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Matrix theory over ringoids

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Abstract. We will study the notion of right distributive ringoids over a field which are neither rings, semi-rings, semi-hyperrings nor near-rings. Matrices over ringoids are defined, and new concepts such as top-row-determinate and down-row-determinate related to 2×2 matrices over a ringoid are introduced. Moreover, we investigate the notions of the (strongly, (very-) weak) orthogonality of vectors over a ringoid. Beside, we discuss the notion of incident vectors and define the concept of $\alpha - K$-sphere on a ringoid, where K is a field and investigate some of their properties. Finally, we show that in a commutative ringoid all of the vectors are strongly orthogonal.

1. Introduction

The theory of groupoids [1, 2] has been introduced by some researchers. The notion of a linear groupoid was applied to the study of Fibonacci sequences in groupoids [3]. Kim et al. [5] introduced the notion of generalized commutative law in algebras, and showed that every pre-commutative BCK-algebra is bounded. In the study of pre-commutativity in groupoids, they proved that if a linear groupoid is left-(right-) pre-commutative, then it is abelian. Hwang et al. [4] discussed some implicativities for groupoids and BCK-algebras. They characterized linear groupoids for implicative groupoids.

Neggers et al. [8] introduced the notion of a ringoid as a generalization of a ring, semi-ring, near-ring, and discussed several properties of d-algebra ringoids, left zero ringoids, (r, s)-ringoids, and left distributive ringoids. Also, they investigated geometric interpretations of the parallelism of vectors in several ringoids.

The notion of a ring is a generalization of a ring of integers, i.e., $(\mathbf{Z}, +, 0)$ is an abelian group and $(\mathbf{Z}, \cdot)$ is a semigroup, and left- and right-distributive laws. If we consider the multiplication of integers, it can be represented by the addition of integers. Moreover, the distributive laws are not necessary in some cases. If we define a binary operation "$*$" on $\mathbf{Z}$ by $x * y := x \cdot (x - y)$, then we obtain $5 * 3 = 5 \cdot (5 - 3) = (5 - 3) + (5 - 3) + (5 - 3) + (5 - 3) + (5 - 3) = 10$. In this calculation we can find that $(\mathbf{Z}, +, 0)$ is an abelian group and $(\mathbf{Z}, *)$ is a groupoid. From this observation, we may construct a notion of a ringoid which can be another generalization of a ring, near-ring, pseudo-ring, semihyperring, etc..

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In this paper, we discuss right distributive ringoids in linear groupoids, and introduce the notion of ∇-product which is similar to the cross product, and investigate several properties in ringoids. By using the notion of a top-row determinant we discuss some properties of matrices over ringoids. Moreover, we discuss the notions of the (strongly, (very-) weak) orthogonality and discuss the notion of incident vectors and define the concept of $\alpha - K$-sphere on a ringoid, where K is a field. Finally, we show that in a commutative ringoid all of the vectors are strongly orthogonal.

2. Preliminaries

A groupoid $(X, *)$, i.e., X is a nonempty set and "$*$" is a binary operation on X, is said to be a *right zero* (resp., *left zero*) *semigroup* if $x * y = y$ (resp., $x * y = x$), for any $x, y \in X$.
Let $\mathbf{R}$ be the set of all real numbers. We define a binary operation "$*$" on $\mathbf{R}$ by

$$x * y := A + Bx + Cy, \tag{1}$$

for all $x, y \in \mathbf{R}$, where $A, B, C \in \mathbf{R}$. We call such a groupoid $(\mathbf{R}, *)$ a *linear groupoid* [4, 5] over real numbers.

Definition 2.1. ([8]) An algebra $(X, *, +, 0)$ of type $(2, 2, 0)$ is said to be a *ringoid* ([8]) if it satisfies the following conditions:

(I) $(X, +, 0)$ is an abelian group,
(II) $(X, *)$ is a groupoid.

Example 2.2. ([8]) *Let $(\mathbf{R}, +, \cdot, 0, 1)$ be the field of real numbers. Define a binary operation "$*$" on $\mathbf{R}$ by $x * y := x \cdot (x - y)$, for all $x, y \in \mathbf{R}$. Then $(\mathbf{R}, *, +, 0)$ is a ringoid, but it is neither a ring nor a recognized type of generalization of a ring such as semi-ring, near-ring, etc..*

Definition 2.3. ([8]) (a) A ringoid $(X, *, +, 0)$ is said to be

- *left distributive* if $x * (y + z) = (x * y) + (x * z)$,
- *right distributive* if $(x + y) * z = (x * z) + (y * z)$,

for all $x, y, z \in X$.
(b) A ringoid $(X, *, +, 0)$ is said to be a *distributive ringoid* if it is both left distributive and right distributive.
(c) A ringoid $(X, *, +, 0)$ is said to be a *left* (resp., *right*) *zero ringoid* if $(X, *)$ is a left (resp., right) zero semigroup.

Given a ringoid $(X, *, +, 0)$, we consider the Cartesian product X^n consisting of vectors $\vec{x} = (x_1, \cdots, x_n)$. For any $\vec{x} = (x_1, \cdots, x_n)$ and $\vec{y} = (y_1, \cdots, y_n) \in X^n$, $\vec{x} = \vec{y}$ if and only if $x_i = y_i$, for all $i \in \{1, \cdots, n\}$, and we have *a natural vector addition*:

$$\vec{x} \oplus \vec{y} = (x_1 + y_1, \cdots, x_n + y_n) \tag{2}$$

which produces an abelian group $(X^n, \oplus, \vec{0})$ where $\vec{0} = (0, \cdots, 0)$, the additive identity of X^n, and a *natural induced product*:

$$\vec{x} \otimes \vec{y} = (x_1 * y_1, \cdots, x_n * y_n). \tag{3}$$

We see that $(X^n, \otimes, \oplus, \vec{0})$ is again a ringoid. Define a *natural scaler product* over a ringoid as follows:

$$\vec{x} \star \vec{y} = x_1 * y_1 + \cdots + x_n * y_n \tag{4}$$

and a *natural projection*:

$$\pi(\vec{x}) := x_1 + \cdots + x_n, \tag{5}$$

for all $\vec{x}, \vec{y} \in X^n$ (see, [8]).

