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Generalized Schechter essential spectrum of the sum of two bounded operators involving measure of non-almost weak noncompactness

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Abstract. In the present paper we introduce a new concept of measuring, called the measure of non-almost weak noncompactness. We use this measure to characterize the almost weakly compact operators and to investigate the generalized Schechter essential spectrum of the sum of two bounded linear operators.

1. Introduction

For X and Y be two Banach spaces, we denote $\mathcal{L}(X,Y)$ the set of all bounded linear operators acting from X into Y. The set of all compact linear operators from X into Y is designed by $\mathcal{K}(X,Y)$. For $T \in \mathcal{L}(X,Y)$, we use denote by $\mathcal{N}(T)$ and $\mathcal{R}(T)$ respectively the kernel and the range of T. The dual (resp., the bidual) is denoted by X^* (resp., X^{**}), T^* is the conjugate of an operator T and T^{**} is the second conjugate. A bounded operator T is said to be weakly compact, if $T(M)$ is relatively weakly compact in Y for every bounded subset M of X. The family of weakly compact operators from X into Y is denoted by $\mathcal{W}(X,Y)$. If $X = Y$, this family of operators, denoted by $\mathcal{W}(X) := \mathcal{W}(X,X)$, is a closed two-sided ideal of $\mathcal{L}(X)$ containing the closed ideal of compact operators (see [9]). Further, an operator T is said to be Tauberian whenever T^{**} preserves the naturel embedding of X into its double dual i.e $x \in X^*$, $T^{**}x \in Y$ implies $X \in X$. It is immediate that if T is a Tauberian operator and $T^{**}x = 0$, $x \in X^{**}$, then $x \in X$. Which implies that $\mathcal{N}(\mathcal{T})$ is reflexive. In particular, if $\mathcal{R}(T)$ is closed, then T is Tauberian if and only if $\mathcal{N}(T)$ is reflexive. Also, T is co-Tauberian if its conjugate T^* is Tauberian. The classes of Tauberian and co-Tauberian operators from X into Y are respectively denoted by $\mathcal{T}(X,Y)$ and $\mathcal{T}^d(X,Y)$. We say that X has the property (H_1) (resp., (H_2)) if every reflexive subspace admits a closed complementary subspace (resp., if every closed subspace with reflexive quotient space admits a closed complementary subspace). We say that X has the property (H), if it satisfies both properties (H_1) and (H_2). For example, the $L_p(0,1)$ spaces for $1 < p < \infty$ have the property (H_1) (see [16]).

In 1953, Atkinson [3] studied the operators which are invertible modulo compact operators on Banach spaces. He proved that an operator $T \in \mathcal{L}(X,Y)$ is invertible modulo compact operators if and only if T is a Fredholm operator. Recall that a bounded operator T is called Fredholm if its kernel $\mathcal{N}(T)$ and co-kernel $X/\mathcal{R}(T)$ are finite-dimensional subspaces. In 1976, K. W. Yang [18] extend some results of [3] to operators invertible modulo weakly compact operators. His aim was to introduce and study the generalized Fredholm theory as an extension of the notion of Fredholm operators in which the reflexive spaces play the role of

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finite-dimensional spaces. The sets of upper and lower generalized semi-Fredholm operators from X into Y, respectively, are defined as:

$$\Phi_{g+}(X,Y) := \{T \in \mathcal{L}(X,Y) \text{ such that } \mathcal{N}(T) \text{ is reflexive and } \mathcal{R}(T) \text{ is closed in } Y\},$$

$$\Phi_{g-}(X,Y) := \{T \in \mathcal{L}(X,Y) \text{ such that } Y/\mathcal{R}(T) \text{ is reflexive and } \mathcal{R}(T) \text{ is closed in } Y\}.$$

We denote by $\Phi_g(X,Y) := \Phi_{g+}(X,Y) \cap \Phi_{g-}(X,Y)$ the set of generalized Fredholm operators in $\mathcal{L}(X,Y)$ and by $\Phi_{g\pm}(X,Y) := \Phi_{g+}(X,Y) \cup \Phi_{g-}(X,Y)$ the set of generalized semi-Fredholm operators. Furthermore, in [18] Yang proved that the set of generalized Fredholm operators is also an intersection of the co-Tauberian operators with closed range and Tauberian operators. If $X = Y$, the sets $\mathcal{L}(X,Y)$, $\mathcal{K}(X,Y)$, $\Phi_g(X,X)$, $\Phi_{g+}(X,Y)$, $\Phi_{g-}(X,Y)$ and $\Phi_{g\pm}(X,Y)$ are replaced by $\mathcal{L}(X)$, $\mathcal{K}(X)$, $\Phi_g(X)$, $\Phi_{g+}(X)$, $\Phi_{g-}(X)$ and $\Phi_{g\pm}(X)$, respectively. Now, for $S \in \mathcal{L}(X,Y)$, $S \neq 0$, a complex number λ is in $\Phi_{g+,S}(T)$, $\Phi_{g-,S}(T)$, $\Phi_{g\pm,S}(T)$ or $\Phi_{g,S}(T)$, if $\lambda S - T$ is in $\Phi_{g+}(X,Y)$, $\Phi_{g-}(X,Y)$, $\Phi_{g\pm}(X,Y)$ or $\Phi_g(X,Y)$, respectively. If $S = I$, then the sets $\Phi_{g+,S}(T)$, $\Phi_{g-,S}(T)$, $\Phi_{g\pm,S}(T)$ and $\Phi_{g,S}(T)$ are simply denoted by $\Phi_{g+}(T)$, $\Phi_{g-}(T)$, $\Phi_{g\pm}(T)$ and $\Phi_g(T)$, respectively. Recently, Azzouz et al. [5, 6] obtained some perturbation results for these classes under the conditions (H_1) and (H_2) and introduced the generalized essential spectrum of a bounded linear operator. In particular, for $S \in \mathcal{L}(X)$ the generalized Wolf essential spectrum and the generalized Gustafson essential spectrum of $T \in \mathcal{L}(X)$ are respectively defined as follow

