x_{0_{r_0,t_0,s_0}} \in \bigcap_{n=1}^{\infty} \overline{C_n}^G$. Since G is a neutrosophically regular subsequential method, it follows from by Lemma 3.21 that $x_{0_{r_0,t_0,s_0}} \in \bigcap_{n=1}^{\infty} \overline{C_n}$.
Let $\{D_k : k \in N\}$ be a countable local base at the point $x_{0_{r_0,t_0,s_0}}$. Take a naturel number k_1 such that $y_{k_{1_{m_{k_1},n_{k_1},p_{k_1}}}}$ is a neutrosophic point in $\mathbf{y}$ that is also a neutrosophic point in $C_1 \cap D_1$. (Clearly, $m_{k_1 \leq T_{C_1}(y_{k_1}}$, $n_{k_1 \leq I_{C_1}(y_{k_1}}$ $p_{k_1 \leq F_{C_1}(y_{k_1})}$. Suppose we have choosen $y_{k_{1_{m_{k_1},n_{k_1},p_{k_1}}}}, y_{k_{2_{m_{k_2},n_{k_2},p_{k_2}}}}, y_{k_{3_{m_{k_3},n_{k_3},p_{k_3}}}}$. We may choose with $k_\alpha > k_{\alpha-1}$ such that $y_{k_{\alpha_{m_{k_\alpha},n_{k_\alpha},p_{k_\alpha}}}} \in C_\alpha \cap D_\alpha$. Inductively, we may construct a generalized neutrosophic subsequence $\mathbf{z} = \{y_{k_{\alpha_{m_{k_\alpha},n_{k_\alpha},p_{k_\alpha}}}}\}$ of $\mathbf{y}$. Clearly $\mathbf{z}$ is also a generalized neutrosophic subsequence of $\mathbf{x}$ and it is neutrosophically convergent to $x_{0_{r_0,t_0,s_0}}$. Since G is neutrosophically regular, $G(\mathbf{z}) = x_{0_{r_0,t_0,s_0}}$. This completes the proof. □

4. Sequential definitions of connectedness

Firstly, we give some new definitions that will be required in defininining G-sequentially connectedness and investigating its properties. Then, main results are presented at the end of this section.

Definition 4.1. Let (X, τ) be a neutrosophic topological space and $Y \subseteq X$. Let H be a neutrosophic set over Y such that

$$T_H(x) = \begin{cases} 1, & x \in Y \\ 0, & x \notin Y \end{cases}, \quad I_H(x) = \begin{cases} 1, & x \in Y \\ 0, & x \notin Y \end{cases}, \quad F_H(x) = \begin{cases} 0, & x \in Y \\ 1, & x \notin Y \end{cases}$$

Let $\tau_Y = \{H \cap F : F \in \tau\}$, then (Y, τ_Y) is called neutrosophic subspace of (X, τ). If $H \in \tau$ (resp. $H^c \in \tau$), then (Y, τ_Y) is called neutrosophic open (resp. closed) subspace of (X, τ).

Example 4.2. Let (X, τ) be a neutrosophic topological space, where $X = \{a, b, c, d\}$ and $\tau = \{0_X, 1_X, \{\langle a, 0.5, 0.5, 0.5\rangle, \langle b, 0.5, 0.5, 0.5\rangle, \langle c, 0.5, 0.5, 0.5\rangle, \langle d, 0.5, 0.5, 0.5\rangle\}\}$. Consider a subset $Y = \{b, c\}$. Then, $\tau_Y = \{0_Y, 1_Y, \{\langle a, 0, 0.1\rangle, \langle b, 0.5, 0.5, 0.5\rangle, \langle c, 0.5, 0.5, 0.5\rangle, \langle d, 0, 0.1\rangle\}\}$,
and (Y, τ_Y) is a neutrosophic subspace of (X, τ), where $1_Y = \{\{\langle a, 0, 0.1\rangle, \langle b, 1, 1, 0\rangle, \langle c, 1, 1, 0\rangle, \langle d, 0, 0.1\rangle\}\}$ and $0_Y = \{\{\langle a, 0, 0.1\rangle, \langle b, 0, 0, 1\rangle, \langle c, 0, 0, 1\rangle, \langle d, 0, 0.1\rangle\}\}$.

Definition 4.3. Non-empty neutrosophic subsets U and V in a neutrosophic topological space (X, τ) is said to be neutrosophically disjoint if $U\tilde{q}V$ and $U^c\tilde{q}V^c$.