Proposition 2.4. ([8]) (*i*) *If* $(X, *)$ *is a left zero semigroup, then* $(X^n, \otimes)$ *is a left zero semigroup.*
(*ii*) *If* $(X, *, +, 0)$ *is a left zero ringoid, then* $(X^n, \otimes, \oplus, \vec{0})$ *is a left zero ringoid.*

Let $(X, *, +, 0)$ be a ringoid. Given $\vec{x}, \vec{y} \in X^n$, Neggers et al. [8] defined two functions as follows:

$$S(\vec{x}, \vec{y}) = (\vec{x} \otimes \vec{y}) \star (\vec{y} \otimes \vec{x}) + (\vec{y} \otimes \vec{x}) \star (\vec{x} \otimes \vec{y})$$

and

$$T(\vec{x}, \vec{y}) = (\vec{x} \otimes \vec{x}) \star (\vec{y} \otimes \vec{y}) + (\vec{y} \otimes \vec{y}) \star (\vec{x} \otimes \vec{x}).$$

They discussed the notion of a parallel, and their applications to several ringoids.

3. Right distributive ringoids in linear groupoids

In this section, we discuss right distributive ringoids related to linear groupoids over a field K, , and show that it is not a ring. We assume that $(K, \cdot, +, 0, 1)$ is a field, and $\alpha, \beta, \gamma \in K$. Consider (1) on K and define binary operation "$\oplus$" on K by

$$x \oplus y := \alpha + \beta x + \gamma y, \tag{6}$$

for all $x, y \in K$.

Lemma 3.1. *Let* $(K, *, \oplus)$ *be an algebra defined by the operations* $*$ *and* $\oplus$*. If it satisfies the right distributive law and* $\beta + \gamma \neq 1$*, then* $x * y = \frac{\alpha(1-B)}{1-(\beta+\gamma)} + Bx$*, for all* $x, y \in K$*.*

Proof. Given $x, y, z \in K$, we have

$$\begin{aligned} (x \oplus y) * z &= (\alpha + \beta x + \gamma y) * z \\ &= A + B(\alpha + \beta x + \gamma y) + Cz \\ &= A + B\alpha + B\beta x + B\gamma y + Cz. \end{aligned} \tag{7}$$

On the other hand,

$$\begin{aligned} (x * z) \oplus (y * z) &= \alpha + \beta(x * z) + \gamma(y * z) \\ &= \alpha + \beta(A + Bx + Cz) + \gamma(A + By + Cz) \\ &= \alpha + A(\beta + \gamma) + B\beta x + B\gamma y + (C\beta + C\gamma)z. \end{aligned} \tag{8}$$

Since the right distributive law holds, by (7) and (8), we obtain

$$A + B\alpha + B\beta x + B\gamma y + Cz = \alpha + A(\beta + \gamma) + B\beta x + B\gamma y + (C\beta + C\gamma)z.$$

It follows that $A + B\alpha = \alpha + A(\beta + \gamma)$ and $C = C\beta + C\gamma$. Thus we obtain $A(1 - (\beta + \gamma)) = \alpha(1 - B)$ and $C(1 - (\beta + \gamma)) = 0$. Since $\beta + \gamma \neq 1$, we obtain $C = 0$ and $A = \frac{\alpha(1-B)}{1-(\beta+\gamma)}$. Hence

$$x * y = \frac{\alpha(1 - B)}{1 - (\beta + \gamma)} + Bx. \tag{9}$$

□

Theorem 3.2. *Let* $(K, *, \oplus, \xi)$ *be a right distributive ringoid. If* $\beta + \gamma \neq 1$*, then* $\beta = \gamma = 1$*,* $\xi = -\alpha$ *and*

$$x * y = \alpha(B - 1) + Bx, \tag{10}$$

for all $x, y \in K$*.*

Proof. If $\beta + \gamma \neq 1$, then we have $x * y = \frac{\alpha(1-B)}{1-(\beta+\gamma)} + Bx$ and $x \oplus y = \alpha + \beta x + \gamma y$, for all $x, y \in K$ by Lemma 3.1. Assume that $(K, \oplus, \xi)$ is an abelian group with zero element ξ. Then $x \oplus \xi = \xi \oplus x = x$, for all $x \in K$. It follows that

$$\alpha + \beta x + \gamma\xi = \alpha + \beta\xi + \gamma x = x, \tag{11}$$

which shows that $\beta x + \gamma\xi = \beta\xi + \gamma x$, and so $\beta(x - \xi) = \gamma(x - \xi)$. Therefore $(\beta - \gamma)(x - \xi) = 0$, for all $x \in K$, and so $\beta = \gamma$. By (11), we get $(x - \alpha)(1 - \beta) = 0$, for all $x \in K$. Hence we obtain $\beta = 1$. Since $\alpha + \beta(x + \xi) = x$, for all $x \in K$, if we take $x := -\xi$, then $\alpha + \beta(-\xi + \xi) = -\xi$, which shows that $\xi = -\alpha$. By Lemma 3.1, we have $x * y = \frac{\alpha(1-B)}{1-(\beta+\gamma)} + Bx = \alpha(B - 1) + Bx$ and $x \oplus y = \alpha + x + y$, for all $x, y \in K$. □