$$\begin{aligned}\sigma_{Se_4,g}(T) &:= \Big\{\lambda \in \mathbb{C} \text{ such that } (\lambda S - T) \notin \Phi_g(X)\Big\} := \mathbb{C}\backslash\Phi_{g,S}(T),\\ \sigma_{Se_1,g}(T) &:= \Big\{\lambda \in \mathbb{C} \text{ such that } (\lambda S - T) \notin \Phi_{g+}(X)\Big\} := \mathbb{C}\backslash\Phi_{g+,S}(T).\end{aligned}$$

Note that if $S = I$, the two previous generalized essential spectrum of T will be respectively replaced by $\sigma_{e_4,g}(T)$ and $\sigma_{e_1,g}(T)$.

Several authors [11, 15, 17] have considered different concepts of measuring associated with bounded linear operators in order to obtain characterizations and perturbation results of various classes of operators of Fredholm theory. For example, M. Schechter in [17] introduced the measure of non-strict singularity of a bounded linear operator which has been applied to characterize the strictly singular operators and to solve such problem concerning the perturbation theory of Fredholm operators. Recall that a strictly singular operator, as defined by T. Kato in [14] is a bounded linear operator T between two Banach spaces X and Y such that the restriction of T to any infinite-dimensional closed subspace of X is not an isomorphism. In 1968, Herman [13] has investigated a class of operators, called almost weakly compact operators, as a bounded linear operator T between two Banach spaces X and Y such that the restriction of T to any non-reflexive closed subspace of X is not an isomorphism. This concept can be seen as a generalization of the concept of strictly singular operators. In general, theses classes are distinct (see Example 3 in [13]). But the question whether the sum of two almost weakly compact operators is almost weakly compact is still open. Therefore, one goal of this paper is to characterize the almost weakly compact operators by defining the so-called measure of non-almost of weak noncompactness. We use the subadditivity property of this measure to prove that the almost weakly compact operators form a closed subspace of the space of bounded linear operators acting on Banach spaces. Moreover, we show that an almost weakly compact perturbation of a generalized Fredholm operator remains generalized Fredholm and has the same generalized index. We will also discuss the generalized Schechter essential spectrum of two bounded linear operators by means measure of non-almost weak noncompactness.

Now, let us outline the content of this paper. In Section 2, we recall some definitions and results needed in the sequel of the paper. In Section 3, we introduce and study a new measure for bounded linear operators acting on Banach spaces using the De Blasi measure of weak noncompactness. Based on this new investigation, we establish a characterization of almost weakly compact operators. In Section 4, we investigate the generalized Schechter essential spectrum of the sum of two bounded operators involving the measure of non-almost weak noncompactness and the concept of left and right weak-Fredholm inverse.

2. Preliminary results

For X a Banach space, let $\mathcal{M}_X$ denote the family of all nonempty bounded subsets of X and $\mathcal{M}_X^w$ the subset of $\mathcal{M}_X^w$ consisting of all relatively weakly compact sets. The following definition will play an important role in our considerations.

Definition 2.1. The weak measure of noncompactness of a bounded subspace D is defined as

$$\omega(D) = \inf\{r > 0 : \text{ there exists } N \in \mathcal{M}_X^w \text{ such that } D \subset N + r\overline{B}_X\},$$

where $\overline{B}_X$ denotes the closed unit ball of X.

This measure was first introduced by De Blasi (see [8]) and has been applied to obtain fixed point theorems. Moreover, it is well known that if X is reflexive, then $\omega(D) = 0$, for all $D \in \mathcal{M}_X$.
Let $T \in \mathcal{L}(X)$ be a bounded linear operator. T is called a weak k-set contraction ($k \geq 0$) if

$$\omega(T(D)) \leq k\omega(D), \text{ for all } D \in \mathcal{M}_X.$$

The quantity

$$\overline{\omega}(T) = \inf\{k \geq 0 \text{ such that } T \text{ is weak } k\text{-set contraction}\}$$

is called the De Blasi measure of weak non compactness of T.
In the next proposition, we recall some properties of $\overline{\omega}(\cdot)$.

