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Non-global nonlinear Lie n-derivations on unital algebras with idempotents

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Abstract. Let $\mathfrak{T}$ be a unital algebra with nontrivial idempotents. For any $s_1, s_2, \ldots, s_n \in \mathfrak{T}$, define $p_1(s_1) = s_1$, $p_2(s_1, s_2) = [s_1, s_2]$ and $p_n(s_1, s_2, \ldots, s_n) = [p_{n-1}(s_1, s_2, \ldots, s_{n-1}), s_n]$ for all integers $n \geq 3$. In the present article, it is shown that if a map $\varphi : \mathfrak{T} \to \mathfrak{T}$ satisfies

$$\varphi(p_n(s_1, s_2, \ldots, s_n)) = \sum_{i=1}^{n} p_n(s_1, \ldots, s_{i-1}, \varphi(s_i), s_{i+1}, \ldots, s_n) \quad (n \geq 3)$$

for all $s_1, s_2, \ldots, s_n \in \mathfrak{T}$ with $s_1 s_2 \cdots s_n = 0$, then $\varphi(s + t) - \varphi(s) - \varphi(t) \in \mathcal{Z}(\mathfrak{T})$ for all $s, t \in \mathfrak{T}$, and under some mild assumptions φ is of the form $\delta + \tau$, where $\delta : \mathfrak{T} \to \mathfrak{T}$ is an additive derivation and $\tau : \mathfrak{T} \to \mathcal{Z}(\mathfrak{T})$ is a map such that $\tau(p_n(s_1, s_2, \ldots, s_n)) = 0$ for all $s_1, s_2, \ldots, s_n \in \mathfrak{T}$ with $s_1 s_2 \cdots s_n = 0$. The above results are then applied to certain special classes of unital algebras, namely triangular algebras, full matrix algebras and algebra of all bounded linear operators.

1. Introduction

Let $\mathfrak{R}$ be a fixed unital commutative ring. In this article, by an algebra we shall mean an algebra over $\mathfrak{R}$ and we further assume that $\frac{1}{2} \in \mathfrak{R}$. Let $\mathfrak{T}$ be an algebra with center $\mathcal{Z}(\mathfrak{T})$. An additive (resp. a linear) map $\delta : \mathfrak{T} \to \mathfrak{T}$ is called an additive (resp. a linear) derivation if $\delta(st) = \delta(s)t + s\delta(t)$ for all $s, t \in \mathfrak{T}$. A linear map $\omega : \mathfrak{T} \to \mathfrak{T}$ is called a Lie derivation (resp. Lie triple derivation) if $\omega([s, t]) = [\omega(s), t] + [s, \omega(t)]$ (resp. $\omega([[s, t], u]) = [[\omega(s), t], u] + [[s, \omega(t)], u] + [[s, t], \omega(u)]$) holds for all $s, t, u \in \mathfrak{T}$. If the map ω is not necessarily linear, then it is called a nonlinear Lie derivation (resp. nonlinear Lie triple derivation). Nonlinear Lie (triple) derivations have been studied extensively on various algebras (see [1, 3, 4, 9, 11, 14, 16, 18, 19, 24, 27] and references therein). In most of the cases it comes out that a nonlinear Lie (triple) derivation $\omega : \mathfrak{T} \to \mathfrak{T}$ is the sum of an additive derivation on $\mathfrak{T}$ and a map from $\mathfrak{T}$ to its center $\mathcal{Z}(\mathfrak{T})$.

One direction of investigation is to study the conditions under which nonlinear Lie (triple) derivations on algebras can be completely determined by the action on some particular subsets of these algebras. In the year 2010, Lu and Jing [16] initiated the study of local actions of Lie derivations of operator algebras

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and proved the following result. Let $\mathfrak{X}$ be a complex Banach space of dimension greater than 2 and $B(\mathfrak{X})$ be the algebra of all bounded linear operators acting on $\mathfrak{X}$. If $\omega : B(\mathfrak{X}) \to B(\mathfrak{X})$ is a linear map satisfying $\omega([s,t]) = [\omega(s),t] + [s,\omega(t)]$ for all $s,t \in B(\mathfrak{X})$ with $st = 0$ (resp. $st = e$, where e is a fixed nontrivial idempotent), then $\omega = \delta + \gamma$, where δ is a derivation of $B(\mathfrak{X})$ and $\gamma : B(\mathfrak{X}) \to \mathbb{C}I$ is a linear map vanishing at commutators $[s,t]$ with $st = 0$ (resp. $st = e$). Motivated by this result, many authors obtained similar results in different directions (see [2, 12, 13, 15, 20–22] and references therein). Let $f : \underbrace{\mathfrak{T} \times \mathfrak{T} \times \cdots \times \mathfrak{T}}_{n-copies} \to \mathfrak{T}$ be a map and $\mathfrak{C}$ be a proper subset of $\mathfrak{T}$. A mapping $\omega : \mathfrak{T} \to \mathfrak{T}$ (not necessarily linear) is called a non-global nonlinear Lie n-derivation if $\omega(p_n(s_1, s_2, \ldots, s_n)) = \sum_{i=1}^{n} p_n(s_1, \ldots, s_{i-1}, \omega(s_i), s_{i+1}, \ldots, s_n)$ for all $s_1, s_2, \ldots, s_n \in \mathfrak{T}$ with $f(s_1, s_2, \ldots, s_n) \in \mathfrak{C}$. Recently, non-global nonlinear Lie 3-derivations of some algebras have been studied in [23, 28, 29]. However, there has not been any study concerning with non-global Lie n-derivations of unital algebras so far in the nonlinear setting. Therefore, it needs to be discussed further. In the present article, we shall investigate non-global nonlinear Lie n-derivations on unital algebras with idempotents. More precisely, we prove that under certain conditions every mapping $\varphi : \mathfrak{T} \to \mathfrak{T}$ satisfying $\varphi(p_n(s_1, s_2, \ldots, s_n)) = \sum_{i=1}^{n} p_n(s_1, \ldots, s_{i-1}, \varphi(s_i), s_{i+1}, \ldots, s_n)$ $(n \geq 3)$ for all $s_1, s_2, \ldots, s_n \in \mathfrak{T}$ with $s_1 s_2 \cdots s_n = 0$, can be expressed as $\varphi = \delta + \tau$, where $\delta : \mathfrak{T} \to \mathfrak{T}$ is an additive derivation and $\tau : \mathfrak{T} \to \mathcal{Z}(\mathfrak{T})$ is an additive mapping vanishing at every $(n-1)$-commutator $p_n(s_1, s_2, \ldots, s_n)$ with $s_1 s_2 \cdots s_n = 0$. Finally, the above is applied to triangular algebras (Corollary 5.1), full matrix algebras (Corollary 5.6) and the algebra of all bounded linear operators (Corollary 5.7).

2. Preliminaries

Let $\mathfrak{T}$ be an algebra with a nontrivial idempotent e_1, and denote the idempotent $e_2 = 1 - e_1$. In this case, $\mathfrak{T}$ can be represented as $\mathfrak{T} = e_1\mathfrak{T}e_1 + e_1\mathfrak{T}e_2 + e_2\mathfrak{T}e_1 + e_2\mathfrak{T}e_2$, where $e_1\mathfrak{T}e_1$ and $e_2\mathfrak{T}e_2$ are subalgebras with identity elements e_1 and e_2, respectively, $e_1\mathfrak{T}e_2$ is an $(e_1\mathfrak{T}e_1, e_2\mathfrak{T}e_2)$-bimodule and $e_2\mathfrak{T}e_1$ is an $(e_2\mathfrak{T}e_2, e_1\mathfrak{T}e_1)$-bimodule. If the bimodule $e_1\mathfrak{T}e_2$ is a faithful $(e_1\mathfrak{T}e_1, e_2\mathfrak{T}e_2)$-bimodule and $e_2\mathfrak{T}e_1 = \{0\}$ then in this case $\mathfrak{T}$ is a triangular algebra. Let us assume that $\mathfrak{T}$ satisfies

$$\begin{cases} e_1 s e_1 \cdot e_1\mathfrak{T}e_2 = \{0\} = e_2\mathfrak{T}e_1 \cdot e_1 s e_1 \quad \text{implies} \quad e_1 s e_1 = 0, \\ e_1\mathfrak{T}e_2 \cdot e_2 s e_2 = \{0\} = e_2 s e_2 \cdot e_2\mathfrak{T}e_1 \quad \text{implies} \quad e_2 s e_2 = 0, \end{cases} \tag{$\spadesuit$}$$

for all $s \in \mathfrak{T}$. The unital algebras with the property ($\spadesuit$) were proposed by Benkovič and Širovnik in [8]. There are many important unital algebras which can be seen as examples of unital algebras with nontrivial idempotents satisfying ($\spadesuit$) such as: triangular algebras, full matrix algebras, (semi-)simple algebras and prime algebras with nontrivial idempotents. Set $\mathfrak{T}_{ij} = e_i\mathfrak{T}e_j$ $(1 \leq i,j \leq 2)$. Then $\mathfrak{T} = \mathfrak{T}_{11} + \mathfrak{T}_{12} + \mathfrak{T}_{21} + \mathfrak{T}_{22}$ and each element $s \in \mathfrak{T}$ can be written as $s = s_{11} + s_{12} + s_{21} + s_{22}$, where $s_{ij} \in \mathfrak{T}_{ij}$, $1 \leq i,j \leq 2$. The center of $\mathfrak{T}$ is given by

$$\mathcal{Z}(\mathfrak{T}) = \{s_{11} + s_{22} \in \mathfrak{T}_{11} + \mathfrak{T}_{22} \,|\, s_{11}x_{12} = x_{12}s_{22},\ x_{21}s_{11} = s_{22}x_{21} \ \text{ for all } \ x_{12} \in \mathfrak{T}_{12},\ x_{21} \in \mathfrak{T}_{21}\}.$$

Furthermore, there exists a unique algebra isomorphism $\xi : \mathcal{Z}(\mathfrak{T})e_1 \to \mathcal{Z}(\mathfrak{T})e_2$ such that $s_{11}s_{12} = s_{12}\xi(s_{11})$ and $s_{21}s_{11} = \xi(s_{11})s_{21}$ for all $s_{11} \in \mathcal{Z}(\mathfrak{T})e_1, s_{12} \in \mathfrak{T}_{12}, s_{21} \in \mathfrak{T}_{21}$ (see [6, Proposition 2.1]). The following remark will be used throughout the article frequently.

Remark 2.1. *Let $s \in \mathfrak{T}$. If $[s, e_1\mathfrak{T}e_2] = 0$ and $[s, e_2\mathfrak{T}e_1] = 0$, then $e_1 s e_1 + e_2 s e_2 \in \mathcal{Z}(\mathfrak{T})$.*

The following condition was introduced by Benkovič and Eremita in [7]. Suppose that $\mathfrak{R}$ is a ring such that for each $x \in \mathfrak{R}$

$$[x, \mathfrak{R}] \in \mathcal{Z}(\mathfrak{R}) \Longrightarrow x \in \mathcal{Z}(\mathfrak{R}). \tag{1}$$

Then we say that $\mathfrak{R}$ satisfies the condition (1). It is easy to verify that every unital algebra satisfying ($\spadesuit$) satisfies (1). We refer the reader to [26, Section 4] and [7, Section 5] where some more interesting examples of rings and algebras satisfying (1) can be seen. We close this section by stating the following elementary lemma below which will be used quite frequently throughout the paper without further mentioning.

Lemma 2.2. *For any $x \in \mathfrak{T}$, we have*

(*i*) $p_n(x, -e_1, -e_1, \ldots, -e_1) = e_1xe_2 + (-1)^{n+1}e_2xe_1$.

(*ii*) $p_n(x, -e_2, -e_2, \ldots, -e_2) = e_2xe_1 + (-1)^{n+1}e_1xe_2$.

3. Almost additivity of non-global nonlinear Lie *n*-derivations

Over the past few decade, a lot of work has been done on the additivity of mappings on various rings and algebras. In the year 1969, Martindale III [17] proved a remarkable result which states that every multiplicative isomorphism from a prime ring containing a nontrivial idempotent onto an arbitrary ring is additive. Daif [10] proved that under certain conditions every multiplicative derivation of a ring containing a nontrivial idempotent is additive. Inspired by these results, in this section we investigate the additivity of non-global nonlinear Lie n-derivations on unital algebras and prove the following result:

Theorem 3.1. *Let $\mathfrak{T}$ be a unital algebra containing a nontrivial idempotent e_1 satisfying the condition (♠). If a map $\varphi : \mathfrak{T} \to \mathfrak{T}$ satisfies*

$$\varphi(p_n(s_1, s_2, \ldots, s_n)) = \sum_{i=1}^{n} p_n(s_1, \ldots, s_{i-1}, \varphi(s_i), s_{i+1}, \ldots, s_n) \ \ (n \geq 3)$$

for all $s_1, s_2, \ldots, s_n \in \mathfrak{T}$ with $s_1s_2 \cdots s_n = 0$, then φ is almost additive, that is, $\varphi(s+t) - \varphi(s) - \varphi(t) \in \mathcal{Z}(\mathfrak{T})$ for all $s, t \in \mathfrak{T}$.