Example 3.3. *Let* $(\mathbf{R}, +, \cdot, 0, 1)$ *be the field of real numbers. If we take* $\alpha := 1, B := 3$ *in* (10)*, then* $x * y = 2 + 3x$ *and* $x \oplus y = 1 + x + y$*, for all* $x, y \in \mathbf{R}$*. It follows that* $(x \oplus y) * z = 5 + 3(x + y) = (x * z) \oplus (y * z)$*, for all* $x, y, z \in K$*. Hence* $(\mathbf{R}, *, \oplus, -1)$ *is a right distributive ringoid, but not a left distributive ringoid, since* $x * (y \oplus z) = 2 + 3x \neq 5 + 6x = x * y \oplus x * z$*.*

Remark 3.4. *The right distributive ringoid* $(K, *, \oplus, -\alpha)$ *described in Theorem 3.2 need not be a ring in general. It is enough to show that* $(K, *)$ *is not a semigroup. Given* $x, y, z \in K$*, we have* $x * (y * z) = (1 - B)\xi + Bx$*. On the other hand,*

$$(x * y) * z = (1 - B)\xi + B(x * y) = (1 - B)\xi + B((1 - B)\xi + Bx) = (1 + B)(1 - B)\xi + B^2x.$$

If we assume that $(K, *)$ *is a semigroup, then*

$$(1 - B)\xi + Bx = (1 + B)(1 - B)\xi + B^2x.$$

It follows that $B(B - 1)(x - \xi) = 0$ *for all* $x \in K$*. Since* K *is a field, we have either* $B = 0$ *or* $B = 1$*. If* $B = 0$*, then* $x * y = -\alpha = \xi$*, for all* $x, y \in K$*, a trivial semigroup. If* $B = 1$*, then* $x * y = x$*, for all* $x, y \in K$*, a left zero semigroup. This shows that if we take* $B \in K$ *so that* $B(B - 1) \neq 0$*, then* $(K, *)$ *can not be a non-trivial semigroup.*

We constructed an example of a right distributive ringoid which is neither a left distributive ringoid nor a ring.

In Lemma 3.1 and Theorem 3.2, we have discussed the case of $\beta + \gamma \neq 1$. From now on, we discuss the case of $\beta + \gamma = 1$.

Theorem 3.5. *Let* $(K, *, \oplus)$ *be an algebra. If* $\beta + \gamma = 1$*, then there is no right distributive ringoid over* K*.*

Proof. Assume that there is a right distributive ringoid $(K, *, \oplus, \xi)$ over a field K satisfying (1) and (6) with $\beta + \gamma = 1$. Since $x * (y * z) = (x * y) * z$, for all $x, y, z \in K$, by (7) and (8), we obtain

$$A + B\alpha + B\beta x + B\gamma y + Cz = \alpha + A(\beta + \gamma) + B\beta x + B\gamma y + C(\beta + \gamma)z.$$

It follows that $\alpha(B - 1) = 0$. Assume that $\alpha = 0$. Since $\beta + \gamma = 1$, we have $x \oplus y = \beta x + \gamma y = \beta x + (1 - \beta)y$. We claim that $(K, \oplus, \xi)$ is not an abelian group. Assume that $(K, \oplus, \xi)$ is an abelian group. Then $x \oplus \xi = x$, for all $x \in K$. It follows that $x = x \oplus \xi = \beta x + (1 - \beta)\xi$, and hence $(1 - \beta)(x - \xi) = 0$, for all $x \in K$. It leads to that $\beta = 1$, and hence $x \oplus y = 1x + (1 - 1)y = x$, for all $x, y \in K$. This shows that $(K, \oplus)$ is a left zero semigroup, but not a group, which is a contradiction.

If we assume $B = 1$, then $x \oplus y = \alpha + \beta x + (1 - \beta)y$, and $x * y = A + x + Cy$, for all $x, y, z \in K$. We claim that $(K, \oplus, \xi)$ can not be an abelian group. Assume that $(K, \oplus, \xi)$ is an abelian group. Then $x \oplus \xi = \xi \oplus x = x$, for all $x \in K$. It follows that

$$\alpha + \beta x + (1 - \beta)\xi = \alpha + \beta\xi + (1 - \beta)x = x, \tag{12}$$

for all $x \in K$. Hence we have $(2\beta - 1)(x - \xi) = 0$, for all $x \in K$, which shows that $\beta = \frac{1}{2}$. If we take $\beta := \frac{1}{2}$ in (12), then

$$\alpha + \frac{1}{2}x + \frac{1}{2}\xi = x, \tag{13}$$

for all $x \in K$. If we take $x := \xi$ in (13), then $\alpha + \xi = \xi$, and hence $\alpha = 0$. Thus $x \oplus y = \frac{1}{2}(x + y)$, for all $x, y \in K$. It follows that $x = x \oplus \xi = \frac{1}{2}(x + \xi)$, for all $x \in K$. Hence $2x = x + \xi$, for all $x \in K$, i.e., $x = \xi$, for all $x \in K$. This shows that $|K| = 1$, which is a contradiction. □

4. Matrices over a ringoid

In this section, we discuss new notions as top-row-determinant (resp., down-row-determinant) of a matrix A (briefly, $TRD(A)$ (resp., $DRD(A)$)) over a ringoid.

Given ringoid $(X, *, \oplus, 0,)$, $\vec{x} = (x_1, \cdots, x_n) \in X^n$ and $a \in X$, we define an operation "$\odot$" by $a \odot \vec{x} = (a * x_1, \cdots, a * x_n)$ and $\vec{x} \odot a = (x_1 * a, \cdots, x_n * a)$.

Example 4.1. *Consider the ringoid* $(\mathbf{R}, *, +, 0)$ *as in Example 2.2. Given* $\vec{x} = (3, -1, 6) \in \mathbf{R}^3$ *and* $5 \in \mathbf{R}$, *we have* $5 \odot \vec{x} = (5*3, 5*(-1), 5*6) = (5\cdot(5-3), 5\cdot(5-(-1)), 5\cdot(5-6)) = (10, 30, -5)$ *and* $\vec{x} \odot 5 = (3*5, (-1)*5, 6*5) = (3\cdot(3-5), (-1)\cdot(-1-5), 6\cdot(6-5)) = (-6, 6, 6)$. *It follows that* $5 \odot \vec{x} \neq \vec{x} \odot 5$.