Proposition 2.2. [8] Let X be a Banach space, T and $S \in \mathcal{L}(X)$ and let $D \in \mathcal{M}_X$. Then, we have the following properties:
(*i*) $\overline{\omega}(T) = 0$ if and only if T is weakly compact.
(*ii*) $\omega(T(D)) \leq \overline{\omega}(T)\omega(D)$.
(*iii*) $\overline{\omega}(TS) \leq \overline{\omega}(T)\overline{\omega}(S)$.
(*iv*) $\overline{\omega}(T + S) \leq \overline{\omega}(T) + \overline{\omega}(S)$.
(*v*) $\overline{\omega}(\lambda T) = |\lambda|\overline{\omega}(T)$, for $\lambda \in \mathbb{C}$.

Let X be a non-reflexive Banach space and $T \in \mathcal{L}(X)$. The De Blasi measure of weak noncompactness of T can be equivalently defined as follows

$$\overline{\alpha}(T) \quad := \quad \sup\left\{\frac{\omega(T(D))}{\omega(D)} \text{ such that } D \in \mathcal{M}_X \text{ and } \omega(D) > 0\right\}.$$

We define the following non-negative quantity as (see [5])

$$\overline{\beta}(T) \quad := \quad \inf\left\{\frac{\omega(T(D))}{\omega(D)} \text{ such that } D \in \mathcal{M}_X \text{ and } \omega(D) > 0\right\}.$$

The quantity $\overline{\beta}(T)$ can be equivalently defined as follows

$$\overline{\beta}(T) \quad := \quad \sup\left\{k \geq 0 \text{ such that } \omega(T(D)) \geq k\omega(D) \text{ for any } D \in \mathcal{M}_X\right\}.$$

The following proposition gives some fundamental properties of $\overline{\alpha}, \overline{\beta}$ already given in [5].

Proposition 2.3. Let X be a Banach space. Let $T, S \in \mathcal{L}(X)$ and $\lambda \in \mathbb{C}$, then the following properties hold:
(*i*) $\overline{\alpha}(\lambda T) = |\lambda|\overline{\alpha}(T)$ and $\overline{\beta}(\lambda T) = |\lambda|\overline{\beta}(T)$.
(*ii*) $|\overline{\alpha}(T) - \overline{\alpha}(S)| \leq \overline{\alpha}(T + S) \leq \overline{\alpha}(T) + \overline{\alpha}(S)$.
(*iii*) $\overline{\beta}(T) - \overline{\alpha}(S) \leq \overline{\beta}(T + S) \leq \overline{\beta}(T) + \overline{\alpha}(S)$.
(*iv*) $\overline{\alpha}(TS) \leq \overline{\alpha}(T)\overline{\alpha}(S)$ and $\overline{\beta}(TS) \geq \overline{\beta}(T)\overline{\beta}(S)$.
(*v*) $\overline{\alpha}(T) = 0$ if and only if T is weakly compact.

Remark 2.4. For $T, S \in \mathcal{L}(X)$, we have $\overline{\omega}(TS) \geq \overline{\beta}(TS) \geq \overline{\omega}(T)\overline{\beta}(S)$.

Next, we recall the following definition.

Definition 2.5. Let X and Y be two Banach spaces.
(i) An operator $T \in \mathcal{L}(X, Y)$ is said to have a left weak-Fredholm inverse if there exists $T_l^w \in \mathcal{L}(Y, X)$ such that $I_X - T_l^w T \in \mathcal{W}(X)$. The operator T_l^w is called left weak-Fredholm inverse of T.
(ii) An operator $T \in \mathcal{L}(X, Y)$ is said to have a right weak-Fredholm inverse if there exists $T_r^w \in \mathcal{L}(Y, X)$ such that $I_Y - TT_r^w \in \mathcal{W}(Y)$. The operator T_r^w is called right weak-Fredholm inverse of T.
(iii) An operator $T \in \mathcal{L}(X, Y)$ is said to have a weak-Fredholm inverse if there exists a map which is both a left and right weak-Fredholm inverse of T.

Now, let us recall a characterization of a generalized Fredholm operator.

Theorem 2.6. [18] Let X and Y be two Banach spaces satisfying the properties (H_1) and (H_2) respectively, and let $T \in \mathcal{L}(X, Y)$. Then the following assertions are equivalent:
(i) T is a generalized Fredholm operator .
(ii) T has a weak-Fredholm inverse, and $\mathcal{R}(T)$ is closed in Y.

Remark 2.7. Note that in [5], the authors proved that if T has a weak-Fredholm inverse, then T is generalized Fredholm, where X is a non-reflexive Banach space having the property (H_1). However, we will show that the result holds true without this conditions on the space X. Indeed, let X be a Banach space and $T \in \mathcal{L}(X)$. Suppose that T has a weak-Fredholm inverse, then there exists $T_0 \in \mathcal{L}(X)$ such that $T_0T = I + W_1$ and $TT_0 = I + W_2$, where W_1 and W_2 are weakly compact. For a bounded subset D of X, we have

$$\begin{aligned}\omega(D) &= \omega((I + W_1)(D)) \\ &= \omega(T_0T(D)) \\ &\leq \|T_0\|\omega(T(D)).\end{aligned}$$

Hence, $\overline{\beta}(T) \geq \frac{1}{\|T_0\|} > 0$. It follows from Lemma 3.4 and Theorem 3.5 in [5] that $T \in \Phi_{g+}(X)$. On the other hand, we have $T_0^*T^* = I + W_2^*$ and $W_2^* \in \mathcal{W}(X)$ (see [9]). Arguing as before we infer that $T^* \in \Phi_{g+}(X^*)$. According to Remark 3.7 in [5] we conclude that $T \in \Phi_{g-}(X)$ and therefore T is generalized Fredholm.