The proof of the above theorem can be achieved via the following series of lemmas.

Lemma 3.2. $\varphi(0) = 0$.

Proof. $\varphi(0) = \varphi(p_n(0, 0, \ldots, 0)) = p_n(\varphi(0), 0, \ldots, 0) + \cdots + p_n(0, 0, \ldots, \varphi(0)) = 0$. □

Lemma 3.3. *For any $s_{ii} \in \mathfrak{T}_{ii}, s_{ij} \in \mathfrak{T}_{ij}, s_{ji} \in \mathfrak{T}_{ji}$ $(1 \leq i \neq j \leq 2)$, we have*
$\varphi(s_{ii} + s_{ij}) - \varphi(s_{ii}) - \varphi(s_{ij}) \in \mathcal{Z}(\mathfrak{T})$ *and* $\varphi(s_{ii} + s_{ji}) - \varphi(s_{ii}) - \varphi(s_{ji}) \in \mathcal{Z}(\mathfrak{T})$.

Proof. Assume that $z = \varphi(s_{11}+s_{12}) - \varphi(s_{11}) - \varphi(s_{12})$. Since $(-e_2)(s_{11}+s_{12})(-e_1)(-e_1)\cdots(-e_1) = 0$ for any $s_{11} \in \mathfrak{T}_{11}$ and $s_{12} \in \mathfrak{T}_{12}$, we obtain

$$\begin{aligned} \varphi(s_{12}) &= \varphi(p_n(-e_2, s_{11}+s_{12}, -e_1, \ldots, -e_1)) \\ &= p_n(\varphi(-e_2), s_{11}+s_{12}, -e_1, \ldots, -e_1) + p_n(-e_2, \varphi(s_{11}+s_{12}), -e_1, \ldots, -e_1) \\ &\quad + p_n(-e_2, s_{11}+s_{12}, \varphi(-e_1), -e_1, \ldots, -e_1) + \cdots + p_n(-e_2, s_{11}+s_{12}, -e_1, \ldots, \varphi(-e_1)). \end{aligned}$$

On the other hand, by invoking Lemma 3.2, we find that

$$\begin{aligned} \varphi(s_{12}) &= \varphi(p_n(-e_2, s_{11}+s_{12}, -e_1, \ldots, -e_1)) \\ &= \varphi(p_n(-e_2, s_{11}, -e_1, \ldots, -e_1)) + \varphi(p_n(-e_2, s_{12}, -e_1, \ldots, -e_1)) \\ &= p_n(\varphi(-e_2), s_{11}+s_{12}, -e_1, \ldots, -e_1) + p_n(-e_2, \varphi(s_{11}) + \varphi(s_{12}), -e_1, \ldots, -e_1) \\ &\quad + p_n(-e_2, s_{11}+s_{12}, \varphi(-e_1), -e_1, \ldots, -e_1) + \cdots + p_n(-e_2, s_{11}+s_{12}, -e_1, \ldots, \varphi(-e_1)). \end{aligned}$$

Combining the above two expressions, we get $p_n(-e_2, z, -e_1, \ldots, -e_1) = 0$ which on simplifying with the help of Lemma 2.2 gives $z_{12} + (-1)^{n+1}z_{21} = 0$. Hence, $z_{12} = z_{21} = 0$.
Since $(-t_{12})(s_{11}+s_{12})(-e_1)\cdots(-e_1) = 0$ for any $t_{12} \in \mathfrak{T}_{12}$, we obtain

$$\begin{aligned} \varphi(s_{11}t_{12}) &= \varphi(p_n(-t_{12}, s_{11}+s_{12}, -e_1, \ldots, -e_1)) \\ &= p_n(\varphi(-t_{12}), s_{11}+s_{12}, -e_1, \ldots, -e_1) + p_n(-t_{12}, \varphi(s_{11}+s_{12}), -e_1, \ldots, -e_1) \\ &\quad + p_n(-t_{12}, s_{11}+s_{12}, \varphi(-e_1), \ldots, -e_1) + \cdots + p_n(-t_{12}, s_{11}+s_{12}, -e_1, \ldots, \varphi(-e_1)). \end{aligned}$$

On the other hand, by invoking Lemma 3.2, we see that

$$\begin{aligned}\varphi(s_{11}t_{12}) &= \varphi(p_n(-t_{12}, s_{11}+s_{12}, -e_1, \ldots, -e_1))\\ &= \varphi(p_n(-t_{12}, s_{11}, -e_1, \ldots, -e_1)) + \varphi(p_n(-t_{12}, s_{12}, -e_1, \ldots, -e_1)])\\ &= p_n(\varphi(-t_{12}), s_{11}+s_{12}, -e_1, \ldots, -e_1) + p_n(-t_{12}, \varphi(s_{11})+\varphi(s_{12}), -e_1, \ldots, -e_1)\\ &\quad + p_n(-t_{12}, s_{11}+s_{12}, \varphi(-e_1), \ldots, -e_1) + \cdots + p_n(-t_{12}, s_{11}+s_{12}, -e_1, \ldots, \varphi(-e_1)).\end{aligned}$$

Comparing these two expressions for $\varphi(s_{11}t_{12})$, we obtain $p_n(-t_{12}, z, -e_1, \ldots, -e_1) = 0$. Simplifying it with the help of Lemma 2.2 and using the fact that $z_{12} = z_{21} = 0$, we conclude that $[z, t_{12}] = 0$.
Since $(s_{11}+s_{12})(-t_{21})(-e_2)\cdots(-e_2) = 0$ for any $t_{21} \in \mathfrak{T}_{21}$, we obtain

$$\begin{aligned}\varphi(t_{21}s_{11}) &= \varphi(p_n(s_{11}+s_{12}, -t_{21}, -e_2, \ldots, -e_2))\\ &= p_n(\varphi(s_{11}+s_{12}), -t_{21}, -e_2, \ldots, -e_2) + p_n(s_{11}+s_{12}, \varphi(-t_{21}), -e_2, \ldots, -e_2)\\ &\quad + p_n(s_{11}+s_{12}, -t_{21}, \varphi(-e_2), \ldots, -e_2) + \cdots + p_n(s_{11}+s_{12}, -t_{21}, -e_2, \ldots, \varphi(-e_2)).\end{aligned}$$

On the other hand, by invoking Lemma 3.2, we find that

$$\begin{aligned}\varphi(t_{21}s_{11}) &= \varphi(p_n(s_{11}+s_{12}, -t_{21}, -e_2, \ldots, -e_2))\\ &= \varphi(p_n(s_{11}, -t_{21}, (-e_2), \ldots, -e_2)) + \varphi(p_n(s_{12}, -t_{21}, -e_2, \ldots, -e_2))\\ &= p_n(\varphi(s_{11})+\varphi(s_{12}), -t_{21}, -e_2, \ldots, -e_2) + p_n(s_{11}+s_{12}, \varphi(-t_{21}), -e_2, \ldots, -e_2)\\ &\quad + p_n(s_{11}+s_{12}, -t_{21}, \varphi(-e_2), \ldots, -e_2) + \cdots + p_n(s_{11}+s_{12}, -t_{21}, -e_2, \ldots, \varphi(-e_2)).\end{aligned}$$

Comparing these two expressions for $\varphi(t_{21}s_{11})$, we see that $p_n(z, -t_{21}, -e_2, \ldots, -e_2) = 0$, which leads to $[z, t_{21}] = 0$ on simplifying with the help of Lemma 2.2. Thus we see that $[z, t_{12}] = 0$ and $[z, t_{21}] = 0$. Using Remark 2.1, we conclude that $z = z_{11} + z_{22} \in \mathcal{Z}(\mathfrak{T})$, that is,

$$z = \varphi(s_{11}+s_{12}) - \varphi(s_{11}) - \varphi(s_{12}) \in \mathcal{Z}(\mathfrak{T}).$$

Symmetrically, we can obtain the other cases. □

Lemma 3.4. *For any* $s_{ij},\ t_{ij} \in \mathfrak{T}_{ij}$ $(1 \leq i \neq j \leq 2)$, *we have* $\varphi(s_{ij}+t_{ij}) = \varphi(s_{ij}) + \varphi(t_{ij})$.

Proof. Observe that $(s_{12}+e_1)(e_2+t_{12})(-e_1)\cdots(-e_1) = 0$ for any $s_{12}, t_{12} \in \mathfrak{T}_{12}$.
Also $p_n(s_{12}+e_1, e_2+t_{12}, -e_1, \ldots, -e_1) = s_{12}+t_{12}$, by invoking Lemmas 3.2 and 3.3, we find that

$$\begin{aligned}&\varphi(s_{12}+t_{12})\\ &= \varphi(p_n(s_{12}+e_1, e_2+t_{12}, -e_1, \ldots, -e_1))\\ &= p_n(\varphi(s_{12}+e_1), e_2+t_{12}, -e_1, \ldots, -e_1) + p_n(s_{12}+e_1, \varphi(e_2+t_{12}), -e_1, \ldots, -e_1)\\ &\quad + p_n(s_{12}+e_1, e_2+t_{12}, \varphi(-e_1), \ldots, -e_1) + \cdots + p_n(s_{12}+e_1, e_2+t_{12}, -e_1, \ldots, \varphi(-e_1))\\ &= p_n(\varphi(s_{12})+\varphi(e_1), e_2+t_{12}, -e_1, \ldots, -e_1) + p_n(s_{12}+e_1, \varphi(e_2)+\varphi(t_{12}), -e_1, \ldots, -e_1)\\ &\quad + p_n(s_{12}+e_1, e_2+t_{12}, \varphi(-e_1), \ldots, -e_1) + \cdots + p_n(s_{12}+e_1, e_2+t_{12}, -e_1, \ldots, \varphi(-e_1))\\ &= p_n(\varphi(s_{12}), e_2, -e_1, \ldots, -e_1) + p_n(s_{12}, \varphi(e_2), -e_1, \ldots, -e_1)\\ &\quad + p_n(s_{12}, e_2, \varphi(-e_1), \ldots, -e_1) + \cdots + p_n(s_{12}, e_2, -e_1, \ldots, \varphi(-e_1))\\ &\quad + p_n(\varphi(s_{12}), t_{12}, -e_1, \ldots, -e_1) + p_n(s_{12}, \varphi(t_{12}), -e_1, \ldots, -e_1)\\ &\quad + p_n(s_{12}, t_{12}, \varphi(-e_1), \ldots, -e_1) + \cdots + p_n(s_{12}, t_{12}, -e_1, \ldots, \varphi(-e_1))\\ &\quad + p_n(\varphi(e_1), e_2, -e_1, \ldots, -e_1) + p_n(e_1, \varphi(e_2), -e_1, \ldots, -e_1)\\ &\quad + p_n(e_1, e_2, \varphi(-e_1), \ldots, -e_1) + \cdots + p_n(e_1, e_2, -e_1, \ldots, \varphi(-e_1))\\ &\quad + p_n(\varphi(e_1), t_{12}, -e_1, \ldots, -e_1) + p_n(e_1, \varphi(t_{12}), -e_1, \ldots, -e_1)\\ &\quad + p_n(e_1, t_{12}, \varphi(-e_1), \ldots, -e_1) + \cdots + p_n(e_1, t_{12}, -e_1, \ldots, \varphi(-e_1))\\ &= \varphi(p_n(s_{12}, e_2, -e_1, \ldots, -e_1)) + \varphi(p_n(s_{12}, t_{12}, -e_1, \ldots, -e_1))\\ &\quad + \varphi(p_n(e_1, e_2, -e_1, \ldots, -e_1)) + \varphi(p_n(e_1, t_{12}, -e_1, \ldots, -e_1))\\ &= \varphi(s_{12}) + \varphi(t_{12}).\end{aligned}$$

Thus, φ is additive on $\mathfrak{T}_{12}$.
Noticing that $(s_{21}+e_2)(e_1+t_{21})(-e_2)\cdots(-e_2)=0$ and $p_n(s_{21}+e_2, e_1+t_{21}, -e_2, \ldots, -e_2)=s_{21}+t_{21}$ for any $s_{21}, t_{21}\in\mathfrak{T}_{21}$, we can prove that φ is additive on $\mathfrak{T}_{21}$. □