Example 4.2. (a) *Let* $(\mathbf{Q}, +, \cdot, 0, 1)$ *be the field of rational numbers. Define two binary operations "$*$" and "$\oplus$" on* $\mathbf{Q}$ *by* $x * y = -\frac{5}{2} - y$ *and* $x \oplus y = 5 + 2x + 3y$, *for all* $x, y \in \mathbf{Q}$. *Then* $(\mathbf{Q}, *, \oplus, -5)$ *is a left distributive ringoid (see,* [8]).

(b) *Let* $(\mathbf{R}, +, \cdot, 0, 1)$ *be the field of real numbers. Define two binary operations "$*$" and "$\oplus$" on* $\mathbf{R}$ *by* $x * y = y$ *and* $x \oplus y = 3 + x + y$, *for all* $x, y \in \mathbf{R}$. *Then* $(\mathbf{R}, *, \oplus, -3)$ *is a right distributive ringoid, since* $(\mathbf{R}, *)$ *is a right zero semigroup, we get* $(\mathbf{R}, *, \oplus, -3)$ *is a right zero ringoid.*

Proposition 4.3. *Let* $(X, *, +, 0)$ *be a right (resp., left) zero ringoid and* $a \in X$. *Then* $a \odot \vec{x} = \vec{x}$ *and* $\vec{x} \odot a = \vec{a}$ *(resp.,* $\vec{x} \odot a = \vec{x}$ *and* $a \odot \vec{x} = \vec{a}$*), for all* $\vec{x} \in X^n$.

Proof. Assume that $(X, *)$ is a right zero semigroup. If $\vec{x} \in X^n$ and $a \in X$, then $a \odot \vec{x} = a \odot (x_1, \cdots, x_n) = (a * x_1, \cdots, a * x_n) = (x_1, \cdots, x_n) = \vec{x}$ and $\vec{x} \odot a = (x_1, \cdots, x_n) \odot a = (x_1 * a, \cdots, x_n * a) = (a, \cdots, a) = \vec{a}$. Similarly, if $(X, *)$ is a left zero semigroup, we obtain $\vec{x} \odot a = \vec{x}$ and $a \odot \vec{x} = \vec{a}$. □

Proposition 4.4. *Let* $(X, *, +, 0)$ *be a left (resp., right) distributive ringoid. If* $a \in X$ *and* $\vec{x}$ *and* $\vec{y}$ *in* X^n, *then*

(*i*) $a \odot (\vec{x} \oplus \vec{y}) = (a \odot \vec{x}) \oplus (a \odot \vec{y})$ *(resp.,* $(\vec{x} \oplus \vec{y}) \odot a = (\vec{x} \odot a) \oplus (\vec{y} \odot a)$*),*

(*ii*) $\vec{x} \odot (a + b) = (\vec{x} \odot a) \oplus (\vec{x} \odot b)$ *(resp.,* $(a + b) \odot \vec{x} = (a \odot \vec{x}) \oplus (b \odot \vec{x})$*),*

(*iii*) $(a \odot \vec{x}) \star (a \odot \vec{y}) = a * (\vec{x} \star \vec{y})$ *(resp.,* $(\vec{x} \odot a) \star (\vec{y} \odot a) = (\vec{x} \star \vec{y}) * a$*),*

(*iv*) *if* π *is natural projection and* $\vec{x} \in \ker \pi$, *then* $\pi(a \odot \vec{x}) = a * \pi(\vec{x}) = a * 0$ *(resp.,* $\pi(\vec{x} \odot a) = \pi(\vec{x}) * a = 0 * a$*).*

Given a ringoid $(X, *, +, 0)$, we consider a 2×2 matrix $A = \begin{bmatrix} x & z \\ y & t \end{bmatrix}$, where $x, y, z, t, a \in X$. The *top-row-determinant* (denoted by $TRD(A)$) of A is defined by

$$TRD(A) := x * t - z * y, \tag{14}$$

and we define a *down-row-determinant* (denoted by $DRD(A)$) by

$$DRD(A) := y * z - t * x. \tag{15}$$

The set of all 2×2 matrices on ringoid $(X, *, +, 0)$ is denoted by $M_{2\times2}(X)$. We see that $(Mat_{2\times2}(X), +_M, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix})$ is an abelian group, where $+_M$ is the ordinary sum of 2×2 matrices. Notice that, if either $A = \begin{bmatrix} x & x \\ x & x \end{bmatrix}$ or $A = \begin{bmatrix} x & x \\ y & y \end{bmatrix}$, then $TRD(A) = DRD(A) = 0$.

Example 4.5. *Consider the ringoid* $(\mathbf{R}, *, +, 0)$ *as in Example 2.2. If* $A = \begin{bmatrix} 2 & -4 \\ 5 & 6 \end{bmatrix}$, *then* $TRD(A) = 2*6-(-4)*5 = -44$ *and* $DRD(A) = 5*(-4) - 6*2 = 22$.

A ringoid $(X, *, +, 0)$ is said to be a commutative ringoid, if $(X, *)$ is a commutative groupoid, i.e., $x * y = y * x$, for all $x, y \in X$.

Example 4.6. (a) *Let* $(\mathbf{R}, +, \cdot, 0, 1)$ *be the field of real numbers. Define a binary operation "$*$" on* $\mathbf{R}$ *by* $x * y = xy$ *and* $x \oplus y = x + y$, *for all* $x, y \in \mathbf{R}$. *Then* $(\mathbf{R}, *, \oplus, 0)$ *is a commutative ringoid.*
(b) *Consider the ringoid* $(\mathbf{R}, *, +, 0)$ *as in Example 2.2. It is not commutative, since*

$$2 * 3 = 2 \cdot (2-3) = -2 \neq 3 = 3 \cdot (3-2) = 3 * 2.$$

Proposition 4.7. *Let* $(X, *, +, 0)$ *be a commutative ringoid* $(X, *, +, 0)$. *Then* $TRD(A) = -DRD(A)$.