From the above remark, the following properties proved in [5] remain valid without assuming that X, Y and Z are non-reflexive spaces.

Theorem 2.8. [5] Let X, Y and Z be three Banach spaces and let $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y, Z)$. Then the following statements hold:
(i) Suppose that X, Y and Z satisfy the properties (H_1), (H) and (H_2) respectively. If $T \in \Phi_g(X, Y)$ and $S \in \Phi_g(Y, Z)$, then $ST \in \Phi_g(X, Z)$.
(ii) Suppose that X and Y satisfy the properties (H_1) and (H_2) respectively. If $T \in \Phi_g(X, Y)$ and $W \in \mathcal{W}(X, Y)$, then $T + W \in \Phi_g(X, Y)$.

3. Measure of non-almost weak noncompactness of a bounded linear operator

Let us start by defining the measure of non-almost weak noncompactness.

Definition 3.1. Let X and Y be two Banach spaces. For $T \in \mathcal{L}(X, Y)$, we make the following definitions

$$\overline{g}_M(T) = \inf_{N \subset M} \overline{\omega}(T_{|N}) \text{ and } \overline{g}(T) = \sup_{M \subset X} \overline{g}_M(T),$$

where M and N are two closed non-reflexive subspace of X and $T_{|N}$ denotes the restriction of T to the subspace N.
The quantity $\overline{g}$ is called measure of non-almost weak noncompactness.

Theorem 3.2. Let X and Y be two Banach spaces. For $T, S \in \mathcal{L}(X, Y)$ we have

$$\overline{g}(T+S) \leq \overline{g}(T) + \overline{g}(S). \tag{1}$$

Proof. We first prove that

$$\overline{g}_M(T+S) \leq \overline{g}(T) + \overline{g}_X(S).$$

To see this let $\varepsilon > 0$ be given and M be a non-reflexive closed subspace of X. Then there exists a non-reflexive subspace N of M such that

$$\overline{\omega}(T_{|N}) \leq \overline{g}_M(T) + \varepsilon.$$

Thus

$$\begin{aligned} \overline{\omega}((T+S)_{|N}) &\leq \overline{\omega}(T_{|N}) + \overline{\omega}(S_{|N}) \\ &\leq \overline{g}_M(T) + \varepsilon + \overline{\omega}(S_{|M}). \end{aligned}$$

This implies

$$\overline{g}_M(T+S) \leq \overline{g}_M(T) + \varepsilon + \overline{\omega}(S_{|M}).$$

Since this is true for every $\varepsilon > 0$, we obtain

$$\begin{aligned} \overline{g}_M(T+S) &\leq \overline{g}_M(T) + \overline{\omega}(S_{|M}) \\ &\leq \overline{g}(T) + \overline{\omega}(S_{|M}). \end{aligned}$$

Hence

$$\inf_{M \subset X} \overline{g}_M(T+S) \leq \overline{g}(T) + \inf_{M \subset X} \overline{\omega}(S_{|M}),$$

and therefore

$$\overline{g}_M(T+S) \leq \overline{g}(T) + \overline{g}_X(S).$$

So, for any closed non-reflexive subspace $M \subset X$ we get

$$\overline{g}_M(T+S) \leq \overline{g}(T) + \overline{g}(S).$$

Consequently Equation (1) holds. □

Proposition 3.3. Let X be a Banach space and let $T \in \mathcal{L}(X)$. Then

$$\overline{\beta}(T) \leq \overline{g}(T).$$

Proof. Let $\varepsilon > 0$ be given. Then for every closed non-reflexive subspace $M \subset X$, we have

$$\overline{g}_M(T) \leq \overline{g}(T),$$

and there exists a closed non-reflexive subspace N of M such that

$$\overline{\omega}(T_{|N}) \leq \overline{g}_M(T) + \varepsilon.$$

Which implies that

$$\inf_{\substack{D \in \mathcal{M}_N \\ \omega(D) > 0}} \frac{\omega(T_{|N}(D))}{\omega(D)} \leq \overline{\omega}(T_{|N}) \leq \overline{g}(T) + \varepsilon,$$

and it follows that

$$\inf_{\substack{D \in \mathcal{M}_X \\ \omega(D) > 0}} \frac{\omega(T(D))}{\omega(D)} \leq \overline{g}(T) + \varepsilon.$$

Thus

$$\overline{\beta}(T) \leq \overline{g}(T) + \varepsilon.$$

Since ε was arbitrary, we obtain $\overline{\beta}(T) \leq \overline{g}(T)$. □

Now, let us recall the following definition.

Definition 3.4. [13] Let X and Y be two Banach spaces and $A \in \mathcal{L}(X, Y)$. A is said almost weakly compact if the restriction of A to any non-reflexive closed subspace of X is not an isomorphism.

Let $\mathcal{AWC}(X, Y)$ denote the set of almost weakly compact operators from X into Y. If $X = Y$, then $\mathcal{AWC}(X, Y)$ will simply be denoted by $\mathcal{AWC}(X)$. It should be noted that the almost weakly compact operators from X into X form a closed subset of $\mathcal{L}(X)$ containing strictly $\mathcal{W}(X)$.