Lemma 3.5. *For any $s_{12}\in\mathfrak{T}_{12}$ and $t_{21}\in\mathfrak{T}_{21}$, $\varphi((-1)^n s_{12}+t_{21})=\varphi((-1)^n s_{12})+\varphi(t_{21})$.*

Proof. Observe that $(e_1-s_{12})(e_1-t_{21})(-e_2)\cdots(-e_2)=0$ and
$p_n(e_1-s_{12}, e_1-t_{21}, -e_2, \ldots, -e_2)=(-1)^n s_{12}+t_{21}$ for any $s_{12}\in\mathfrak{T}_{12}, t_{21}\in\mathfrak{T}_{21}$. It follows from Lemma 3.3 that

$$\begin{aligned}
&\varphi((-1)^n s_{12}+t_{21})\\
&= \varphi(p_n(e_1-s_{12}, e_1-t_{21}, -e_2, \ldots, -e_2))\\
&= p_n(\varphi(e_1)+\varphi(-s_{12}), e_1-t_{21}, -e_2, \ldots, -e_2)+p_n(e_1-s_{12}, \varphi(e_1)+\varphi(-t_{21}), -e_2, \ldots, -e_2)\\
&\quad +p_n(e_1-s_{12}, e_1-t_{21}, \varphi(-e_2), \ldots, -e_2)+\cdots+p_n(e_1-s_{12}, e_1-t_{21}, -e_2, \ldots, \varphi(-e_2))\\
&= p_n(\varphi(e_1), e_1, -e_2, \ldots, -e_2)+p_n(e_1, \varphi(e_1), -e_2, \ldots, -e_2)\\
&\quad +p_n(e_1, e_1, \varphi(-e_2), \ldots, -e_2)+\cdots+p_n(e_1, e_1, -e_2, \ldots, \varphi(-e_2))\\
&\quad +p_n(\varphi(-s_{12}), e_1, -e_2, \ldots, -e_2)+p_n(-s_{12}, \varphi(e_1), -e_2, \ldots, -e_2)\\
&\quad +p_n(-s_{12}, e_1, -e_2, \ldots, -e_2)+\cdots+p_n(-s_{12}, e_1, -e_2, \ldots, \varphi(-e_2))\\
&\quad +p_n(\varphi(-s_{12}), -t_{21}, -e_2, \ldots, -e_2))+p_n(-s_{12}, \varphi(-t_{21}), -e_2, \ldots, -e_2))\\
&\quad +p_n(-s_{12}, -t_{21}, \varphi(-e_2), \ldots, -e_2))+\cdots+p_n(-s_{12}, -t_{21}, -e_2, \ldots, \varphi(-e_2))\\
&\quad +p_n(\varphi(e_1), -t_{21}, -e_2, \ldots, -e_2)+p_n(e_1, \varphi(-t_{21}), -e_2, \ldots, -e_2)\\
&\quad +p_n(e_1, -t_{21}, \varphi(-e_2), \ldots, -e_2)+\cdots+p_n(e_1, -t_{21}, -e_2, \ldots, \varphi(-e_2))\\
&= \varphi(p_n(e_1, e_1, -e_2, \ldots, -e_2))+\varphi(p_n(-s_{12}, e_1, -e_2, \ldots, -e_2))\\
&\quad +\varphi(p_n(-s_{12}, -t_{21}, -e_2, \ldots, -e_2))+\varphi(p_n(e_1, -t_{21}, -e_2, \ldots, -e_2))\\
&= \varphi((-1)^n s_{12})+\varphi(t_{21}).
\end{aligned}$$

□

Lemma 3.6. *For any $s_{ii},\ t_{ii}\in\mathfrak{T}_{ii}\ (i=1,2)$, we have $\varphi(s_{ii}+t_{ii})-\varphi(s_{ii})-\varphi(t_{ii})\in\mathcal{Z}(\mathfrak{T})$.*

Proof. Let us set $z=\varphi(s_{11}+t_{11})-\varphi(s_{11})-\varphi(t_{11})$.
Note that $(s_{11}+t_{11})(-e_2)\cdots(-e_2)=0$ for any $s_{11}, t_{11}\in\mathfrak{T}_{11}$. By invoking Lemma 3.2, we obtain

$$\begin{aligned}
0 &= \varphi(p_n(s_{11}+t_{11}, -e_2, \ldots, -e_2))\\
&= p_n(\varphi(s_{11}+t_{11}), -e_2, \ldots, -e_2)+p_n(s_{11}+t_{11}, \varphi(-e_2), \ldots, -e_2)+\cdots+p_n(s_{11}+t_{11}, -e_2, \ldots, \varphi(-e_2))
\end{aligned}$$

and

$$\begin{aligned}
0 &= \varphi(p_n(s_{11}+t_{11}, -e_2, \ldots, -e_2))\\
&= \varphi(p_n(s_{11}, -e_2, \ldots, -e_2))+\varphi(p_n(t_{11}, -e_2, \ldots, -e_2))\\
&= p_n(\varphi(s_{11})+\varphi(t_{11}), -e_2, \ldots, -e_2)+p_n(s_{11}+t_{11}, \varphi(-e_2), \ldots, -e_2)+\cdots+p_n(s_{11}+t_{11}, -e_2, \ldots, \varphi(-e_2)).
\end{aligned}$$

It follows from the above two expressions that $p_n(z, -e_2, \ldots, -e_2)=0$ which on simplification gives $z_{21}+(-1)^{n+1}z_{12}=0$. That is, $z_{12}=0$ and $z_{21}=0$.
Since $u_{12}(s_{11}+t_{11})(-e_1)\cdots(-e_1)=0$, we obtain

$$\begin{aligned}
&\varphi(p_n(u_{12}, s_{11}+t_{11}, -e_1, \ldots, -e_1))\\
&= p_n(\varphi(u_{12}), s_{11}+t_{11}, -e_1, \ldots, -e_1)+p_n(u_{12}, \varphi(s_{11}+t_{11}), -e_1, \ldots, -e_1)\\
&\quad +p_n(u_{12}, s_{11}+t_{11}, \varphi(-e_1), \ldots, -e_1)+\cdots+p_n(u_{12}, s_{11}+t_{11}, -e_1, \ldots, \varphi(-e_1)).
\end{aligned}$$

On the other hand, by invoking Lemma 3.4, we find that

$$\begin{aligned}
&\varphi(p_n(u_{12}, s_{11}+t_{11}, -e_1, \dots, -e_1)) \\
&= \varphi(-s_{11}u_{12} - t_{11}u_{12}) \\
&= \varphi(-s_{11}u_{12}) + \varphi(-t_{11}u_{12}) \\
&= \varphi(p_n(u_{12}, s_{11}, -e_1, \dots, -e_1)) + \varphi(p_n(u_{12}, t_{11}, -e_1, \dots, -e_1)) \\
&= p_n(\varphi(u_{12}), s_{11}+t_{11}, -e_1, \dots, -e_1) + p_n(u_{12}, \varphi(s_{11})+\varphi(t_{11}), -e_1, \dots, -e_1) \\
&\quad + p_n(u_{12}, s_{11}+t_{11}, \varphi(-e_1), \dots, -e_1) + \cdots + p_n(u_{12}, s_{11}+t_{11}, -e_1, \dots, \varphi(-e_1)).
\end{aligned}$$

Comparing the above two expressions, we obtain $p_n(u_{12}, z, -e_1, \dots, -e_1) = 0$, which leads to $[z, u_{12}] = 0$. Also, $(s_{11}+t_{11})u_{21}(-e_2)\cdots(-e_2) = 0$ for any $u_{21} \in \mathfrak{T}_{21}$. Thus, we obtain

$$\begin{aligned}
&\varphi(p_n(s_{11}+t_{11}, u_{21}, -e_2, \dots, -e_2)) \\
&= p_n(\varphi(s_{11}+t_{11}), u_{21}, -e_2, \dots, -e_2) + p_n(s_{11}+t_{11}, \varphi(u_{21}), -e_2, \dots, -e_2) \\
&\quad + p_n(s_{11}+t_{11}, u_{21}, \varphi(-e_2), \dots, -e_2) + \cdots + p_n(s_{11}+t_{11}, u_{21}, -e_2, \dots, \varphi(-e_2)).
\end{aligned}$$

On the other hand, by invoking Lemma 3.4, we find that

$$\begin{aligned}
&\varphi(p_n(s_{11}+t_{11}, u_{21}, -e_2, \dots, -e_2)) \\
&= \varphi(-u_{21}s_{11} - u_{21}t_{11}) \\
&= \varphi(-u_{21}s_{11}) + \varphi(-u_{21}t_{11}) \\
&= \varphi(p_n(s_{11}, u_{21}, -e_2, \dots, -e_2)) + \varphi(p_n(t_{11}, u_{21}, -e_2, \dots, -e_2)) \\
&= p_n(\varphi(s_{11})+\varphi(t_{11}), u_{21}, -e_2, \dots, -e_2) + p_n(s_{11}+t_{11}, \varphi(u_{21}), -e_2, \dots, -e_2) \\
&\quad + p_n(s_{11}+t_{11}, u_{21}, \varphi(-e_2), \dots, -e_2) + \cdots + p_n(s_{11}+t_{11}, u_{21}, -e_2, \dots, \varphi(-e_2)).
\end{aligned}$$

Comparing the above two expressions for $\varphi(p_n(s_{11}+t_{11}, u_{21}, -e_2, \dots, -e_2))$ we obtain $p_n(z, u_{21}, -e_2, \dots, -e_2) = 0$, which leads to $[z, u_{21}] = 0$. Thus we see that $[z, u_{12}] = 0$ and $[z, u_{21}] = 0$. Using Remark 2.1, we conclude that $z = z_{11} + z_{22} \in \mathcal{Z}(\mathfrak{T})$, that is, $\varphi(s_{11}+t_{11}) - \varphi(s_{11}) - \varphi(t_{11}) \in \mathcal{Z}(\mathfrak{T})$.
Symmetrically, one can prove the other case. □

Lemma 3.7. *For any $s_{11} \in \mathfrak{T}_{11}$, $t_{12} \in \mathfrak{T}_{12}$, $u_{21} \in \mathfrak{T}_{21}$, $v_{22} \in \mathfrak{T}_{22}$, we have*

$$\varphi(s_{11}+t_{12}+u_{21}+v_{22}) - \varphi(s_{11}) - \varphi(t_{12}) - \varphi(u_{21}) - \varphi(v_{22}) \in \mathcal{Z}(\mathfrak{T}).$$

Proof. Let us set $z = \varphi(s_{11}+t_{12}+u_{21}+v_{22}) - \varphi(s_{11}) - \varphi(t_{12}) - \varphi(u_{21}) - \varphi(v_{22})$ and $r = s_{11}+t_{12}+u_{21}+v_{22}$. Note that $(s_{11}+t_{12}+u_{21}+v_{22})(-e_1)(-e_2)\cdots(-e_2) = 0$. Hence, we have

$$\begin{aligned}
&\varphi(p_n(r, -e_1, -e_2, \dots, -e_2)) \\
&= p_n(\varphi(r), -e_1, -e_2, \dots, -e_2) + p_n(r, \varphi(-e_1), -e_2, \dots, -e_2) + p_n(r, -e_1, \varphi(-e_2), \dots, -e_2) \\
&\quad + \cdots + p_n(r, -e_1, -e_2, \dots, \varphi(-e_2)).
\end{aligned}$$

On the other hand, by invoking Lemmas 3.2 and 3.5, we find that

$$\begin{aligned}
&\varphi(p_n(r, -e_1, -e_2, \dots, -e_2)) \\
&= \varphi(-u_{21} + (-1)^n t_{12}) \\
&= \varphi(-u_{21}) + \varphi((-1)^n t_{12}) \\
&= \varphi(p_n(s_{11}, -e_1, -e_2, \dots, -e_2)) + \varphi(p_n(t_{12}, -e_1, -e_2, \dots, -e_2)) \\
&\quad + \varphi(p_n(u_{21}, -e_1, -e_2, \dots, -e_2)) + \varphi(p_n(v_{22}, -e_1, -e_2, \dots, -e_2)) \\
&= p_n(\varphi(s_{11}) + \varphi(t_{12}) + \varphi(u_{21}) + \varphi(v_{22}), -e_1, -e_2, \dots, -e_2) \\
&\quad + p_n(r, \varphi(-e_1), -e_2, \dots, -e_2) + p_n(r, -e_1, \varphi(-e_2), \dots, -e_2) \\
&\quad + \cdots + p_n(r, -e_1, -e_2, \dots, \varphi(-e_2)).
\end{aligned}$$