We extend the definition of $\odot$ to the set $M_{2\times 2}(X)$ of all 2×2 matrices as follows:

$$a \odot A = \begin{bmatrix} a*x & a*z \\ a*y & a*t \end{bmatrix} \text{ and } A \odot a = \begin{bmatrix} x*a & z*a \\ y*a & t*a \end{bmatrix}. \tag{16}$$

Proposition 4.8. *Let* $(X, *, +, 0)$ *be a left (resp., right) distributive ringoid, and let* $a \in X$ *and* $A \in M_{2\times 2}(X)$. *Then* $TRD(a \odot A) = a * TRD(A)$ *(resp.,* $TRD(A \odot a) = TRD(A) * a$*).*

Proof. Assume that $(X, *, +, 0)$ is a left distributive ringoid, $a \in X$ and $A \in M_{2\times 2}(X)$. Then

$$\begin{aligned} TRD(a \odot A) &= (a*x)*(a*t) - (a*z)(a*y) \\ &= a*(x*t) - a*(z*y) \\ &= a*(x*t - z*y) \\ &= a*TRD(A). \end{aligned}$$

Similarly, if $(X, *, +, 0)$ is a right distributive ringoid, then $TRD(A \odot a) = TRD(A) * a$. □

Proposition 4.9. *Let* $(X, *, +, 0)$ *be a right (resp., left) zero ringoid,* $A \in M_{2\times 2}(X)$ *and* $a \in X$. *Then*

(*i*) $a \odot A = A$ *and* $A \odot a = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$ *(resp.,* $A \odot a = A$ *and* $a \odot A = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$*),*

(*ii*) $TRD(a \odot A) = TRD(A)$ *and* $TRD(A \odot a) = 0$ *(resp.,* $TRD(A \odot a) = TRD(A)$ *and* $TRD(a \odot A) = 0$*).*

Proof. (i) Given $a \in X$ and $A = \begin{bmatrix} x & z \\ y & t \end{bmatrix}$, where $x, y, z, t \in X$, since $(X, *)$ is a right zero semigroup, i.e., $x * y = y$, for any $x, y \in X$, we obtain

$$a \odot A = a \odot \begin{bmatrix} x & z \\ y & t \end{bmatrix} = \begin{bmatrix} a*x & a*z \\ a*y & a*t \end{bmatrix} = \begin{bmatrix} x & z \\ y & t \end{bmatrix} = A$$

and

$$A \odot a = \begin{bmatrix} x & z \\ y & t \end{bmatrix} \odot a = \begin{bmatrix} x*a & z*a \\ y*a & t*a \end{bmatrix} = \begin{bmatrix} a & a \\ a & a \end{bmatrix}.$$

Similarly, if $(X, *)$ is a left zero semigroup, we prove that $A \odot a = A$ and $a \odot A = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$. (ii) Using (i), the proof is obvious. □

Example 4.10. *Let* $(\mathbf{Q}, +, \cdot, 0, 1)$ *be the field of rational numbers. Define two binary operations "$*$" and "$\oplus$" on* $\mathbf{Q}$ *by* $x * y = x$ *and* $x \oplus y = x \cdot y$ *(resp.,* $x \oplus y = x + y$*), for all* $x, y \in \mathbf{Q}$. *Then* $(\mathbf{Q}, *, \oplus, 1)$ *(resp.,* $(\mathbf{Q}, *, \oplus, 0)$*) is a left zero ringoid.*

Given a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Mat_{2\times 2}(X)$, we define a matrix $A^S := \begin{bmatrix} a & b \\ c & d \end{bmatrix}^S = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$. It follows that

$$TRD(A^S) = TRD(\begin{bmatrix} a & b \\ c & d \end{bmatrix}^S) = TRD(\begin{bmatrix} c & d \\ a & b \end{bmatrix}) = c * b - d * a = DRD(A). \tag{17}$$

Notice that $(A^S)^S = A$. Generally speaking, for any $n \in \mathbb{N}$, we define the following:

$$\begin{bmatrix} x_l & x_k \\ y_l & y_k \end{bmatrix}^{S^n} := \begin{cases} \begin{bmatrix} x_l & x_k \\ y_l & y_k \end{bmatrix} & \text{if } n \text{ is even}, \\ \\ \begin{bmatrix} y_l & y_k \\ x_l & x_k \end{bmatrix} & \text{if } n \text{ is odd}. \end{cases} \tag{18}$$

Example 4.11. *In Example 4.5,* $TRD(A^S) = 5 * (-4) - 6 * 2 = 5 \cdot (5 - (-4)) - 6 \cdot (6 - 2) = 21$.

We introduce a new notion, ∇-product, defined on X^n, which is similar to the cross product on $\mathbf{R}^3$ as follows: Given $\vec{x}, \vec{y} \in X^n$, $(n \geq 3)$, we define a ∇-product $\vec{x} \nabla \vec{y}$, i.e.,

$$\vec{x} \nabla \vec{y} = (x_1, \cdots, x_n) \nabla (y_1, \cdots, y_n) = \vec{z} = (z_1, \cdots, z_n), \tag{19}$$

where

$$z_i = (-1)^i \left(\sum_{\substack{l<k, \\ l,k \neq i}} (-1)^{l+k}\, TRD \begin{bmatrix} x_l & x_k \\ y_l & y_k \end{bmatrix}^{S^{i-1}} \right). \tag{20}$$