Example 3.5. For $1 < p < \infty$, let J_p be the James spaces which are defined as follows:

$$J_p = \{x \in C_0 \text{ such that } \|x\|_{J_p} < \infty\},$$

where

$$\|x\|_{J_p}^p = \sup_s \sup_{j_1 < \cdots < j_n} \sum_{i=1}^{s} |x_{j_i} - x_{j_{i-1}}|.$$

Now, let us consider the natural inclusion i from the James space J_2 into the James space J_3. It is proved in [2] that it is strictly singular operator and therefore it defines an almost weakly compact operators. However, it is not weakly compact. Indeed, we denote by $(e_n)_{n\in\mathbb{N}}$ the canonical basis. The sequence $x_m = \sum_{j=1}^{m} e_j$ is bounded in J_2 but admits no weakly convergent subsequence in J_3.

Remark 3.6. (*i*) If X is a reflexive space, then

$$\mathcal{AWC}(X) = \mathcal{W}(X) = \mathcal{L}(X).$$

(*ii*) On the space l_1, the classes of almost weakly compact and strictly singular operators coincide from the fact that every infinite-dimensional subspace of l_1 is non-reflexive.

Lemma 3.7. If X is a non-reflexive Banach space having the property (H_1) and $T \in \mathcal{L}(X)$, then the following implication holds:

$$T \in \Phi_{g+}(X) \Longrightarrow T \notin \mathcal{AWC}(X)$$

Proof. Assume that $T \in \Phi_{g+}(X)$. Since X satisfies the property (H_1), for some closed subspace M of X we have $X = \mathcal{N}(T) \oplus M$. The restriction $T_{|M} : M \longrightarrow \mathcal{R}(T)$ is an isomorphism between non-reflexive closed subspaces. Therefore $T \notin \mathcal{AWC}(X)$. □

Now, we present the following useful lemma.

Lemma 3.8. [13] If T is almost weakly compact, then for all M an infinite-dimensional closed subspace of X there exists a closed infinite-dimensional subspace $N \subset M$ such that $T_{|N}$ is a weakly compact operator.

The next theorem shows that the almost weakly compact operators can be characterize by the measure of non-almost weak noncompactness.

Theorem 3.9. Let X and Y be two Banach spaces and let $A \in \mathcal{L}(X, Y)$. Then the following assertions are equivalent:
(*i*) A is almost weakly compact.
(*ii*) $\overline{g}_M(A) = 0$ for all closed subspace $M \subset X$.
(*iii*) $\overline{g}(A) = 0$.

Proof. (*i*) ⇒ (*ii*) Let A be an almost weakly compact operator and M be an infinite-dimensional subspace of X. According to Lemma 3.8, there exists a closed infinite-dimensional subspace N of M such that $A_{|N}$ is a weakly compact and therefore $\overline{\omega}(A_{|N}) = 0$. Which implies that

$$\overline{g}_M(A) = 0 \text{ for each closed subspace } M \subset X.$$

$(ii) \Rightarrow (i)$ Assume that A is not almost weakly compact. Then there exists a non-reflexive subspace N of X such that $A_{|N}$ has a bounded inverse S. By hypothesis, we have $\overline{g}_N(A) = 0$. Hence, for all $\varepsilon > 0$ there exists a non-reflexive subspace N_ε of N such that

$$\overline{\omega}(A_{|N_\varepsilon}) \leq \varepsilon.$$

On the other hand, we have

$$\begin{aligned}\overline{\omega}(I_{|N}) &= \overline{\omega}(SA_{|N_\varepsilon}) \\ &\leq \overline{\omega}(S)\overline{\omega}(A_{|N_\varepsilon}) \\ &\leq \overline{\omega}(S)\varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0$, we get $\overline{\omega}(I_{|N}) = 0$, which is a contradiction.
$(ii) \Leftrightarrow (iii)$ Follows from the definition. □

An immediate consequence of above theorem.

Corollary 3.10. Let $T, S \in \mathcal{L}(X, Y)$ be two almost weakly compact operators. Then $T + S$ is almost weakly compact.

Proof. Since T and S are both almost weakly compact, by Theorem 3.9, we infer that $\overline{g}(T) = \overline{g}(S) = 0$. It follows from Equation (1) that $\overline{g}(T + S) = 0$ and therefore we can deduce by Theorem 3.9 that $T + S$ is almost weakly compact. □

For almost weakly compact operators the analogue of Theorem 3.3 in [17] is as follows.

Theorem 3.11. Let $A \in \mathcal{L}(X, Y)$. Then A is almost weakly compact if and only if

$$\overline{g}(T + A) = \overline{g}(T) \text{ for all } T \in \mathcal{L}(X, Y). \tag{2}$$

Proof. Suppose that A is almost weakly compact, then from Equation (1) and Theorem 3.9 we have

$$\overline{g}(T + A) \leq \overline{g}(T) + \overline{g}(A) = \overline{g}(T),$$

for every $T \in \mathcal{L}(X, Y)$. For the same reasons we prove that

$$\overline{g}(T) \leq \overline{g}(T + A) + \overline{g}(A) = \overline{g}(T + A).$$

Hence Equation (2) holds.
Conversely, if $\overline{g}(T + A) = \overline{g}(T)$ for every $T \in \mathcal{L}(X, Y)$, then we get $\overline{g}(A) = \overline{g}(0 + A) = \overline{g}(0) = 0$. By applying Theorem 3.9, we deduce that A is almost weakly compact. □

Definition 3.12. Let X and Y be two Banach spaces and let $A \in \mathcal{L}(X, Y)$. We say that A is bounded below if $\|x\| \leq c\|Ax\|$ for all $x \in X$, for some $c > 0$.