Example 4.4. Let (X, τ) be a neutrosophic topological space, where $X = \{a, b\}$. Consider neutrosophic subsets U and V, where $U = \{\langle a, 0.5, 0.5, 0.5\rangle, \langle b, 0.5, 0.5, 0.5\rangle\}$ and $V = \{\langle a, 0.3, 0.3, 0.7\rangle, \langle b, 0.3, 0.3, 0.7\rangle\}$. Then, U and V are neutrosophically disjoint.

Definition 4.5. A non-empty neutrosophic subset A in a neutrosophic topological space (X, τ) is called *neutrosophically G-sequentially connected* if there are no non-null neutrosophically disjoint and neutrosophically G-sequentially closed subsets U and V such that $A \subseteq U \cup V$ where $A \cap U$ and $A \cap V$ are non-null neutrosophic subsets. In particular, X is called neutrosophically G-sequentially connected if there are no non-null neutrosophically G-sequentially closed subsets in (X, τ) that are neutrosophically disjoint.

We give the following definition before presenting some characterization of neutrosophically connectedness of a subset.

Definition 4.6. Let A be a neutrosophic subset in (X, τ). A neutrosophic subset F of A is called neutrosophically *G-sequentially closed* in A if there exists a neutrosophically G-sequentially closed subset U in (X, τ) such that $F = U \cap A$. We say that a neutrosophic subset V of A is neutrosophically *G-sequentially open* in A if $A \cap V^c$ is neutrosophically G-sequentially closed in A.

Here, we note that a neutrosophic subset B of A is neutrosophically G-sequentially open in A if and only if there exists a neutrosophically G-sequentially open subset V in (X, τ) such that $B = A \cap V$.

Now, we give the following lemma.

Lemma 4.7. *For a neutrosophic subset A in (X, τ), the following are equivalent:*

1. *A is neutrosophically G-sequentially connected;*

2. *A can not be written as a union of non-null neutrosophically disjoint neutrosophically G-sequentially closed subsets in A;*

3. *A can not be written as a union of non-null neutrosophically disjoint neutrosophically G-sequentially open subsets in A;*

4. *There is no neutrosophically G-sequentially open and closed proper subset in A.*

The proof is straightforward by Definition 4.5 and Definition 4.6 and is therefore omitted.

Definition 4.8. ([1]) A function $f : X \to X$ is *neutrosophic G-sequentially continuous* at a neutrosophic point $u_{r,t,s}$, if, for any given a sequence $\mathbf{x} = \left\{x_{n_{r_n,t_n,s_n}}\right\}_{n\in N}$ of neutrosophic points in X, $G(\mathbf{x}) = u_{r,t,s}$ implies that $G(f(\mathbf{x})) = f(u_{r,t,s})$. For a neutrosophic subset D of X, f continuous at every $u_{r,t,s} \in D$ and is neutrosophic G-sequentially continuous, if it is neutrosophic G-sequentially continuous on X.

Theorem 4.9. *A neutrosophically G-sequentially continuous image of any neutrosophically G-sequentially connected subset in (X, τ) is neutrosophically G-sequentially connected.*

Proof. Suppose that $f(A)$ is not neutrosophically G-sequentially connected so that $f(A)$ can be covered as a union $U \cup V$ of some non-null, neutrosophically disjoint neutrosophically G-sequentially closed subsets U and V of X. both meeting $f(A)$. Because the inverse image of a neutrosophically G-sequentially closed subset in (X, τ) is neutrosophically G-sequentially closed for the neutrosophically G-sequentially continuous function f, $f^{-1}(U)$ and $f^{-1}(V)$ are non-null, neutrosophically disjoint neutrosophically G-sequentially closed subsets in (X, τ) and cover A. This statement implies that A is not neutrosophically G-sequentially connected. This contradiction completes the proof of the theorem. □

Definition 4.10. Let G be a neutosophic sequential method and A, B be neutrosophic subsets in (X, τ) such that $B \subseteq A$. A neutrosophic point $a_{r,t,s} \in A$ is said to be in the *G-sequential closure of B* in A if there is a neutrosophic sequence x of points in B such that $G(\mathbf{x}) = a_{r,t,s}$.

We denote the neutrosophic G-sequential closure of B in A by $\overline{B_A}^G$.