For example, if $n = 3$, then

$$z_1 = (-1)^1 \left((-1)^{2+3} TRD \begin{bmatrix} x_2 & x_3 \\ y_2 & y_3 \end{bmatrix} \right) = x_2 * y_3 - x_3 * y_2,$$

$$z_2 = (-1)^2 \left((-1)^{1+3} TRD \begin{bmatrix} x_1 & x_3 \\ y_1 & y_3 \end{bmatrix}^S \right) = y_1 * x_3 - y_3 * x_1,$$

$$z_3 = (-1)^3 \left((-1)^{1+2} TRD \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}^{S^2} \right) = x_1 * y_2 - x_2 * y_1.$$

It follows that

$$(x_1, x_2, x_3) \nabla (y_1, y_2, y_3) = (x_2 * y_3 - x_3 * y_2,\ y_1 * x_3 - y_3 * x_1,\ x_1 * y_2 - x_2 * y_1).$$

We may define another product, called a "$\times$-product", on a ringoid $(X, *, +, 0)$ as follows:

$$\begin{aligned} (x_1, x_2, x_3) \times (y_1, y_2, y_3) &:= (TRD \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}, -TRD \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix}, TRD \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}) \\ &= (x_2 * y_3 - x_3 * y_2,\ x_3 * y_1 - x_1 * y_3,\ x_1 * y_2 - x_2 * y_1). \end{aligned}$$

Proposition 4.12. *Let* $(X, *, +, 0)$ *be a ringoid. Then* $(X^n, \nabla, \oplus, \vec{0})$ *is a ringoid.*

Proposition 4.13. *Let* $(X, *, +, 0)$ *be a commutative ringoid. Then*

(i) $\vec{x} \nabla \vec{y} = -(\vec{y} \nabla \vec{x})$, *for all* $\vec{x}, \vec{y} \in X^n$ *and* $n \geq 3$,

(ii) $\vec{x} \nabla \vec{x} = \vec{0}$, *for all* $\vec{x} \in X^n$ *and* $n \geq 3$, (*ii*) $\vec{x}\ \nabla\ \vec{y} = \vec{x} \times \vec{y}$, *for all* $\vec{x}, \vec{y} \in X^3$.

Corollary 4.14. *If* $(X, *, +, 0)$ *is a commutative ringoid, then* $(X^n, \nabla, \oplus, \vec{0})$ *is not a commutative ringoid.*

The following example shows that the commutative law, in Proposition 4.13 is necessary, and so we do not remove it.

Example 4.15. *Consider the ringoid* $(\mathbf{R}, *, +, 0)$ *in Example 4.6. Given* $\vec{x} = (-1, 2, 4)$ *and* $\vec{y} = (7, 1, 3)$ *in* $\mathbf{R}^3$*, we obtain* $z_1 = 2 * 3 - 4 * 1 = -22$, $z_2 = 7 * 4 - 3 * (-1) = 9$ *and* $z_3 = (-1) * 1 - 2 * 7 = 12$. *It follows that* $(-1, 2, 4) \nabla (7, 1, 3) = (-22, 9, 12)$ *and* $(7, 1, 3) \nabla (-1, 2, 4) = (-6, 16, 33)$, *but* $(-1, 2, 4) \times (7, 1, 3) = (-22, 31, 12)$. *Further, we see that* $(-1, 2, 4)\nabla(-1, 2, 4) = (-12, -15, -3) \neq \vec{0}$.

We discuss the ∇-product on X^4, where $(X, *, +, 0)$ is a ringoid. Let $\vec{x} := (x_1, x_2, x_3, x_4)$, $\vec{y} := (y_1, y_2, y_3, y_4) \in X^4$. Let $\vec{z} := \vec{x}\ \nabla\ \vec{y}$. Then

$$\begin{aligned}
z_1 &= (-1)^i \left(\sum_{\substack{l<k,\\ l,k\neq i}} (-1)^{l+k}\, TRD \begin{bmatrix} x_l & x_k \\ y_l & y_k \end{bmatrix}^{S^{i-1}} \right) \\
&= -(-1)^{2+3}\, TRD \begin{bmatrix} x_2 & x_3 \\ y_2 & y_3 \end{bmatrix} - (-1)^{2+4}\, TRD \begin{bmatrix} x_2 & x_4 \\ y_2 & y_4 \end{bmatrix} - (-1)^{3+4}\, TRD \begin{bmatrix} x_3 & x_4 \\ y_3 & y_4 \end{bmatrix} \\
&= TRD \begin{bmatrix} x_2 & x_3 \\ y_2 & y_3 \end{bmatrix} - TRD \begin{bmatrix} x_2 & x_4 \\ y_2 & y_4 \end{bmatrix} + TRD \begin{bmatrix} x_3 & x_4 \\ y_3 & y_4 \end{bmatrix} \\
&= (x_2 * y_3 - x_3 * y_2) - (x_2 * y_4 - x_4 * y_2) + (x_3 * y_4 - x_4 * y_3).
\end{aligned}$$

Similarly, we obtain $z_2 = (y_1 * x_3 - y_3 * x_1) - (y_1 * x_4 - y_4 * x_1) - (y_3 + x_4 - y_4 * x_3)$, $z_3 = x_1 * y_2 - x_2 * y_1 + x_1 * y_4 - x_4 * y_1 - x_2 * y_4 + x_4 * y_2$ and $z_4 = -y_1 * x_2 + y_2 * x_1 + y_1 * x_3 - y_3 * x_1 - y_2 * x_3 + y_3 * x_2$.