The following proposition characterizes the almost weakly compact operators.

Proposition 3.13. Let X and Y be two Banach spaces. A bounded operator $A : X \to Y$ between Banach spaces is almost weakly compact if and only if A is not bounded below on any non-reflexive closed subspace.

Proof. Suppose that there is a closed non-reflexive subspace $N \subset X$ such that $A_{|N}$ is bounded below. Then, there exists $c > 0$ such that $\|x\| \leq c\|A_{|N}(x)\|$ for all $x \in N$. Let $x_n \in N$, with $A_{|N}(x_n) \to y \in Y$. Hence $\|x_n - x_m\| \leq c\|A_{|N}(x_n) - A_{|N}(x_m)\| \to 0$ as $n, m \to \infty$, the sequence $(x_n)_{n\in\mathbb{N}^*}$ is cauchy. So, by completeness of N, $x_n \to x \in N$ and hence $A_{|N}(x_n) \to A_{|N}(x)$. Which implies that $y = A_{|N}(x)$. Therefore $\mathcal{R}(A_{|N})$ is closed. Since $\mathcal{N}(A_{|N}) = \{0\}$, $A_{|N} : N \to \mathcal{R}(A_{|N})$ is an homomorphism and consequently A is not almost weakly compact.
Conversely, if A is not almost weakly compact, then there exists a non-reflexive subspace N of X such that $A_{|N}$ has a bounded inverse, which means that it is bounded below. □

Theorem 3.14. Let X and Y be two Banach spaces. Then the class of almost weakly compact operators taking X into X forms a closed left ideal in $\mathcal{L}(X)$.

Proof. By using Theorem 9 in [13] and Corollary 3.10 we infer that $\mathcal{AWC}(X)$ is a closed subgroup of $\mathcal{L}(X)$. Now let $A \in \mathcal{AWC}(X)$, we have to show that TA is in $\mathcal{AWC}(X)$ for all $T \in \mathcal{L}(X)$. To see this, suppose that TA is not almost weakly compact, then there exists a closed non-reflexive subspace N such that $(TA)_{|N}$ has a bounded inverse B. Hence for all $y = TA(x)$, where $x \in N$, there exists $c > 0$ such that

$$\|B(y)\| \leq c\|y\|.$$

This implies

$$\begin{aligned} \|x\| &\leq c\|TA(x)\| \\ &\leq c\|T\|\|A(x)\|. \end{aligned}$$

Therefore

$$\|A(x)\| \geq \frac{1}{c\|T\|}\|x\|.$$

Hence A is bounded below on N. This means from Proposition 3.13 that A cannot almost weakly compact and $A : N \to A(N)$ is an isomorphism, which is a contradiction. Thus TA is almost weakly compact for all $T \in \mathcal{L}(X)$ and consequently $\mathcal{AWC}(X)$ is a closed left ideal of $\mathcal{L}(X)$. □

4. Generalized Schechter essential spectrum of the sum of two bounded operators

We begin this section by the following definition.

Definition 4.1. [4] Let X be a Banach space satisfying the property (H). We define the generalized Fredholm index on the semigroup $\Phi_g(X)$ by the semigroup homomorphism $i_g : \Phi_g(X) \to \mathbb{Z}$ such that the following conditions hold:
(*i*) $i_g(T) = 0$ for all invertible elements T in $\mathcal{L}(X)$, and
(*ii*) $i_g(I + W) = 0$ for all W in $\mathcal{W}(X)$.

The set of generalized Fredholm operators defines the corresponding generalized Schechter essential spectrum:

$$\sigma_{e_5,g}(T) \quad := \quad \mathbb{C}\backslash\Big\{\lambda \in \Phi_g(T) \text{ such that } i_g(\lambda - T) = 0\Big\}.$$

For the next we need the following lemmas.

Lemma 4.2. [10] Let X and Y be two Banach spaces and $Z \subset X$. For an operator $T \in \mathcal{L}(X, Y)$, the following statements are equivalent:
(*i*) T is co-Tauberian.
(*ii*) Every operator $S \in \mathcal{L}(Y, Z)$ is weakly compact whenever ST is weakly compact.

Lemma 4.3. Let X be a Banach space having the property (H_1) and let $T \in \mathcal{L}(X)$. If $T \in \Phi_{g+}(X)$, then $T_{|M}$, the restriction of T to any closed subspace M of X is generalized upper semi-Fredholm on M.