Example 4.11. Let (X,τ) be a neutrosophic topological space, where $X = \{a,b\}$. Consider family of neutrosophic sets $\beta = \{U_k : U_k = \{\langle a, 0.5 - \frac{1}{k+2}, 0.5 - \frac{1}{k+2}, 0.5 + \frac{1}{k+2}\rangle, \langle b, 0.5 - \frac{1}{k+2}, 0.5 - \frac{1}{k+2}, 0.5 + \frac{1}{k+2}\rangle\} and k \in Z^+\}$ a neutrosophic method G such that $G(y) = a_{0.5,0.5,0.5}$ for any convergent neutrosophic sequence y, neutrosophic subsets $A = \{\langle a, 0.5, 0.5, 0.5\rangle, \langle b, 0.5, 0.5, 0.5\rangle\}$, and $B = \bigcup_{U_k \in \beta} U_k$. Since $\mathbf{x} = \left\{x_{n_{r_n,t_n,s_n}}\right\}_{n\in N}$ defined as $x_n = a$, $r_n = t_n = 0.5 - \frac{1}{n+2}, s_n = 0.5 + \frac{1}{n+2}$ is a convergent neutrosophic sequence of points in B and $G(x) \in A$, $a_{0.5,0.5,0.5} \in \overline{B_A}^G$.

Lemma 4.12. *Let G be a neutosophic sequential method and A, B be neutrosophic subsets in (X,τ) such that $B \subseteq A$. Then, $\overline{B_A}^G = \overline{B}^G \cap A$.*

Proof. The proof is straightforward and is therefore omitted. □

Lemma 4.13. *Let G be a neutosophic sequential method and A, B be neutrosophic subsets in (X,τ) such that $B \subseteq A$. If A is neutrosophically G-sequentially closed in (X,τ), and B is neutrosophically G-sequentially closed in A, then B is neutrosophically G-sequentially closed in (X,τ).*

Proof. Let $a_{r,t,s}$ be a neutrosophic point in the neutrosophic G-sequential closure of B in (X,τ). Then there exists a neutrosophic sequence x of points in B such that $G(x) = a_{r,t,s}$. $A \subseteq B$ implies $\overline{B}^G \subseteq \overline{A}^G$, and thus $a_{r,t,s} \in \overline{A}^G$. Because A is neutrosophically G-sequentially closed in (X,τ), we have $a_{r,t,s} \in A$. Therefore, $a_{r,t,s}$ is a neutrosophic point in the neutrosophic G-sequential closure of B in A. Because B is neutrosophically G-sequentially closed in A, we have that $a_{r,t,s} \in B$. This conclusion completes the proof of the lemma. □

Lemma 4.14. *Let A be a neutrosophically G-sequentially connected subset in (X,τ). If U and V are non-null neutrosophically disjoint and neutrosophically G-sequentially closed subsets in (X,τ) such that $A \subseteq U \cup V$, then either $A \subseteq U^c$ or $A \subseteq V^c$.*

Proof. Let (X,τ) be a neutrosophic topological space. Consider a neutrosophically G-sequentially connected subset A and non-null neutrosophically disjoint and neutrosophically G-sequentially closed subsets U and V in (X,τ) such that $A \subseteq U \cup V$. Suppose that $A \nsubseteq U^c$ and $A \nsubseteq V^c$. This means that AqU and AqV. This implies that A is not neutrosophically G-sequentially connected subset in (X,τ). From this contradiction, it is obvious that $A \subseteq U^c$ or $A \subseteq V^c$. □

Lemma 4.15. *Let G be a neutrosophic sequential method, A be a neutrosophic subset in (X,τ), and U a neutrosophically G-sequentially open and neutrosophically G-sequentially closed subset in (X,τ) that is neutrosophically disjoint with its complement and $A \subseteq U \cup U^c$. If A is neutrosophically G-sequentially connected, then either $A \subseteq U$ or $A \subseteq U^c$.*

Proof. If $U = 0_X$ or $U = 1_X$, the proof is obvious. Suppose that $U \neq 0_X$ and $U \neq 1_X$. Since $A \subseteq U \cup U^c$, and so by Lemma 4.14, either $A \subseteq U^c$ or $A \subseteq (U^c)^c$. So, $A \subseteq U^c$ or $A \subseteq U$. □

Theorem 4.16. *Let G be a neutrosophic regular sequential method, $B \subseteq 1_X$, and $B \subseteq A \subseteq \overline{B}^G$. If B is neutrosophic G-sequentially connected, then so is A.*