The above two expressions result in $p_n(z, -e_1, -e_2, \ldots, -e_2) = 0$, which leads to $z_{12} = z_{21} = 0$. Since $(s_{11} + t_{12} + u_{21} + v_{22})w_{12}(-e_1) \cdots (-e_1) = 0$, we have

$$\begin{aligned}\varphi(p_n(r, w_{12}, -e_1, \ldots, -e_1) &= p_n(\varphi(r), w_{12}, -e_1, \ldots, -e_1) + p_n(r, \varphi(w_{12}), -e_1, \ldots, -e_1)\\ &+p_n(r, w_{12}, \varphi(-e_1), \ldots, -e_1) + \cdots + p_n(r, w_{12}, -e_1, \ldots, \varphi(-e_1)).\end{aligned}$$

On the other hand, using Lemmas 3.2 and 3.4, we find that

$$\begin{aligned}&\varphi(p_n(r, w_{12}, -e_1, \ldots, -e_1)\\ &= \varphi(s_{11}w_{12} - w_{12}v_{22})\\ &= \varphi(s_{11}w_{12}) + \varphi(-w_{12}v_{22})\\ &= \varphi(p_n(s_{11}, w_{12}, -e_1, \ldots, -e_1)) + \varphi(p_n(t_{12}, w_{12}, -e_1, \ldots, -e_1))\\ &\quad +\varphi(p_n(u_{21}, w_{12}, -e_1, \ldots, -e_1)) + \varphi(p_n(v_{22}, w_{12}, -e_1, \ldots, -e_1))\\ &= p_n(\varphi(s_{11}) + \varphi(t_{12}) + \varphi(u_{21}) + \varphi(v_{22}), w_{12}, -e_1, \ldots, -e_1) + p_n(r, \varphi(w_{12}), -e_1, \ldots, -e_1)\\ &\quad +p_n(r, w_{12}, \varphi(-e_1), \ldots, -e_1) + \cdots + p_n(r, w_{12}, -e_1, \ldots, \varphi(-e_1)).\end{aligned}$$

Combining the above two expressions, we obtain $p_n(z, w_{12}, -e_1, \ldots, -e_1) = 0$, which leads to $[z, w_{12}] = 0$. Observe that $(s_{11} + t_{12} + u_{21} + v_{22})w_{21}(-e_2) \cdots (-e_2) = 0$, we have

$$\begin{aligned}\varphi(p_n(r, w_{21}, -e_2, \ldots, -e_2)) &= p_n(\varphi(r), w_{21}, -e_2, \ldots, -e_2) + p_n(r, \varphi(w_{21}), -e_2, \ldots, -e_2)\\ &+p_n(r, w_{21}, \varphi(-e_2), \ldots, -e_2) + \cdots + p_n(r, w_{21}, -e_2, \ldots, \varphi(-e_2)).\end{aligned}$$

On the other hand, using Lemmas 3.2 and 3.4, we find that

$$\begin{aligned}&\varphi(p_n(r, w_{21}, -e_2, \ldots, -e_2))\\ &= \varphi(-w_{21}s_{11} + v_{22}w_{21})\\ &= \varphi(-w_{21}s_{11}) + \varphi(v_{22}w_{21})\\ &= \varphi(p_n(s_{11}, w_{21}, -e_2, \ldots, -e_2)) + \varphi(p_n(t_{12}, w_{21}, -e_2, \ldots, -e_2))\\ &\quad +\varphi(p_n(u_{21}, w_{21}, -e_2, \ldots, -e_2)) + \cdots + \varphi(p_n(v_{22}, w_{21}, -e_2, \ldots, -e_2))\\ &= p_n(\varphi(s_{11}) + \varphi(t_{12}) + \varphi(u_{21}) + \varphi(v_{22}), -e_2, \ldots, -e_2) + p_n(r, \varphi(w_{21}), -e_2, \ldots, -e_2)\\ &\quad +p_n(r, w_{21}, \varphi(-e_2), \ldots, -e_2) + \cdots + p_n(r, w_{21}, -e_2, \ldots, \varphi(-e_2)).\end{aligned}$$

Combining the above two expressions for $\varphi(p_n(r, w_{21}, -e_2, \ldots, -e_2))$, we obtain $p_n(z, w_{21}, -e_2, \ldots, -e_2) = 0$, which leads to $[z, w_{21}] = 0$. Thus we have that $[z, w_{12}] = 0$ and $[z, w_{21}] = 0$. Using Remark 2.1 we conclude that $z = z_{11} + z_{22} \in \mathcal{Z}(\mathfrak{T})$, that is,

$$\varphi(s_{11} + t_{12} + u_{21} + v_{22}) - \varphi(s_{11}) - \varphi(t_{12}) - \varphi(u_{21}) - \varphi(w_{22}) \in \mathcal{Z}(\mathfrak{T}). \quad \square$$

Proof. [Proof of Theorem 3.1] Consider any two arbitrary elements $s = s_{11}+s_{12}+s_{21}+s_{22}$ and $t = t_{11}+t_{12}+t_{21}+t_{22}$ in $\mathfrak{T}$. Using Lemmas 3.4, 3.6 and 3.7, we see that

$$\begin{aligned}\varphi(s + t) &= \varphi((s_{11} + t_{11}) + (s_{12} + t_{12}) + (s_{21} + t_{21}) + (s_{22} + t_{22}))\\ &= \varphi(s_{11} + t_{11}) + \varphi(s_{12} + t_{12}) + \varphi(s_{21} + t_{21}) + \varphi(s_{22} + t_{22}) + z_1\\ &= \varphi(s_{11}) + \varphi(t_{11}) + z_2 + \varphi(s_{12}) + \varphi(t_{12}) + \varphi(s_{21}) + \varphi(t_{21})\\ &\quad +\varphi(s_{22}) + \varphi(t_{22}) + z_3 + z_1\\ &= \varphi(s_{11} + s_{12} + s_{21} + s_{22}) + z_4 + z_2 + \varphi(t_{11} + t_{12} + t_{21} + t_{22}) + z_5 + z_3 + z_1\\ &= \varphi(s) + \varphi(t) + z_1 + z_2 + z_3 + z_4 + z_5,\end{aligned}$$

for some $z_i \in \mathcal{Z}(\mathfrak{T})$ $(i = 1, \ldots, 5)$. Thus, $\varphi(s + t) - \varphi(s) - \varphi(t) \in \mathcal{Z}(\mathfrak{T})$, that is, φ is almost additive. The proof of Theorem 3.1 is complete. $\square$

4. Characterization of non-global nonlinear Lie n-derivations

In the present section, we consider the question of characterizing non-global nonlinear Lie n-derivation on unital algebra with a nontrivial idempotent and obtain the following result.

Theorem 4.1. *Let $\mathfrak{T}$ be a $(n-1)$-torsion free unital algebra with a nontrivial idempotent e_1 satisfying (♠). Let us assume that*

(*i*) $\mathcal{Z}(e_1\mathfrak{T}e_1) = \mathcal{Z}(\mathfrak{T})e_1$ *and* $\mathcal{Z}(e_2\mathfrak{T}e_2) = \mathcal{Z}(\mathfrak{T})e_2$;

(*ii*) *either* $\mathfrak{T}_{11}$ *or* $\mathfrak{T}_{22}$ *does not contain nonzero central ideals;*

(*iii*) *either* $\mathfrak{T}_{11}$ *or* $\mathfrak{T}_{22}$ *satisfies the condition* (1);

(*iv*) *for each* $u_{ij} \in \mathfrak{T}_{ij}$, *the condition* $u_{ij}\mathfrak{T}_{ji} = \{0\} = \mathfrak{T}_{ji}u_{ij}$ *implies* $u_{ij} = 0$ $(1 \leq i \neq j \leq 2)$.

If a map $\varphi : \mathfrak{T} \to \mathfrak{T}$ *satisfies* $\varphi(p_n(s_1, s_2, \ldots, s_n)) = \sum_{i=1}^{n} p_n(s_1, \ldots, s_{i-1}, \varphi(s_i), s_{i+1}, \ldots, s_n)$ $(n \geq 3)$ *for all* $s_1, s_2, \ldots, s_n \in \mathfrak{T}$ *with* $s_1s_2\cdots s_n = 0$, *then* $\varphi = \delta + \tau$, *where* $\delta : \mathfrak{T} \to \mathfrak{T}$ *is an additive derivation and* $\tau : \mathfrak{T} \to \mathcal{Z}(\mathfrak{T})$ *is a map vanishing at* $p_n(s_1, s_2, \ldots, s_n)$ *for all* $s_1, s_2, \ldots, s_n \in \mathfrak{T}$ *with* $s_1s_2\cdots s_n = 0$.

The proof of the above theorem can be achieved via a series of the following lemmas.

Lemma 4.2. $e_1\varphi(-e_i)e_1 + e_2\varphi(-e_i)e_2 \in \mathcal{Z}(\mathfrak{T})$ $\quad (i = 1, 2)$.

Proof. Since $s_{12}(-e_1)\cdots(-e_1) = 0$ for any $s_{12} \in \mathfrak{T}_{12}$, we have

$$\begin{aligned}
\varphi(s_{12}) &= \varphi(p_n(s_{12}, -e_1, \ldots, -e_1)) \\
&= p_n(\varphi(s_{12}), -e_1, \ldots, -e_1) + p_n(s_{12}, \varphi(-e_1), \ldots, -e_1) \\
&\quad + \cdots + p_n(s_{12}, -e_1, \ldots, \varphi(-e_1)) \\
&= e_1\varphi(s_{12})e_2 + (-1)^{n+1}e_2\varphi(s_{12})e_1 + (n-1)\{s_{12}\varphi(-e_1)e_2 - e_1\varphi(-e_1)s_{12}\} \\
&\quad + s_{12}\varphi(-e_1)e_1 - e_2\varphi(-e_1)s_{12}.
\end{aligned}$$

Multiplying by e_1 on the left and by e_2 on the right and using $(n-1)$-torsion freeness of $\mathfrak{T}$, we get

$$e_1\varphi(-e_1)e_1s_{12} = s_{12}e_2\varphi(-e_1)e_2. \tag{2}$$

Since $s_{21}(-e_2)\cdots(-e_2) = 0$ for any $s_{21} \in \mathfrak{T}_{21}$, we obtain

$$\begin{aligned}
\varphi(s_{21}) &= \varphi(p_n(s_{21}, -e_2, \ldots, -e_2)) \\
&= p_n(\varphi(s_{21}), -e_2, \ldots, -e_2) + p_n(s_{21}, \varphi(-e_2), \ldots, -e_2) \\
&\quad + \cdots + p_n(s_{21}, -e_2, \ldots, \varphi(-e_2)) \\
&= e_2\varphi(s_{21})e_1 + (-1)^{n+1}e_1\varphi(s_{21})e_2 + (n-1)\{s_{21}\varphi(-e_2)e_1 - e_2\varphi(-e_2)s_{21}\} \\
&\quad + s_{21}\varphi(-e_2)e_2 - e_1\varphi(-e_2)s_{21}.
\end{aligned}$$

Multiplying by e_2 on the left and by e_1 on the right and using $(n-1)$-torsion freeness of $\mathfrak{T}$, we obtain

$$s_{21}e_1\varphi(-e_2)e_1 = e_2\varphi(-e_2)e_2s_{21}. \tag{3}$$

Also, $(-e_1)s_{21}(-e_2)\cdots(-e_2) = 0$ for any $s_{21} \in \mathfrak{T}_{21}$, we have

$$\begin{aligned}
\varphi(s_{21}) &= \varphi(p_n(-e_1, s_{21}, -e_2, \ldots, -e_2)) \\
&= p_n(\varphi(-e_1), s_{21}, -e_2, \ldots, -e_2) + p_n(-e_1, \varphi(s_{21}), -e_2, \ldots, -e_2) \\
&\quad + p_n(-e_1, s_{21}, \varphi(-e_2), \ldots, -e_2) + \cdots + p_n(-e_1, s_{21}, -e_2, \ldots, \varphi(-e_2)) \\
&= e_2\varphi(-e_1)s_{21} - s_{21}\varphi(-e_1)e_1 + e_2\varphi(s_{21})e_1 + (-1)^{n+1}e_1\varphi(s_{21})e_2) \\
&\quad + (n-1)\{s_{21}\varphi(-e_2)e_1 - e_2\varphi(-e_2)s_{21}\} + s_{21}\varphi(-e_2)e_2 - e_1\varphi(-e_2)s_{21}.
\end{aligned}$$