Proof. Suppose that $T \in \Phi_{g+}(X)$. We observe that $\mathcal{N}(T_{|M}) = \mathcal{N}(T) \cap M$. Since $\mathcal{N}(T)$ is reflexive and M is a closed subspace, the intersection $\mathcal{N}(T) \cap M$ is also a reflexive subspace. Now, from the fact that X satisfies the property (H_1), there exists a closed subspace M_1 of X such that $X = \mathcal{N}(T) \oplus M_1$. Clearly, the restriction $T_{|M_1}$ is injective and has closed range since $\mathcal{R}(T) = T(M_1)$. On the other hand, we have $M = \mathcal{N}(T) \cap M \oplus M_1 \cap M$. Hence $T(M) = T(M_1 \cap M)$. Since $T_{|M_1}$ is bounded below, then $T(M_1 \cap M)$ is closed. Consequently $T_{|M_1} \in \Phi_{g+}(M)$. □

Theorem 4.4. Let X be a non-reflexive Banach space having the property (H) and let $T, A \in \mathcal{L}(X)$. If T is generalized Fredholm, A is almost weakly compact and $\mathcal{R}(T+A)$ is closed, then

$$T + A \in \Phi_g(X) \text{ and } i_g(T+A) = i_g(T).$$

Proof. First, suppose that $T+A \notin \Phi_{g+}(X)$, then $\mathcal{N}(T+A)$ is not reflexive since $\mathcal{R}(T+A)$ is closed. Hence $T+A$ is not Tauberian. By using Remark 4.6, there exists a non-reflexive subspace $M \subset X$ such that $(T+A)_{|M}$, the restriction of $T+A$ to M is compact and hence it is almost weakly compact. Thus, by Theorem 3.9 we obtain $\overline{g}((T+A)_{|M}) = 0$. By Theorem 3.11 we have

$$\begin{array}{rcl} \overline{g}(T_{|M}) & = & \overline{g}((T+A)_{|M} - A_{|M}) \\ & \leq & \overline{g}((T+A)_{|M}) + \overline{g}(A_{|M}). \end{array}$$

It follows that $\overline{g}(T_{|M}) = 0$ and hence $T_{|M}$ is an almost weakly compact operator. According to Lemma 3.7, we get $T_{|M} \notin \Phi_{g+}(X)$. But this impossible since $T \in \Phi_{g+}(X)$, see Lemma 4.3. Next, using Theorem 6.7 in [18], it remains to prove that $T+A$ is co-Tauberian. So let $B \in \mathcal{L}(X)$ such that $B(T+A)$ is weakly compact. By using Remark 2.4 we have

$$\overline{\omega}(B(T+A)) \geq \overline{\omega}(B)\overline{\beta}(T+A).$$

Since $\overline{\beta}(T+A) > 0$ and $\overline{\omega}(B(T+A)) = 0$, $\overline{\omega}(B) = 0$ and it follows from Lemma 4.2 that $T+A$ is co-Tauberian. Consequently $T+A \in \Phi_{g-}(X)$ Now, since λA is an almost weakly compact operator for each scalar λ, it follows from the last conclusion that $T+\lambda A$ is also generalized Fredholm for each $\lambda \in \mathbb{R}$. Finally, in view of Theorem 3.1 in [4] the mapping $\lambda \to i_g(T+\lambda A)$ from $\mathbb{R}$ to $\mathbb{Z}$ is continuous and hence it must be constant. This implies that $i_g(T+\lambda A) = i_g(T)$ for each $\lambda \in \mathbb{R}$. Letting $\lambda = 1$, yields that $i_g(T+A) = i_g(T)$. □

We now present a characterization of Tauberian operators proved in [12].

Lemma 4.5. An operator $T : X \to Y$ is Tauberian if and only if $\mathcal{N}(T+K)$ is reflexive for every compact operator $K : X \to Y$.

Remark 4.6. From the preceding lemma we remark that if T is not Tauberian, then there exists a non-reflexive subspace $M \subset X$ such that $T_{|M}$ is compact.

Theorem 4.7. Let X be a Banach space satisfying the property (H) and let $T \in \mathcal{L}(X)$. If $\overline{g}(T) < 1$ and $\mathcal{R}(I-\mu T)$ is a closed subspace of X for all $\mu \in [0,1]$, then $I-T$ is a generalized Fredholm operator and $i_g(I-T) = 0$.

Proof. Suppose that $I-T \notin \Phi_{g+}(X)$ and hence $I-T$ is not Tauberian. According to Remark 4.6, there exists a non-reflexive subspace $M \subset X$ such that $(I-T)_{|M}$, the restriction of $I-T$ to M is compact and hence it is almost weakly compact. Then $\overline{g}((I-T)_{|M}) = 0$. This means from Theorem 3.11 that

$$\overline{g}(I_{|M}) = \overline{g}(T_{|M}) < 1.$$

Which is a contradiction since M is not reflexive and hence we must have $I-T \in \Phi_{g+}(X)$. So, by applying Lemma 4.2, it is sufficient to prove that $\lambda I - T$ is co-Tauberian. So, let $D \in \mathcal{L}(X)$ such that $D(I-T) \in \mathcal{W}(X)$. On the other hand we have

$$\overline{\omega}(D(I-T)) > \overline{\omega}(D)\overline{\beta}(I-T).$$

By using Theorem 3.6 in [5], we have $\overline{\beta}(I-T) > 0$, then $\overline{\omega}(T) = 0$. Hence $I-T$ is a co-Tauberian operator and since $\mathcal{R}(I-T)$ is closed, we deduce that $I-T \in \Phi_g(X)$. Now, since $\mathcal{R}(I-\mu T)$ is closed for all $\mu \in [0,1]$, $I-\mu T \in \Phi_g(X)$. Hence, by constancy of the generalized index we have

$$i_g(I-T) = i_g(I) = 0.$$

Consequently $I-T \in \Phi_g(X)$ and $i_g(I-T) = 0$. □

We now give a generalization of a result in [1] which is a characterization of the generalized Schechter essential spectrum of the sum of two bounded operators by means of measure of non-almost weak non-compactness.