Proof. If $B \subseteq A \subseteq \overline{B}^G$, then $A \subseteq \overline{B}^G \cap A = \overline{B_A}^G$. On the other hand, $\overline{B_A}^G \subseteq A$. Therefore, $\overline{B}_A = A$ where $\overline{B_A}^G$ is the neutrosophic G-sequential closure of B in A. Now, conversely, suppose that A is not neutrosophically G-sequentially connected. So there are non-null and neutrosophically disjoint G-sequentially closed subsets U and V in (X,τ) such that $A \subseteq U \cup V$, and AqU and AqV . Because B is connected by Lemma 4.14, either $B \subseteq U^c$ or $B \subseteq V^c$. If $B \subseteq U^c$, then $B \subseteq V$ and $\overline{B}^G \subseteq \overline{V}^G$, and so $\overline{B_A}^G \subseteq \overline{V}^G \cap A$. Because G is neutrosophic disjoint and V is neutrosophic G-sequentially closed in (X,τ), we have that $\overline{V}^G = V$. So we have that $A = \overline{B_A}^G \subseteq A \cap V$, which implies that $A = A \cap V$. Similarly, if $B \subseteq U$, then $A = A \cap U$. This contradiction completes the proof. □

Corollary 4.17. *If G is a neuosophic regular sequential method, and A is a neutrosophic G-sequentially connected subset in (X,τ), then so is $\overline{A}^G$.*

We know by Theorem 2 in [1] that for a neutrosophic G-regular method and a neutrosophic subset A, $\overline{A}^G = \overline{A}$ if and only if G is a neutrosophic subsequential method. Here, $\overline{A}$ denotes the usual closure of A. Thus, we can state the following corollary.

Corollary 4.18. *Let G be a neutrosophic regular subsequential method. If there exists a neutrosophic G-sequentially connected and dense neutrosophic subset in (X,τ), then (X,τ) is Neutrosophically G-sequentially connected space.*

Theorem 4.19. *Let $\{A_i \mid i \in I\}$ be a class of neutrosophically G-sequentially connected subsets in (X,τ). If $\bigcap_{i\in I} A_i$ is non-null $\bigcup_{i\in I} A_i$, then is neutrosophically G-sequentially connected.*

Proof. Suppose that A is not neutrosophically G-sequentially connected, so that there exist non-null neutrosophically disjoint G-sequentially closed subsets U and V in (X,τ) such that $A \subseteq U \cup V$. Because each A_i is neutrosophically G-sequentially connected, by Lemma 4.14, either $A_i \subseteq U^c$ or $A_i \subseteq V^c$. If $A_i \subseteq U^c$ and $A_j \subseteq V^c$ for $i \neq j$, then $A_i \subseteq U$ and $A_j \subseteq V$ for $i \neq j$. So, A_i and A_j are neutrosophically disjoint sets. Because $\bigcap_{i\in I} A_i$ is non-null, for all $i \in I$, either $A_i \subseteq U$ or $A_i \subseteq V$. Therefore, either $A \subseteq U$ or $A \subseteq V$. $A \subseteq U$, then $A = A \cap U$. If $A \subseteq V$, then $A = A \cap V$, which is a contradiction. Thus, A is neutrosophically G-sequentially connected. □

Corollary 4.20. *Let $\{A_i \mid i \in I\}$ be a class of neutrosophically G-sequentially connected subsets in (X,τ). Let B be neutrosophically G-sequentially connected subsets in (X,τ) such that $B \cap A_i$ is non-null for each $i \in I$. Then $B \cup (\bigcup_{i\in I} A_i)$ is also neutrosophically G-sequentially connected.*

Proof. Because $B \cap A_i$ is non-null, $B_i = B \cup A_i$ is neutrosophically G-sequentially connected for each $i \in I$, and $\bigcap_{i\in I} B_i$ is non-null. Therefore, by Theorem 4.19, the union $\bigcup_{i\in I} B_i = B \cup (\bigcap_{i\in I} A_i)$ is neutrosophically G-sequentially connected. □

Definition 4.21. Let (X,Δ) be a neutrosophic group in a neutrosophic topological space (X,τ), where Δ is a binary operation defined on (X,τ) and Y be a non-null subset of X. If (Y,Δ) is a neutrosophic group then (Y,Δ) is said to be a subgroup of (X,Δ) in (X,τ).

Definition 4.22. Let (X,Δ) be a neutrosophic group in a neutrosophic topological space (X,τ), where Δ is a binary operation defined on (X,τ) and A, B be a non-null subsets of X. Then,

i. $a\Delta B = \{a\Delta b \mid b \in B\}$, where $a \in X$,

ii. $A\Delta B = \{a\Delta b \mid a \in A, b \in B\}$,

iii. $A^{-1} = \{a^{-1} \mid a \in A\}$.