Left multiplication by e_2 and right multiplication by e_1 yields

$$e_2\varphi(-e_1)s_{21} - s_{21}\varphi(-e_1)e_1 + (n-1)\{s_{21}\varphi(-e_2)e_1 - e_2\varphi(-e_2)s_{21}\} = 0.$$

Using (3), we have

$$s_{21}e_1\varphi(-e_1)e_1 = e_2\varphi(-e_1)e_2s_{21}. \tag{4}$$

Using the relations (2) and (4) together with Remark 2.1, we find that

$$e_1\varphi(-e_1)e_1 + e_2\varphi(-e_1)e_2 \in \mathcal{Z}(\mathfrak{T}).$$

Again, observe that $(-e_2)s_{12}(-e_1)\cdots(-e_1) = 0$ for any $s_{12} \in \mathfrak{T}_{12}$. Thus we obtain

$$\begin{aligned}
\varphi(s_{12}) &= \varphi(p_n(-e_2, s_{12}, -e_1, \ldots, -e_1)) \\
&= p_n(\varphi(-e_2), s_{12}, -e_1, \ldots, -e_1) + p_n(-e_2, \varphi(s_{12}), -e_1, \ldots, -e_1) \\
&\quad + p_n(-e_2, s_{12}, \varphi(-e_1), \ldots, -e_1) + \cdots + p_n(-e_2, s_{12}, -e_1, \ldots, \varphi(-e_1)) \\
&= e_1\varphi(-e_2)s_{12} - s_{12}\varphi(-e_2)e_2 + e_1\varphi(s_{12})e_2 + (-1)^{n+1}e_2\varphi(s_{12})e_1 \\
&\quad + (n-1)\{s_{12}\varphi(-e_1)e_2 - e_1\varphi(-e_1)s_{12}\} + s_{12}\varphi(-e_1)e_1 - e_2\varphi(-e_1)s_{12}.
\end{aligned}$$

Multiplying by e_1 on the left and by e_2 on the right and using (2), we get

$$e_1\varphi(-e_2)e_1s_{12} = s_{12}e_2\varphi(-e_2)e_2. \tag{5}$$

Using the relations (3) and (5) together with Remark 2.1, we obtain

$$e_1\varphi(-e_2)e_1 + e_2\varphi(-e_2)e_2 \in \mathcal{Z}(\mathfrak{T}). \quad \square$$

In view of Lemma 4.2, we define a map $\rho(s) = \varphi(s) + [s, e_1\varphi(-e_1)e_2 - e_2\varphi(-e_1)e_1]$. Then ρ is a nonlinear mapping satisfying $\rho(p_n(s_1, s_2, \ldots, s_n)) = \sum_{i=1}^{n} p_n(s_1, \ldots, s_{i-1}, \rho(s_i), s_{i+1}, \ldots, s_n)$ $(n \geq 3)$ for all $s_1, s_2, \ldots, s_n \in \mathfrak{T}$ with $s_1s_2\cdots s_n = 0$, and $\rho(-e_1) \in \mathcal{Z}(\mathfrak{T})$.

Lemma 4.3. $\rho(-e_2) \in \mathcal{Z}(\mathfrak{T})$.

Proof. Since $(-e_2)(-e_1)\cdots(-e_1) = 0$ and $\rho(-e_1) \in \mathcal{Z}(\mathfrak{T})$, we have $0 = \rho(p_n(-e_2, -e_1, \ldots, -e_1)) = p_n(\rho(-e_2), -e_1, \ldots, -e_1) = e_1\rho(-e_2)e_2 + (-1)^{n+1}e_2\rho(-e_2)e_1$. Using the definition of ρ, we have

$$e_1\rho(-e_2)e_1 = e_1\varphi(-e_2)e_1 \text{ and } e_2\rho(-e_2)e_2 = e_2\varphi(-e_2)e_2.$$

By invoking Lemma 4.2, we find that

$$\rho(-e_2) = e_1\rho(-e_2)e_1 + e_2\rho(-e_2)e_2 = e_1\varphi(-e_2)e_1 + e_2\varphi(-e_2)e_2 \in \mathcal{Z}(\mathfrak{T}). \quad \square$$

Lemma 4.4. $\rho(\mathfrak{T}_{ij}) \subseteq \mathfrak{T}_{ij}$ $\quad (1 \leq i \neq j \leq 2)$.

Proof. We only give the proof for $\mathfrak{T}_{12}$, the proof of the other case is similar.
Since $s_{12}(-e_1)\cdots(-e_1) = 0$ for any $s_{12} \in \mathfrak{T}_{12}$ and $\rho(-e_1) \in \mathcal{Z}(\mathfrak{T})$. For any $s_{12} \in \mathfrak{T}_{12}$, we obtain

$$\begin{aligned}
\rho(s_{12}) &= \rho(p_n(s_{12}, -e_1, \ldots, -e_1)) \\
&= p_n(\rho(s_{12}), -e_1, \ldots, -e_1) \\
&= e_1\rho(s_{12})e_2 + (-1)^{n+1}e_2\rho(s_{12})e_1.
\end{aligned} \tag{6}$$

We show that $e_2\rho(s_{12})e_1 = 0$. If n is even then $\rho(s_{12}) = e_1\rho(s_{12})e_2 - e_2\rho(s_{12})e_1$. Multiplying by e_2 on the left and by e_1 on the right, we obtain $2e_2\rho(s_{12})e_1 = 0$. Thus $e_2\rho(s_{12})e_1 = 0$ and hence the $\rho(s_{12}) \in \mathfrak{T}_{12}$ for all $s_{12} \in \mathfrak{T}_{12}$. Next, we assume that n is odd.

Observe that $s_{12}t_{12}u_{12}(-e_1)\cdots(-e_1) = s_{12}t_{12}u_{21}(-e_2)\cdots(-e_2) = 0$ for any $t_{12}, u_{12} \in \mathfrak{T}_{12}$ and $u_{21} \in \mathfrak{T}_{21}$. Using the fact that $\rho(0) = 0$, we arrive at

$$\begin{aligned} 0 &= \rho(p_n(s_{12}, t_{12}, u_{12}, -e_1, \ldots, -e_1)) \\ &= p_n(\rho(s_{12}), t_{12}, u_{12}, -e_1, \ldots, -e_1) + p_n(s_{12}, \rho(t_{12}), u_{12}, -e_1, \ldots, -e_1) \\ &= p_{n-1}([\rho(s_{12}), t_{12}] + [s_{12}, \rho(t_{12})], u_{12}, -e_1, \ldots, -e_1) \\ &= p_{n-1}(v, u_{12}, -e_1, \ldots, -e_1) \\ &= p_{n-2}([v, u_{12}], -e_1, \ldots, -e_1) \\ &= [v, u_{12}] \end{aligned} \tag{7}$$

and

$$\begin{aligned} 0 &= \rho(p_n(s_{12}, t_{12}, u_{21}, -e_2, \ldots, -e_2)) \\ &= p_n(\rho(s_{12}), t_{12}, u_{21}, -e_2, \ldots, -e_2) + p_n(s_{12}, \rho(t_{12}), u_{21}, -e_2, \ldots, -e_2) \\ &= p_{n-1}([\rho(s_{12}), t_{12}] + [s_{12}, \rho(t_{12})], u_{21}, -e_2, \ldots, -e_2) \\ &= p_{n-1}(v, u_{21}, -e_2, \ldots, -e_2) \\ &= p_{n-2}([v, u_{21}], -e_2, \ldots, -e_2) \\ &= [v, u_{21}], \end{aligned} \tag{8}$$

where $v = [\rho(s_{12}), t_{12}] + [s_{12}, \rho(t_{12})] \in \mathfrak{T}_{11} + \mathfrak{T}_{22}$. Thus (7) and (8) together with Remark 2.1 gives $v_{11} + v_{22} \in \mathcal{Z}(\mathfrak{T})$. Therefore, $v = v_{11} + v_{22} \in \mathcal{Z}(\mathfrak{T})$.
Let $[\rho(s_{12}), t_{12}] + [s_{12}, \rho(t_{12})] = z \in \mathcal{Z}(\mathfrak{T})$ for some $z \in \mathcal{Z}(\mathfrak{T})$. Using (4.5), we find that

$$\begin{aligned} [e_2\rho(s_{12})e_1, t_{12}] &= [\rho(s_{12}), t_{12}] \\ &= z - [s_{12}, \rho(t_{12})] \\ &= z + p_n(-e_1, s_{12}, -e_1, \ldots, -e_1, \rho(t_{12})) \\ &= z + \rho(p_n(-e_1, s_{12}, -e_1, \ldots, -e_1, t_{12})) - p_n(-e_1, \rho(s_{12}), -e_1, \ldots, -e_1, t_{12}) \\ &\quad - p_n(-e_1, s_{12}, -e_1, \ldots, -e_1, \rho(t_{12})) + p_n(-e_1, s_{12}, -e_1, \ldots, -e_1, \rho(t_{12})) \\ &= z - p_n(-e_1, \rho(s_{12}), -e_1, \ldots, -e_1, t_{12}) \\ &= z - p_{n-1}(-e_1\rho(s_{12}) + \rho(s_{12})e_1, -e_1, \ldots, -e_1, t_{12}) \\ &= z - p_{n-1}(e_2\rho(s_{12})e_1, -e_1, \ldots, -e_1, t_{12}) \\ &= z - [p_{n-2}(e_2\rho(s_{12})e_1, -e_1, \ldots, -e_1), t_{12}] \\ &= z - [(-1)^{n-1}e_2\rho(s_{12})e_1, t_{12}]. \end{aligned}$$

Since n is odd, so $n-1$ is even and hence $[e_2\rho(s_{12})e_1, t_{12}] = z - [e_2\rho(s_{12})e_1, t_{12}]$. Therefore, $2[e_2\rho(s_{12})e_1, t_{12}] = z \in \mathcal{Z}(\mathfrak{T})$, that is, $[e_2\rho(s_{12})e_1, t_{12}] \in \mathcal{Z}(\mathfrak{T})$ for all $t_{12} \in \mathfrak{T}_{12}$. Consequently, $\mathfrak{T}_{12}e_2\rho(s_{12})e_1 \in \mathcal{Z}(\mathfrak{T}_{11})$ and $e_2\rho(s_{12})e_1\mathfrak{T}_{12} \in \mathcal{Z}(\mathfrak{T}_{22})$. Without loss of generality, we assume that $\mathfrak{T}_{11}$ does not contain nonzero central ideal. Since $\mathfrak{T}_{12}e_2\rho(s_{12})e_1$ is a central ideal of $\mathfrak{T}_{11}$, we have $\mathfrak{T}_{12}e_2\rho(s_{12})e_1 = \{0\}$. Therefore, $e_2\rho(s_{12})e_1\mathfrak{T}_{12} = \{0\}$. That is, $e_2\rho(s_{12})e_1\mathfrak{T}_{12} = \mathfrak{T}_{12}e_2\rho(s_{12})e_1 = \{0\}$. Making use of condition (*iv*) of Theorem 4.1, we get $e_2\rho(s_{12})e_1 = 0$. Therefore, $\rho(s_{12}) = e_1\rho(s_{12})e_2 \in \mathfrak{T}_{12}$ for all $s_{12} \in \mathfrak{T}_{12}$ and hence $\rho(\mathfrak{T}_{12}) \subseteq \mathfrak{T}_{12}$. □

Lemma 4.5. *$\rho(\mathfrak{T}_{ii}) \subseteq \mathfrak{T}_{11} + \mathfrak{T}_{22}$ and there exists a map $f_i : \mathfrak{T}_{ii} \to \mathcal{Z}(\mathfrak{T})$ such that $\rho(s_{ii}) - f_i(\mathfrak{T}_{ii}) \in \mathfrak{T}_{ii}$, $i = 1, 2$.*

Proof. Let $s_{ii} \in \mathfrak{T}_{ii}$ $(i = 1, 2)$. Since $s_{11}(-e_2)\cdots(-e_2) = 0$, using Lemma 4.3, we obtain

$$0 = \rho(p_n(s_{11}, -e_2, \ldots, -e_2)) = p_n(\rho(s_{11}), -e_2, \ldots, -e_2) = e_2\rho(s_{11})e_1 + (-1)^{n+1}e_1\rho(s_{11})e_2.$$