Theorem 4.8. Let X be a Banach space satisfying the property (H) and let A and T be two operators in $\mathcal{L}(X)$. Then the following assertions hold:

(i) If for each $\lambda \in \Phi_g(A)$, there exists a left weak-Fredholm inverse $A^w_{\lambda l}$ of $\lambda - A$ such that $\overline{g}(TA^w_{\lambda l}) < 1$ and $\mathcal{R}(I - \mu TA^w_{\lambda l})$ is a closed subspace of X for all $\mu \in [0, 1]$, then

$$\sigma_{e5,g}(A + T) \subset \sigma_{e5,g}(A).$$

(ii) If for each $\lambda \in \Phi_g(A)$, there exists a right weak-Fredholm inverse $A^w_{\lambda r}$ of $\lambda - A$ such that $\overline{g}(A^w_{\lambda r}T) < 1$ and $\mathcal{R}(I - \mu A^w_{\lambda r}T)$ is a closed subspace of X for all $\mu \in [0, 1]$, then

$$\sigma_{e5,g}(A + T) \subset \sigma_{e5,g}(A).$$

Proof. (i) Suppose that $\lambda \notin \sigma_{e5,g}(A)$, then $\lambda - A \in \Phi_g(X)$ and $i_g(\lambda - A) = 0$. Since $A^w_{\lambda l}$ is a left weak-Fredholm inverse of $\lambda - A$, then there exists $W \in \mathcal{W}(X)$ such that

$$A^w_{\lambda l}(\lambda - A) = I - W \text{ on } X. \tag{3}$$

It follows from Equation (3) that

$$\begin{aligned} \lambda - A - T &= \lambda - A - T(A^w_{\lambda l}(\lambda - A) + W) \\ &= (I_X - TA^w_{\lambda l})(\lambda - A) - TW. \end{aligned}$$

Since $\overline{g}(TA^w_{\lambda l}) < 1$ and $\mathcal{R}(I - \mu TA^w_{\lambda l})$ is closed for all $\mu \in [0, 1]$, by applying Theorem 4.7 we obtain

$$I_X - TA^w_{\lambda l} \in \Phi_g(X) \text{ and } i_g(I_X - TA^w_{\lambda l}) = i_g(I_X) = 0.$$

Hence, from Theorem 2.8 we have

$$(I_X - TA^w_{\lambda l})(\lambda - A) \in \Phi_g(X),$$

and

$$\begin{aligned} i_g((I_X - TA^w_{\lambda l})(\lambda - A)) &= i_g(I_X - TA^w_{\lambda l}) + i_g(\lambda - A) \\ &= i_g(\lambda - A). \end{aligned}$$

From the fact that $TW \in \mathcal{W}(X)$, we infer that $\lambda - A - T \in \Phi_g(X)$ and $i_g(\lambda - A - T) = i_g(\lambda - A) = 0$. Thus, $\lambda \notin \sigma_{e5,g}(A + T)$.

(ii) Let $\lambda \in \mathbb{C}$ and $A^w_{\lambda r}$ be right weak-Fredholm inverse of $\lambda - A$. Then there exists a weakly compact operator $F \in \mathcal{W}(X)$, such that $(\lambda - A)A^w_{\lambda r} = I - F$ on X. The operator $\lambda - A - T$ can be written as follows:

$$\begin{aligned} \lambda - A - T &= \lambda - A - ((\lambda - T)A^w_{\lambda r} + F)T \\ &= (\lambda - A)(I_X - A^w_{\lambda r}T) - FT. \end{aligned}$$

Reasoning as above, we can easily obtain the rest of the proof of this assertion in the same way as (i). □

Corollary 4.9. Let X be a Banach space satisfying the property (H) and let A and T be two operators in $\mathcal{L}(X)$. Then the following assertions hold:

(i) If for each $\lambda \in \Phi_g(A)$, there exists a left weak-Fredholm inverse $A^w_{\lambda l}$ of $\lambda - A$ such that $TA^w_{\lambda l} \in \mathcal{AWC}(X)$, then

$$\sigma_{e5,g}(A + T) \subset \sigma_{e5,g}(A).$$

(ii) If for each $\lambda \in \Phi_g(A)$, there exists a right weak-Fredholm inverse $A^w_{\lambda r}$ of $\lambda - A$ such that $A^w_{\lambda r}T \in \mathcal{AWC}(X)$, then

$$\sigma_{e5,g}(A + T) \subset \sigma_{e5,g}(A).$$

Proof. The proof is similar to the proof of Theorem 4.8, it suffices to replace Theorem 4.7 by Theorem 4.4. □

Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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