Theorem 4.16. *Let* $(X, *, +, 0)$ *be a left (resp., right) distributive ringoid and let* $a \in X$. *If* $\vec{x}$ *and* $\vec{y}$ *in* X^n $(n \geq 3)$, *then* $(a \odot \vec{x}) \nabla (a \odot \vec{y}) = a \odot (\vec{x}\ \nabla\ \vec{y})$ *(resp.,* $(\vec{x} \odot a) \nabla (\vec{y} \odot a) = (\vec{x}\ \nabla\ \vec{y}) \odot a$*).*

Proof. Assume that $(X, *, +, 0)$ is a left distributive ringoid and $a \in X$. Given $\vec{x}, \vec{y} \in X^n$, we let $\vec{z} := \vec{x}\ \nabla\ \vec{y}$, i.e.,

$$\vec{x}\ \nabla\ \vec{y} = (x_1, \cdots, x_n)\nabla(y_1, \cdots, y_n) = \vec{z} = (z_1, \cdots, z_n).$$

It follows that $a \odot (\vec{x}\ \nabla\ \vec{y}) = a \odot \vec{z} = (a * z_1, \cdots, a * z_n)$, where

$$z_i = (-1)^i \left(\sum_{\substack{l<k,\\ l,k\neq i}} (-1)^{l+k} TRD \begin{bmatrix} x_l & x_k \\ y_l & y_k \end{bmatrix}^{S^{i-1}} \right) \tag{21}$$

and

$$(a \odot \vec{x}) \nabla (a \odot \vec{y}) = (a * x_1, \cdots, a * x_n)\nabla(a * y_n, \cdots, a * y_n) = \vec{w} = (w_1, \cdots, w_n),$$

where

$$w_i = (-1)^i \left(\sum_{l<k, l,k\neq i} (-1)^{l+k} TRD \begin{bmatrix} a * x_l & a * x_k \\ a * y_l & a * y_k \end{bmatrix}^{S^{i-1}} \right), \tag{22}$$

for any $i \in \{1, \cdots, n\}$. Using Proposition 4.8 and the left distributivity law, we obtain

$$\begin{aligned} w_i &= (-1)^i \left(\sum_{\substack{l<k,\\ l,k\neq i}} (-1)^{l+k} TRD \begin{bmatrix} a*x_l & a*x_k \\ a*y_l & a*y_k \end{bmatrix}^{S^{i-1}} \right) \\ &= (-1)^i \left(\sum_{\substack{l<k,\\ l,k\neq i}} (-1)^{l+k} TRD \, (a \odot \begin{bmatrix} x_l & x_k \\ y_l & y_k \end{bmatrix}^{S^{i-1}}) \right) \\ &= (-1)^i \left(\sum_{\substack{l<k,\\ l,k\neq i}} (-1)^{l+k} a * (TRD \begin{bmatrix} x_l & x_k \\ y_l & y_k \end{bmatrix}^{S^{i-1}}) \right) \\ &= a * \Big[(-1)^i \Big(\sum_{\substack{l<k,\\ l,k\neq i}} (-1)^{l+k} (TRD \begin{bmatrix} x_l & x_k \\ y_l & y_k \end{bmatrix}^{S^{i-1}}) \Big] \\ &= a * z_i. \end{aligned}$$

Hence we prove that $(a \odot \vec{x}) \nabla (a \odot \vec{y}) = a \odot (\vec{x} \nabla \vec{y})$. Similarly, if $(X, *, +, 0)$ is a right distributive ringoid, then $(\vec{x} \odot a) \nabla (\vec{y} \odot a) = (\vec{x} \nabla \vec{y}) \odot a$. □

Proposition 4.17. *Let $(X, *, +, 0)$ be a right (resp., left) zero ringoid and $a \in X$. Then $(a \odot \vec{x}) \nabla (a \odot \vec{y}) = \vec{x} \nabla \vec{y}$ and $(\vec{x} \odot a) \nabla (\vec{y} \odot a) = \vec{a} \nabla \vec{a}$ (resp., $(\vec{x} \odot a) \nabla (\vec{y} \odot a) = \vec{x} \nabla \vec{y}$ and $(a \odot \vec{x}) \nabla (a \odot \vec{y}) = \vec{a} \nabla \vec{a}$), for all $\vec{x}, \vec{y} \in X^n$ $(n \geq 3)$.*

Proof. It follows from Proposition 4.3. □

Theorem 4.18. *Let $(X, *, +, 0)$ be a ringoid. Assume $(x_l, y_l) \star (y_k, x_k) = (y_k, x_k) \star (x_l, y_l)$, for all $l < k$. If $\vec{x}, \vec{y} \in X^n$ $(n \geq 3)$, then*

$$\vec{x} \nabla \vec{y} \oplus \vec{y} \nabla \vec{x} = \vec{0}.$$

Proof. Given $\vec{x}, \vec{y} \in X^n$, if we let $\vec{z} := \vec{x} \nabla \vec{y}$ and $\vec{w} := \vec{y} \nabla \vec{x}$, then

$$z_i = (-1)^i \left(\sum_{\substack{l<k,\\ l,k\neq i}} (-1)^{l+k} TRD \begin{bmatrix} x_l & x_k \\ y_l & y_k \end{bmatrix}^{S^{i-1}} \right) \text{ and } w_i = (-1)^i \left(\sum_{\substack{l<k,\\ l,k\neq i}} (-1)^{l+k} TRD \begin{bmatrix} x_k & x_l \\ y_k & y_l \end{bmatrix}^{S^{i-1}} \right). \tag{23}$$

If we define $\vec{v} := \vec{z} \oplus \vec{w}$, then

$$v_i = z_i + w_i = (-1)^i \left(\sum_{\substack{l<k,\\ l,k\neq i}} (-1)^{l+k} \{ TRD \begin{bmatrix} x_l & x_k \\ y_l & y_k \end{bmatrix}^{S^{i-1}} + TRD \begin{bmatrix} y_l & y_k \\ x_l & x_k \end{bmatrix}^{S^{i-1}} \} \right).$$

Since $(x_l, y_l) \star (y_k, x_k) = (y_k, x_k) \star (x_l, y_l)$, for all $l < k$, we have $x_l * y_k + y_l * x_k = y_k * x_l + x_k * y_l$, and hence

$$x_l * y_k - y_k * x_l = -(y_l * x_k - x_k * y_l). \tag{24}$$

It follows that

$$TRD \begin{bmatrix} x_l & x_k \\ y_l & y_k \end{bmatrix} + TRD \begin{bmatrix} y_l & y_k \\ x_l & x_k \end{bmatrix} = x_l * y_k - y_k * x_l + y_l * x_k - x_k * y_l = 0.$$