That is, $\rho(\mathfrak{T}_{11}) \subseteq \mathfrak{T}_{11} + \mathfrak{T}_{22}$. Symmetrically, $\rho(\mathfrak{T}_{22}) \subseteq \mathfrak{T}_{11} + \mathfrak{T}_{22}$.
Observe that $s_{11}s_{22}u_{12}(-e_1)\cdots(-e_1) = s_{11}s_{22}u_{21}(-e_2)\cdots(-e_2) = 0$ for any $u_{12} \in \mathfrak{T}_{12}$ and $u_{21} \in \mathfrak{T}_{21}$. Thus, we

obtain

$$\begin{aligned} 0 &= \rho(p_n(s_{11}, s_{22}, u_{12}, -e_1, \ldots, -e_1)) \\ &= p_n(\rho(s_{11}), s_{22}, u_{12}, -e_1, \ldots, -e_1) + p_n(s_{11}, \rho(s_{22}), u_{12}, -e_1, \ldots, -e_1) \\ &= p_{n-1}([\rho(s_{11}), s_{22}] + [s_{11}, \rho(s_{22})], u_{12}, -e_1, \ldots, -e_1) \\ &= p_{n-1}(v, u_{12}, -e_1, \ldots, -e_1) \\ &= p_{n-2}([v, u_{12}], -e_1, \ldots, -e_1) \\ &= [v, u_{12}] \end{aligned} \tag{9}$$

and

$$\begin{aligned} 0 &= \rho(p_n(s_{11}, s_{22}, u_{21}, -e_2, \ldots, -e_2)) \\ &= p_n(\rho(s_{11}), s_{22}, u_{21}, -e_2, \ldots, -e_2) + p_n(s_{11}, \rho(s_{22}), u_{21}, -e_2, \ldots, -e_2) \\ &= p_n(\rho(s_{11}), s_{22}, u_{21}, -e_2, \ldots, -e_2) + p_n(s_{11}, \rho(s_{22}), u_{21}, -e_2, \ldots, -e_2) \\ &= p_{n-1}([\rho(s_{11}), s_{22}] + [s_{11}, \rho(s_{22})], u_{21}, -e_2, \ldots, -e_2) \\ &= p_{n-1}(v, u_{21}, -e_2, \ldots, -e_2) \\ &= p_{n-2}([v, u_{21}], -e_2, \ldots, -e_2) \\ &= [v, u_{21}], \end{aligned} \tag{10}$$

where $v = [\rho(s_{11}), s_{22}] + [s_{11}, \rho(s_{22})] \in \mathfrak{T}_{11} + \mathfrak{T}_{22}$. Thus (9) and (10) together with Remark 2.1 implies that $v = [\rho(s_{11}), s_{22}] + [s_{11}, \rho(s_{22})] \in \mathcal{Z}(\mathfrak{T})$. Multiplication by e_2 on both sides yields $[e_2\rho(s_{11})e_2, s_{22}] \in \mathcal{Z}(\mathfrak{T})e_2 = \mathcal{Z}(\mathfrak{T}_{22})$. Without loss of generality, we assume that $\mathfrak{T}_{22}$ satisfies (1). Invoking condition (*iii*) of Theorem 4.1, we obtain that $[e_2\rho(s_{11})e_2, s_{22}] = 0$. That is, $e_2\rho(s_{11})e_2 \in \mathcal{Z}(\mathfrak{T}_{22}) = \mathcal{Z}(\mathfrak{T})e_2$. Let $e_2\rho(s_{11})e_2 = ze_2$ for some $z \in \mathcal{Z}(\mathfrak{T})$. Define $f_1 : \mathfrak{T}_{11} \to \mathcal{Z}(\mathfrak{T})$ such that $f_1(s_{11})e_2 = e_2\rho(s_{11})e_2$. Thus,
$\rho(s_{11}) = e_1\rho(s_{11})e_1 + e_2\rho(s_{11})e_2 = e_1\rho(s_{11})e_1 + f_1(s_{11})e_2 = e_1\rho(s_{11})e_1 + f_1(s_{11}) - f_1(s_{11})e_1$.
That is, $\rho(s_{11}) - f_1(s_{11}) \in \mathfrak{T}_{11}$. Symmetrically, there exists $f_2 : \mathfrak{T}_{22} \to \mathcal{Z}(\mathfrak{T})$ such that $\rho(s_{22}) - f_2(s_{22}) \in \mathfrak{T}_{22}$. □

Let us now define mappings $\partial : \mathfrak{T} \to \mathfrak{T}$ and $\tau : \mathfrak{T} \to \mathcal{Z}(\mathfrak{T})$ by $\partial(s) = \rho(s_{11}) + \rho(s_{12}) + \rho(s_{21}) + \rho(s_{22}) - (f_1(s_{11}) + f_2(s_{22}))$ and $\tau(s) = \rho(s) - \partial(s)$, respectively. Then, it is easy to verify the following.

Lemma 4.6. *Let* $s_{ij} \in \mathfrak{T}_{ij} \quad (1 \leq i, j \leq 2)$.

(*i*) $\partial(\mathfrak{T}_{ij}) \subseteq \mathfrak{T}_{ij}$,

(*ii*) $\partial(s_{ij}) = \rho(s_{ij}) \quad (i \neq j)$,

(*iii*) $\partial(s_{11} + s_{12} + s_{21} + s_{22}) = \partial(s_{11}) + \partial(s_{12}) + \partial(s_{21}) + \partial(s_{22})$,

(*iv*) ∂ *is additive on* $\mathfrak{T}_{ij}$ $(i \neq j)$.

Lemma 4.7. *For any* $s_{ii} \in \mathfrak{T}_{ii}$, $t_{ij} \in \mathfrak{T}_{ij}$ $(1 \leq i \neq j \leq 2)$, *we have*

$$\partial(s_{ii}t_{ij}) = \partial(s_{ii})t_{ij} + s_{ii}\partial(t_{ij}) \quad \text{and} \quad \partial(t_{ij}s_{jj}) = \partial(t_{ij})s_{jj} + t_{ij}\partial(s_{jj}).$$

Proof. Let $s_{11} \in \mathfrak{T}_{11}, t_{12} \in \mathfrak{T}_{12}$. Since $s_{11}t_{12}(-e_1)\cdots(-e_1) = 0$, using Lemma 4.6 and the fact that $\rho(-e_1) \in \mathcal{Z}(\mathfrak{T})$, we find that

$$\begin{aligned} \partial(s_{11}t_{12}) &= \rho(s_{11}t_{12}) \\ &= \rho(p_n(s_{11}, t_{12}, -e_1, \ldots, -e_1)) \\ &= p_n(\rho(s_{11}), t_{12}, -e_1, \ldots, -e_1) + p_n(s_{11}, \rho(t_{12}), -e_1, \ldots, -e_1) \\ &= p_n(\partial(s_{11}), t_{12}, -e_1, \ldots, -e_1) + p_n(s_{11}, \partial(t_{12}), -e_1, \ldots, -e_1) \\ &= p_{n-1}([\partial(s_{11}), t_{12}], -e_1, \ldots, -e_1) + p_{n-1}([s_{11}, \partial(t_{12})], -e_1, \ldots, -e_1) \\ &= [\partial(s_{11}), t_{12}] + [s_{11}, \partial(t_{12})] \\ &= \partial(s_{11})t_{12} + s_{11}\partial(t_{12}). \end{aligned}$$

Also, since $s_{22}t_{21}(-e_2)\cdots(-e_2)=0$, using Lemmas 4.3 and 4.6, we get

$$\begin{aligned}
\partial(s_{22}t_{21}) &= \rho(s_{22}t_{21}) \\
&= \rho(p_n(s_{22},t_{21},-e_2,\dots,-e_2)) \\
&= p_n(\rho(s_{22}),t_{21},-e_2,\dots,-e_2)+p_n(s_{22},\rho(t_{21}),-e_2,\dots,-e_2) \\
&= p_n(\partial(s_{22}),t_{21},-e_2,\dots,-e_2)+p_n(s_{22},\partial(t_{21}),-e_2,\dots,-e_2) \\
&= p_{n-1}([\partial(s_{22}),t_{21}],-e_2,\dots,-e_2)+p_{n-1}([s_{22},\partial(t_{21})],-e_2,\dots,-e_2) \\
&= [\partial(s_{22}),t_{21}]+[s_{22},\partial(t_{21})] \\
&= \partial(s_{22})t_{21}+s_{22}\partial(t_{21}).
\end{aligned}$$

Symmetrically, the other cases can be proved. □

Lemma 4.8. *Let $s_{ii},t_{ii}\in\mathfrak{T}_{ii}$ $(i=1,2)$. Then $\partial(s_{ii}t_{ii})=\partial(s_{ii})t_{ii}+s_{ii}\partial(t_{ii})$.*

Proof. Let $s_{11},t_{11}\in\mathfrak{T}_{11}$ and $u_{12}\in\mathfrak{T}_{12}$. Using Lemma 4.7, we obtain

$$\partial(s_{11}t_{11}u_{12})=\partial(s_{11}t_{11})u_{12}+s_{11}t_{11}\partial(u_{12}).$$

On the other hand,

$$\begin{aligned}
\partial(s_{11}t_{11}u_{12}) &= \partial(s_{11})t_{11}u_{12}+s_{11}\partial(t_{11}u_{12}) \\
&= \partial(s_{11})t_{11}u_{12}+s_{11}\partial(t_{11})u_{12}+s_{11}t_{11}\partial(u_{12}).
\end{aligned}$$

Combining the above two expressions for $\partial(s_{11}t_{11}u_{12})$, we have

$$(\partial(s_{11}t_{11})-\partial(s_{11})t_{11}-s_{11}\partial(t_{11}))u_{12}=0.$$

Analogously, we obtain for any $u_{21}\in\mathfrak{T}_{21}$ that

$$u_{21}(\partial(s_{11}t_{11})-\partial(s_{11})t_{11}-s_{11}\partial(t_{11}))=0.$$

Thus for any $u_{12}\in\mathfrak{T}_{12}, u_{21}\in\mathfrak{T}_{21}$, we have

$$(\partial(s_{11}t_{11})-\partial(s_{11})t_{11}-s_{11}\partial(t_{11}))u_{12}=0=u_{21}(\partial(s_{11}t_{11})-\partial(s_{11})t_{11}-s_{11}\partial(t_{11})).$$

Application of the condition (♠) implies that $\partial(s_{11}t_{11})=\partial(s_{11})t_{11}+s_{11}\partial(t_{11})$.
Symmetrically, we can obtain the other case. □

Lemma 4.9. *∂ is additive on $\mathfrak{T}_{ii}$ $(i=1,2)$.*

Proof. Let $s_{11},t_{11}\in\mathfrak{T}_{11}$ and $u_{12}\in\mathfrak{T}_{12}$. Using Lemmas 4.6 and 4.7, we obtain

$$\partial((s_{11}+t_{11})u_{12})=\partial(s_{11}+t_{11})u_{12}+(s_{11}+t_{11})\partial(u_{12}).$$

On the other hand,

$$\begin{aligned}
\partial((s_{11}+t_{11})u_{12}) &= \partial(s_{11}u_{12})+\partial(t_{11}u_{12}) \\
&= \partial(s_{11})u_{12}+s_{11}\partial(u_{12})+\partial(t_{11})u_{12}+t_{11}\partial(u_{12}) \\
&= (\partial(s_{11})+\partial(t_{11}))u_{12}+(s_{11}+t_{11})\partial(u_{12}).
\end{aligned}$$

Combining the above two equations, we get

$$(\partial(s_{11}+t_{11})-\partial(s_{11})-\partial(t_{11}))u_{12}=0 \quad \text{for all} \quad u_{12}\in\mathfrak{T}_{12}.$$

Analogously, we obtain that

$$u_{21}(\partial(s_{11}+t_{11})-\partial(s_{11})-\partial(t_{11}))=0 \quad \text{for all} \quad u_{21}\in\mathfrak{T}_{12}.$$

Thus for any $u_{12}\in\mathfrak{T}_{12}$ and $u_{21}\in\mathfrak{T}_{21}$, we have

$$(\partial(s_{11}+t_{11})-\partial(s_{11})-\partial(t_{11}))u_{12}=0=u_{21}(\partial(s_{11}+t_{11})-\partial(s_{11})-\partial(t_{11})).$$

Applying the condition (♠), we conclude that $\partial(s_{11}+t_{11})=\partial(s_{11})+\partial(t_{11})$.
Symmetrically, one can prove that $\partial(s_{22}+t_{22})=\partial(s_{22})+\partial(t_{22})$. □

The following lemma can be obtained immediately using Lemmas 4.6 and 4.9.