Definition 4.23. Let (X,Δ) be a neutrosophic group in a neutrosophic topological space (X,τ) and (Y,Δ) be a subgroup of (X,Δ) such that $x\Delta Y = Y\Delta x$ for each $x \in X$. Then, (Y,Δ) is said to be *a normal subgroup* of (X,Δ).

Lemma 4.24. *Let G be a neutrosophic regular method, and let A and B be neutrosophic subsets in (X,τ). Then the following are satisfied:*

i. *If $A \subset B$, then $\overline{A}^G \subset \overline{B}^G$;*

ii. $\overline{A}^G \Delta \overline{B}^G \subset \overline{A\Delta B}^G$;

iii. $(\overline{A}^G)^{-1} = \overline{A^{-1}}^G$.

Theorem 4.25. *Let G be a neutrosophic regular sequential method. If H is a neutrosophic G-sequentially connected (normal) subgroup in (X, τ), then so is $\overline{H}^G$.*

Proof. Let H be a neutrosophically G-sequentially connected subgroup in (X, τ). Then, by Corollary 4.17, $\overline{H}^G$ is neutrosophically G-sequentially connected because H is a neutrosophic subgroup $H - H \subseteq H$. By Lemma 4.24, $\overline{H}^G - \overline{H}^G \subseteq \overline{H - H}^G \subseteq \overline{H}^G$. Therefore, $\overline{H}^G - \overline{H}^G \subseteq \overline{H}^G$, and so $\overline{H}^G$ is a neutrosophic subgroup in (X, τ). Furthermore, if H is neutrosophic normal, $a_{r,t,s} + H - a_{r,t,s} \subseteq H$ for each neutrosophic point $a_{r,t,s}$ in (X, τ). Thus, by Lemma 4.24,

$$\overline{a_{r,t,s}}^G + \overline{H}^G - \overline{a_{r,t,s}}^G \subseteq \overline{a_{r,t,s} + H - a_{r,t,s}}^G \subseteq \overline{H}^G.$$

Because G is neutrosophic regular, $\{a\} \subseteq \overline{\{a\}}$, and so $\{a\} + \overline{H}^G - \{a\} \subseteq \overline{H}^G$. Therefore, $\overline{H}^G$ is neutrosophic normal. □

Lemma 4.26. *Let G be a neutrosophic regular sequential method and U a symmetric neighbourhood of identity element. If U is G-sequentially connected, then so is $U + U$.*

Proof. If U is neutrosophically G-sequentially connected, then, by Theorem 4.9, for each $a_{r,t,s} \in U$, the set $a_{r,t,s} + U$ is neutrosophically G-sequentially connected, and as U is symmetric, $a_{r,t,s} + U$ includes the identity element. Because $U + U = \bigcup_{a_{r,t,s} \in U} a_{r,t,s} + U$ by Theorem 4.19, the set $U + U$ is neutrosophically G-sequentially connected. □

Theorem 4.27. *Let G be a neutrosophic regular sequential method, and let H be a neutrosophic subgroup in (X, τ). If H is neutrosophic G-sequentially open, then it is neutrosophic G-sequentially closed.*

Proof. Let H be a neutrosophic G-sequentially open subgroup in (X, τ). Then $a_{r,t,s} + H$ is neutrosophic G-sequentially open for each neutrosophic point $a_{r,t,s}$ in (X, τ). On the other hand, because $H^c = \bigcup_{a_{r,t,s} \in H^c} a_{r,t,s} + H$ and the union of neutrosophic G-sequentially open subsets is open, H^c becomes neutrosophic G-sequentially open. Therefore, H is neutrosophically sequentially closed. □

5. Conclusion

We gave a different identity to the concept of compactness by using a different function definition in neutrosophic spaces to sequential compactness, which had been the basis of many previous studies. We also introduced terms that had never been used before in neutrosophic spaces. At the end of this research, we introduced neutrosophic G-sequentially connected spaces using a type of function called the neutrosophic method. Besides, using this function type, we presented neutrosophic G-sequentially continuity and neutrosophically G-sequentially closedness in neutrosophic spaces. Then, we examined the relationships between these new concepts that we presented in this study and in what situations how their behaviors were shaped. We hope that this study will contribute to many studies that will be conducted in different disciplines. Also, we expect that this study also will help to overcome the handicaps faced by many scientists working in the rapidly advancing technology world and create new fields of study for scientists to contribute to the science. One of our expectations is that the achievements in this study inspire scientists studying in mathematics and other sciences to come up with new ideas, and use these ideas for the benefit of human life.

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