Lemma 4.10. *∂ is additive on $\mathfrak{T}$.*

Lemma 4.11. *For any $s_{ij}\in\mathfrak{T}_{ij}$, $t_{ji}\in\mathfrak{T}_{ji}$ $(1\le i\ne j\le 2)$, we have*

$$\partial(s_{ij}t_{ji})=\partial(s_{ij})t_{ji}+s_{ij}\partial(t_{ji}).$$

Proof. Since $t_{21}s_{12}u_{12}(-e_1)\cdots(-e_1)=0$, using Lemmas 4.6, 4.7 and 4.10, we have

$$\begin{aligned}
&\rho(p_n(t_{21},s_{12},u_{12},-e_1,\ldots,-e_1))\\
&\quad= p_n(\rho(t_{21}),s_{12},u_{12},-e_1,\ldots,-e_1)+p_n(t_{21},\rho(s_{12}),u_{12},-e_1,\ldots,-e_1)\\
&\qquad+p_n(t_{21},s_{12},\rho(u_{12}),-e_1,\ldots,-e_1)\\
&\quad= p_n(\partial(t_{21}),s_{12},u_{12},-e_1,\ldots,-e_1)+p_n(t_{21},\partial(s_{12}),u_{12},-e_1,\ldots,-e_1)\\
&\qquad+p_n(t_{21},s_{12},\partial(u_{12}),-e_1,\ldots,-e_1)\\
&\quad= p_{n-2}([[\partial(t_{21}),s_{12}],u_{12}],-e_1,\ldots,-e_1)+p_{n-2}([[t_{21},\partial(s_{12})],u_{12}],-e_1,\ldots,-e_1)\\
&\qquad+p_{n-2}([[t_{21},s_{12}],\partial(u_{12})],-e_1,\ldots,-e_1)\\
&\quad= -s_{12}\partial(t_{21})u_{12}-u_{12}\partial(t_{21})s_{12}-\partial(s_{12})t_{21}u_{12}-u_{12}t_{21}\partial(s_{12})-s_{12}t_{21}\partial(u_{12})\\
&\qquad-\partial(u_{12})t_{21}s_{12}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\rho(p_n(t_{21},s_{12},u_{12},-e_1,\ldots,-e_1))\\
&\quad= \rho(-s_{12}t_{21}u_{12}-u_{12}t_{21}s_{12})\\
&\quad= \rho(-s_{12}t_{21}u_{12})+\rho(-u_{12}t_{21}s_{12})\\
&\quad= -\partial(s_{12}t_{21}u_{12})-\partial(u_{12}t_{21}s_{12})\\
&\quad= -\partial(s_{12}t_{21})u_{12}-s_{12}t_{21}\partial(u_{12})-\partial(u_{12})t_{21}s_{12}-u_{12}\partial(t_{21}s_{12}).
\end{aligned}$$

The above two expressions yield

$$(\partial(s_{12}t_{21})-\partial(s_{12})t_{21}-s_{12}\partial(t_{21}))u_{12}=u_{12}(\partial(t_{21}s_{12})-\partial(t_{21})s_{12}-t_{21}\partial(s_{12})),$$

which can be re-written as

$$[(\partial(s_{12}t_{21})-\partial(s_{12})t_{21}-s_{12}\partial(t_{21}))-(\partial(t_{21}s_{12})-\partial(t_{21})s_{12}-t_{21}\partial(s_{12})),u_{12}]=0. \tag{11}$$

Since $s_{12}t_{21}u_{21}(-e_2)\cdots(-e_2)=0$, using Lemmas 4.6, 4.7 and 4.10, we get

$$\begin{aligned}
&\rho(p_n(s_{12},t_{21},u_{21},-e_2,\ldots,-e_2))\\
&\quad= p_n(\rho(s_{12}),t_{21},u_{21},-e_2,\ldots,-e_2)+p_n(s_{12},\rho(t_{21}),u_{21},-e_2,\ldots,-e_2)\\
&\qquad+p_n(s_{12},t_{21},\rho(u_{21}),-e_2,\ldots,-e_2)\\
&\quad= p_n(\partial(s_{12}),t_{21},u_{21},-e_2,\ldots,-e_2)+p_n(s_{12},\partial(t_{21}),u_{21},-e_2,\ldots,-e_2)\\
&\qquad+p_n(s_{12},t_{21},\partial(u_{21}),-e_2,\ldots,-e_2)\\
&\quad= p_{n-2}([[\partial(s_{12}),t_{21}],u_{21}],-e_2,\ldots,-e_2)+p_{n-2}([[s_{12},\partial(t_{21})],u_{21}],-e_2,\ldots,-e_2)\\
&\qquad+p_{n-2}([[s_{12},t_{21}],\partial(u_{21})],-e_2,\ldots,-e_2)\\
&\quad= -t_{21}\partial(s_{12})u_{21}-u_{21}\partial(s_{12})t_{21}-\partial(t_{21})s_{12}u_{21}-u_{21}s_{12}\partial(t_{21})-t_{21}s_{12}\partial(u_{21})\\
&\qquad-\partial(u_{21})s_{12}t_{21}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\rho(p_n(s_{12},t_{21},u_{21},-e_2,\ldots,-e_2))\\
&\quad= \rho(-t_{21}s_{12}u_{21}-u_{21}s_{12}t_{21})\\
&\quad= \rho(-t_{21}s_{12}u_{21})+\rho(-u_{21}s_{12}t_{21})\\
&\quad= -\partial(t_{21}s_{12}u_{21})-\partial(u_{21}s_{12}t_{21})\\
&\quad= -\partial(t_{21}s_{12})u_{21}-t_{21}s_{12}\partial(u_{21})-\partial(u_{21})s_{12}t_{21}-u_{21}\partial(s_{12}t_{21}).
\end{aligned}$$

The above two equations yield

$$u_{21}(\partial(s_{12}t_{21})-\partial(s_{12})t_{21}-s_{12}\partial(t_{21}))=(\partial(t_{21})s_{12}+t_{21}\partial(s_{12})-\partial(t_{21}s_{12}))u_{21},$$

which can be re-written as

$$[(\partial(s_{12}t_{21}) - \partial(s_{12})t_{21} - s_{12}\partial(t_{21})) - (\partial(t_{21}s_{12}) - \partial(t_{21})s_{12} - t_{21}\partial(s_{12})), u_{21}] = 0. \tag{12}$$

Thus (11) and (12) together with Remark 2.1 yields

$$(\partial(s_{12}t_{21}) - \partial(s_{12})t_{21} - s_{12}\partial(t_{21})) - (\partial(t_{21}s_{12}) - \partial(t_{21})s_{12} - t_{21}\partial(s_{12})) \in \mathcal{Z}(\mathfrak{T})$$

Assume that

$$\eta(s_{12}, t_{21}) = \partial(s_{12}t_{21}) - \partial(s_{12})t_{21} - s_{12}\partial(t_{21}).$$

Then $\eta(s_{12}, t_{21}) \in \mathcal{Z}(\mathfrak{T}_{11})$. For any $u_{11} \in \mathfrak{T}_{11}$, using Lemma 4.8, we have

$$\begin{aligned}
\eta(u_{11}s_{12}, t_{21}) &= \partial(u_{11}s_{12}t_{21}) - \partial(u_{11}s_{12})t_{21} - u_{11}s_{12}\partial(t_{21}) \\
&= \partial(u_{11})s_{12}t_{21} + u_{11}\partial(s_{12}t_{21}) - \partial(u_{11})s_{12}t_{21} \\
&\quad - u_{11}\partial(s_{12}t_{21}) - u_{11}s_{12}\partial(t_{21}) \\
&= u_{11}(\partial(s_{12}t_{21}) - \partial(s_{12})t_{21} - s_{12}\partial(t_{21})) \\
&= u_{11}\eta(s_{12}, t_{21}).
\end{aligned}$$

That is, $\mathfrak{T}_{11}\eta(s_{12}, t_{21})$ is a central ideal in $\mathfrak{T}_{11}$. Without loss of generality, we assume that $\mathfrak{T}_{11}$ contains no nonzero central ideal. Therefore, $\eta(s_{12}, t_{21}) = 0$ and hence $\partial(s_{12}t_{21}) = \partial(s_{12})t_{21} + s_{12}\partial(t_{21})$.
Similarly, we can prove that $\partial(t_{21}s_{12}) = \partial(t_{21})s_{12} + t_{21}\partial(s_{12})$. □

Lemma 4.12. *∂ is an additive derivation on $\mathfrak{T}$.*

Proof. Using Lemmas 4.7,4.8, 4.10 and 4.11, the proof follows immediately. □

Proof. [Proof of Theorem 4.1] Define a map $\delta : \mathfrak{T} \to \mathfrak{T}$ by $\delta(s) = \partial(s) - [s, e_1\varphi(-e_1)e_2 - e_2\varphi(-e_1)e_1]$. Then, it is easy to check that δ is an additive derivation on $\mathfrak{T}$. Moreover,

$$\begin{aligned}
\varphi(s) &= \rho(s) - [s, e_1\varphi(-e_1)e_2 - e_2\varphi(-e_1)e_1] \\
&= \partial(s) + \tau(s) + [s, e_1\varphi(-e_1)e_2 - e_2\varphi(-e_1)e_1] \\
&= \delta(s) + [s, e_1\varphi(-e_1)e_2 - e_2\varphi(-e_1)e_1] + \tau(s) - [s, e_1\varphi(-e_1)e_2 - e_2\varphi(-e_1)e_1] \\
&= \delta(s) + \tau(s)
\end{aligned}$$

for all $s \in \mathfrak{T}$. Further, for all $s_1, s_2, s_3 \ldots, s_n \in \mathfrak{T}$ with $s_1s_2 \cdots s_n = 0$, we obtain

$$\begin{aligned}
&\tau(p_n(s_1, s_2, s_3, \ldots, s_n)) \\
&= \rho(p_n(s_1, s_2, s_3, \ldots, s_n)) - \partial(p_n(s_1, s_2, s_3, \ldots, s_n)) \\
&= p_n(\rho(s_1), s_2, s_3, \ldots, s_n) + p_n(s_1, \rho(s_2), s_3, \ldots, s_n) + \cdots + p_n(s_1, s_2, s_3, \ldots, \rho(s_n)) \\
&\quad - \partial(p_n(s_1, s_2, s_3, \ldots, s_n)) \\
&= p_n(\partial(s_1) + \tau(s_1), s_2, s_3, \ldots, s_n) + p_n(s_1, \partial(s_2) + \tau(s_2), s_3, \ldots, s_n) \\
&\quad + \cdots + p_n(s_1, s_2, s_3, \ldots, \partial(s_n) + \tau(s_n)) - \partial(p_n(s_1, s_2, s_3, \ldots, s_n)) \\
&= p_n(\partial(s_1), s_2, s_3, \ldots, s_n) + p_n(s_1, \partial(s_2), s_3, \ldots, s_n) + \cdots + p_n(s_1, s_2, s_3, \ldots, \partial(s_n)) \\
&\quad - \partial(p_n(s_1, s_2, s_3, \ldots, s_n)) \\
&= \partial(p_n(s_1, s_2, s_3, \ldots, s_n)) - \partial(p_n(s_1, s_2, s_3, \ldots, s_n)) \\
&= 0.
\end{aligned}$$

□

5. Applications

In this section, we apply Theorems 3.1 and 4.1 to obtain the almost additivity and characterization of non-global nonlinear Lie n-derivations on some classical examples of unital algebras namely triangular algebras (hence, upper triangular algebras, block upper triangular algebras and nest algebras), full matrix algebras and the algebra of all bounded linear operators.

Triangular algebras:

Let $\mathfrak{T}$ be a unital algebra with a nontrivial idempotent e_1 such that $e_1\mathfrak{T}e_2$ is a faithful $(e_1\mathfrak{T}e_1, e_2\mathfrak{T}e_2)$-bimodule and $e_2\mathfrak{T}e_1 = \{0\}$. Then $\mathfrak{T} = e_1\mathfrak{T}e_1 + e_1\mathfrak{T}e_2 + e_2\mathfrak{T}e_2$ is a triangular algebra. Since Lemma 4.11 holds trivially in case of triangular algebra, we can omit the assumption (*ii*) of Theorem 4.1. Also, as $e_2\mathfrak{T}e_1 = \{0\}$, we can delete the condition (*iv*) of Theorem 4.1. Therefore, as a consequence of Theorems 3.1 and 4.1, we obtain the following result:

Corollary 5.1. *Let $\mathfrak{T} = e_1\mathfrak{T}e_1 + e_1\mathfrak{T}e_2 + e_2\mathfrak{T}e_2$ be a $(n-1)$-torsion free triangular algebra. Suppose that*

(i) $\mathcal{Z}(e_1\mathfrak{T}e_1) = \mathcal{Z}(\mathfrak{T})e_1$ and $\mathcal{Z}(e_2\mathfrak{T}e_2) = \mathcal{Z}(\mathfrak{T})e_2$;

(ii) either $e_1\mathfrak{T}e_1$ or $e_2\mathfrak{T}e_2$ satisfies (1).

If a map $\varphi : \mathfrak{T} \to \mathfrak{T}$ satisfies $\varphi(p_n(s_1, s_2, \ldots, s_n)) = \sum_{i=1}^{n} p_n(s_1, \ldots, s_{i-1}, \varphi(s_i), s_{i+1}, \ldots, s_n)$ $(n \geq 3)$ for all $s_1, s_2, \ldots, s_n \in \mathfrak{T}$ with $s_1s_2\cdots s_n = 0$, then $\varphi(s+t) - \varphi(s) - \varphi(t) \in \mathcal{Z}(\mathfrak{T})$ for all $s, t \in \mathfrak{T}$ and $\varphi = \delta + \tau$, where $\delta : \mathfrak{T} \to \mathfrak{T}$ is an additive derivation and $\tau : \mathfrak{T} \to \mathcal{Z}(\mathfrak{T})$ is a map vanishing at $p_n(s_1, s_2, \ldots, s_n)$ for all $s_1, s_2, \ldots, s_n \in \mathfrak{T}$ with $s_1s_2\cdots s_n = 0$.

In particular, for $n = 3$ we have the following result which was obtained by Zhao in [28].

Corollary 5.2. *Let $\mathfrak{T} = e_1\mathfrak{T}e_1 + e_1\mathfrak{T}e_2 + e_2\mathfrak{T}e_2$ be a 2-torsion free triangular algebra. Suppose that*

(i) $\mathcal{Z}(e_1\mathfrak{T}e_1) = \mathcal{Z}(\mathfrak{T})e_1$ and $\mathcal{Z}(e_2\mathfrak{T}e_2) = \mathcal{Z}(\mathfrak{T})e_2$;

(ii) either $e_1\mathfrak{T}e_1$ or $e_2\mathfrak{T}e_2$ satisfies (1).

If a map $\varphi : \mathfrak{T} \to \mathfrak{T}$ satisfies $\varphi([[s_1, s_2], s_3]) = [[\varphi(s_1), s_2], s_3] + [[s_1, \varphi(s_2)], s_3] + [[s_1, s_2], \varphi(s_3)]$ for all $s_1, s_2, s_3 \in \mathfrak{T}$ with $s_1s_2s_3 = 0$, then $\varphi(s+t) - \varphi(s) - \varphi(t) \in \mathcal{Z}(\mathfrak{T})$ for all $s, t \in \mathfrak{T}$ and $\varphi = \delta + \tau$, where $\delta : \mathfrak{T} \to \mathfrak{T}$ is an additive derivation and $\tau : \mathfrak{T} \to \mathcal{Z}(\mathfrak{T})$ is a map vanishing at $[[s_1, s_2], s_3]$ for all $s_1, s_2, s_3 \in \mathfrak{T}$ with $s_1s_2s_3 = 0$.

Some examples of triangular algebras are: upper triangular matrix algebras, block upper triangular matrix algebras, nest algebras (see [26] for details). Thus, in view of Corollary 5.1, the following results follows immediately.

Corollary 5.3. *Let $T_r(\mathfrak{T})$ be an upper triangular matrix algebra over a $(n-1)$-torsionfree unital algebra $\mathfrak{T}$. If a map $\varphi : T_r(\mathfrak{T}) \to T_r(\mathfrak{T})$ satisfies $\varphi(p_n(s_1, s_2, \ldots, s_n)) = \sum_{i=1}^{n} p_n(s_1, \ldots, s_{i-1}, \varphi(s_i), s_{i+1}, \ldots, s_n)$ $(n \geq 3)$ for all $s_1, s_2, \ldots, s_n \in \mathfrak{T}$ with $s_1s_2\cdots s_n = 0$, then $\varphi(s+t) - \varphi(s) - \varphi(t) \in \mathcal{Z}(\mathfrak{T})I_r$ and $\varphi = \delta + \tau$, where $\delta : T_r(\mathfrak{T}) \to T_r(\mathfrak{T})$ is an additive derivation and $\tau : T_r(\mathfrak{T}) \to \mathcal{Z}(\mathfrak{T})I_r$, I_r is the identity element in $T_r(\mathfrak{T})$, is a map vanishing at $p_n(s_1, s_2, \ldots, s_n)$ for all $s_1, s_2, \ldots, s_n \in \mathfrak{T}$ with $s_1s_2\cdots s_n = 0$.*

Corollary 5.4. *Let $B_r^{\bar{k}}(\mathfrak{T})$ be an upper block triangular matrix algebra over a $(n-1)$-torsionfree unital algebras $\mathfrak{T}$ with $B_r^{\bar{k}}(\mathfrak{T}) \neq M_r(\mathfrak{T})$. If a map $\varphi : B_r^{\bar{k}}(\mathfrak{T}) \to B_r^{\bar{k}}(\mathfrak{T})$ satisfies $\varphi(p_n(s_1, s_2, \ldots, s_n)) = \sum_{i=1}^{n} p_n(s_1, \ldots, s_{i-1}, \varphi(s_i), s_{i+1}, \ldots, s_n)$ $(n \geq 3)$ for all $s_1, s_2, \ldots, s_n \in \mathfrak{T}$ with $s_1s_2\cdots s_n = 0$, then $\varphi(s+t) - \varphi(s) - \varphi(t) \in \mathcal{Z}(\mathfrak{T})I_r$ and $\varphi = \delta + \tau$, where $\delta : B_r^{\bar{k}}(\mathfrak{T}) \to B_r^{\bar{k}}(\mathfrak{T})$ is an additive derivation and $\tau : B_r^{\bar{k}}(\mathfrak{T}) \to \mathcal{Z}(\mathfrak{T})I_r$, I_r is the identity element in $B_r^{\bar{k}}(\mathfrak{T})$, is a map vanishing at $p_n(s_1, s_2, \ldots, s_n)$ for all $s_1, s_2, \ldots, s_n \in \mathfrak{T}$ with $s_1s_2\cdots s_n = 0$.*

Corollary 5.5. *Let $\mathcal{N}$ be a nontrivial nest of a complex Hilbert space $\mathcal{H}$ and $\mathcal{T}(\mathcal{N})$ be a nest algebra. If a map $\varphi : \mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{N})$ satisfies $\varphi(p_n(s_1, s_2, \ldots, s_n)) = \sum_{i=1}^{n} p_n(s_1, \ldots, s_{i-1}, \varphi(s_i), s_{i+1}, \ldots, s_n)$ $(n \geq 3)$ for all $s_1, s_2, \ldots, s_n \in \mathfrak{T}$ with $s_1s_2\cdots s_n = 0$, then $\varphi(s+t) - \varphi(s) - \varphi(t) \in \mathbb{C}\mathbf{1}$ and $\varphi = \delta + \tau$, where $\delta : \mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{N})$ is an additive derivation and $\tau : \mathcal{T}(\mathcal{N}) \to \mathbb{C}\mathbf{1}$ is a map vanishing at $p_n(s_1, s_2, \ldots, s_n)$ for all $s_1, s_2, \ldots, s_n \in \mathfrak{T}$ with $s_1s_2\cdots s_n = 0$.*

Full matrix algebras:

A full matrix algebra $M_r(\mathfrak{T})$ is the algebra of all $r \times r$ matrices over a unital algebra $\mathfrak{T}$, where $r \geq 2$ is an integer. The full matrix algebra $M_r(\mathfrak{T})$ can be viewed as

$$M_r(\mathfrak{T}) = \begin{bmatrix} \mathfrak{T} & M_{1\times(r-1)}(\mathfrak{T}) \\ M_{(r-1)\times 1}(\mathfrak{T}) & M_{r-1}(\mathfrak{T}) \end{bmatrix}$$

Let $e_1 = e_{11}$ be a matrix unit and $e_2 = 1 - e_1$. In this case, it is easy to observe that the condition (♠) holds good. Moreover, $e_1M_r(\mathfrak{T})e_1 \cong \mathfrak{T}$, $e_2M_r(\mathfrak{T})e_2 \cong M_{(r-1)}(\mathfrak{T})$, $e_1M_r(\mathfrak{T})e_2 \cong M_{(1)\times r-1}(\mathfrak{T})$ is faithful $(e_1M_r(\mathfrak{T})e_1, e_2M_r(\mathfrak{T})e_2)$-bimodule and $e_2M_r(\mathfrak{T})e_1 \cong M_{(r-1)\times 1}(\mathfrak{T})$ is faithful $(e_2M_r(\mathfrak{T})e_2, e_1M_r(\mathfrak{T})e_1)$-bimodule. Note that $\mathcal{Z}(M_r(\mathfrak{T})) = \mathcal{Z}(\mathfrak{T}) \cdot I_r$, where I_r is the identity element of $M_r(\mathfrak{T})$. Thus, the condition (*i*) of Theorem 4.1 is satisfied. Furthermore, $M_r(\mathfrak{T})(r \geq 2)$ does not contain nonzero central ideals (see [11, Lemma 1]). Hence, the condition (*ii*) of Theorem 4.1 holds if $n \geq 3$. Also, note that $M_{r-1}(A) \cong e_2M_r(\mathfrak{T})e_2$ satisfies the condition (1) for $r - 1 \geq 2$. (see [7, Example 5.6]) Thus, the condition (*iii*) of Theorem 4.1 also holds for all $r \geq 3$. The condition (*iv*) also holds (see [25, Lemma 1]). Therefore, as a direct consequence of Theorems 3.1 and 4.1, we have:

Corollary 5.6. *Let $\mathfrak{T}$ be a $(n-1)$-torsion free unital algebra and $M_r(\mathfrak{T})$ be a full matrix algebra with $r \geq 3$. If a map $\varphi : M_r(\mathfrak{T}) \to M_r(\mathfrak{T})$ satisfies $\varphi(p_n(s_1, s_2, \ldots, s_n)) = \sum_{i=1}^{n} p_n(s_1, \ldots, s_{i-1}, \varphi(s_i), s_{i+1}, \ldots, s_n)$ $(n \geq 3)$ for all $s_1, s_2, \ldots, s_n \in \mathfrak{T}$ with $s_1s_2 \cdots s_n = 0$,then $\varphi(s+t) - \varphi(s) - \varphi(t) \in \mathcal{Z}(\mathfrak{T})I_r$ and $\varphi = \delta + \tau$, where $\delta : M_r(\mathfrak{T}) \to M_r(\mathfrak{T})$ is an additive derivation and $\tau : M_r(\mathfrak{T}) \to \mathcal{Z}(\mathfrak{T})I_r$ is a map vanishing at $p_n(s_1, s_2, \ldots, s_n)$ for all $s_1, s_2, \ldots, s_n \in \mathfrak{T}$ with $s_1s_2 \cdots s_n = 0$.*

Algebra of all bounded linear operators:

Let $\mathcal{B} = \mathfrak{B}(\mathfrak{X})$ be the algebra of all bounded linear operators on a complex Banach space $\mathfrak{X}$ with $dim(\mathfrak{X}) \geq 2$. It is easy to observe that $\mathcal{B}$ contains a nontrivial idempotent e_1 and hence can be expressed as $\mathcal{B} = e_1\mathcal{B}e_1 + e_1\mathcal{B}e_2 + e_2\mathcal{B}e_1 + e_2\mathcal{B}e_2$. Note that $\mathcal{B}$ is a prime algebra and $\mathcal{B}, e_1\mathcal{B}e_1, e_2\mathcal{B}e_2$ are central algebras over $\mathbb{C}$. Thus, all the assumptions of Theorems 3.1 and 4.1 are fulfilled (see [5] for details). Therefore, we have the following result:

Corollary 5.7. *Let $\mathcal{B} = \mathfrak{B}(\mathfrak{X})$ be the algebra of all bounded linear operators on a complex Banach space $\mathfrak{X}$ with $dim(\mathfrak{X}) \geq 2$. If a map $\varphi : \mathcal{B} \to \mathcal{B}$ satisfies $\varphi(p_n(s_1, s_2, \ldots, s_n)) = \sum_{i=1}^{n} p_n(s_1, \ldots, s_{i-1}, \varphi(s_i), s_{i+1}, \ldots, s_n)$ $(n \geq 3)$ for all $s_1, s_2, \ldots, s_n \in \mathfrak{T}$ with $s_1s_2 \cdots s_n = 0$, then $\varphi(s+t) - \varphi(s) - \varphi(t) \in \mathbb{C}1$ and $\varphi = \delta + \tau$, where $\delta : \mathcal{B} \to \mathcal{B}$ is an additive derivation and $\tau : \mathcal{B} \to \mathbb{C}1$ is a map vanishing at $p_n(s_1, s_2, \ldots, s_n)$ for all $s_1, s_2, \ldots, s_n \in \mathfrak{T}$ with $s_1s_2 \cdots s_n = 0$.*

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