Similarly, we obtain $TRD \left[\begin{smallmatrix} x_l & x_k \\ y_l & y_k \end{smallmatrix}\right] + TRD \left[\begin{smallmatrix} y_l & y_k \\ x_l & x_k \end{smallmatrix}\right] = 0$, proving the theorem. □

Corollary 4.19. *Let $(X, *, +, 0)$ be a left zero ringoid. Assume $x_l + y_l = x_k + y_k$, for all $l < k$. If $\overrightarrow{x}, \overrightarrow{y} \in X^n$ $(n \geq 3)$, then*

$$\overrightarrow{x} \;\nabla\; \overrightarrow{y} \;\oplus\; \overrightarrow{y} \;\nabla\; \overrightarrow{x} = \overrightarrow{0}\,.$$

Proof. Since $(X, *)$ is a left zero semigroup and $x_l + y_l = x_k + y_k$ holds, for all $l < k$, the condition (25) also holds. □

5. (Strongly (very-)weak) orthogonality of vectors over a ringoid

Let $(X, *, +, 0)$ be a ringoid. Two vectors $\overrightarrow{x}$ and $\overrightarrow{y}$ are said to be *orthogonal* if

$$\overrightarrow{x} \star \overrightarrow{y} + \overrightarrow{y} \star \overrightarrow{x} = 0. \tag{25}$$

Example 5.1. (a) *Consider ringoid $(\mathbf{R}, *, +, 0)$ in Example 2.2. Given $\vec{x}, \vec{y} \in \mathbf{R}^n$, two vectors $\overrightarrow{x}$ and $\overrightarrow{y}$ are orthogonal if and only if $\overrightarrow{x} = \overrightarrow{y}$, since*

$$0 = \overrightarrow{x} \star \overrightarrow{y} + \overrightarrow{y} \star \overrightarrow{x} = \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n}(x_i \cdot y_i) + \sum_{i=1}^{n} y_i^2 - \sum_{i=1}^{n}(y_i \cdot x_i) = \sum_{i=1}^{n}(x_i - y_i)^2.$$

Thus $x_i - y_i = 0$, and so $x_i = y_i$, for all $i \in \{1, \cdots, n\}$, i.e., $\overrightarrow{x} = \overrightarrow{y}$.

(b) *Consider right zero ringoid $(\mathbf{R}, *, \oplus, 0)$ in Example 4.2(b). Then x and $-x$, for all $x \in \mathbf{R}$, are orthogonal, since $x * (-x) + (-x) * x = -x + x = 0$ (Notice that, if $n = 1$, then $\star := *$).*

Also, in Example 4.2(a), x and $-5 - x$ are orthogonal, for all $x \in \mathbf{Q}$.

(c) *Consider commutative ringoid $(\mathbf{R}, *, \oplus, 0)$ in Example 4.6. Then x and 0 are orthogonal, for all $x \in \mathbf{R}$, since $x * 0 + 0 * x = 0 + 0 = 0$. Further, we can see that if $x \neq 0$ and $y \neq 0$ are not orthogonal, since $x * y + y * x = xy + yx = 2xy \neq 0$.*

(d) ([8]) *Consider the interval $[0, 1]$ of real numbers. Define binary operations "+" and "$*$" on it as follows: for all $x, y \in [0, 1]$,*

$$x + y = \begin{cases} x + y & \text{if } x + y < 1, \\ x + y - 1 & \text{if } x + y \geq 1, \end{cases} \quad \text{and} \quad x * y = \begin{cases} 1 & \text{if } x = y = 0, \\ y^x & \text{otherwise.} \end{cases}$$

Then $([0, 1], *, +, 0)$ is a ringoid. Then there are not any orthogonal elements in $[0, 1]$.

Proposition 5.2. *Let $(X, *, +, 0)$ be a left (resp., right) zero distributive ringoid and $a \in X$ and let $(X, *)$ be a right (resp., left) zero semigroup. If $\overrightarrow{x}$ and $\overrightarrow{y}$ are orthogonal, then $\overrightarrow{x} \odot a$ and $\overrightarrow{y} \odot a$ (resp., $a \odot \overrightarrow{x}$ and $a \odot \overrightarrow{y}$) are also orthogonal.*

Proof. Let $(X, *, +, 0)$ be a left distributive ringoid and let $(X, *)$ be a right zero semigroup. If $\overrightarrow{y}$ are orthogonal and if $a \in X$, then $\overrightarrow{x} \star \overrightarrow{y} + \overrightarrow{y} \star \overrightarrow{x} = 0$. Using Proposition 4.4(iii) and the left distributivity, we obtain

$$\begin{aligned} (a \odot \overrightarrow{x}) \star (a \odot \overrightarrow{y}) + (a \odot \overrightarrow{y}) \star (a \odot \overrightarrow{x}) &= a * (\overrightarrow{x} \star \overrightarrow{y}) + a * (\overrightarrow{y} \star \overrightarrow{x}) \\ &= a * (\overrightarrow{x} \star \overrightarrow{y} + \overrightarrow{y} \star \overrightarrow{x}) \\ &= a * 0 \\ &= 0. \end{aligned}$$

Let $(X, *, +, 0)$ be a right distributive ringoid and let $(X, *)$ be a left zero semigroup. If $\overrightarrow{y}$ are orthogonal and if $a \in X$, then $\overrightarrow{x} \star \overrightarrow{y} + \overrightarrow{y} \star \overrightarrow{x} = 0$. Using Proposition 4.4(iii) and the right distributivity, we obtain

$$\begin{aligned}(\vec{x}\odot a)\star(\vec{y}\odot a)+(\vec{y}\odot a)\star(\vec{x}\odot a) &= (\vec{x}\star\vec{y})*a+(\vec{y}\star\vec{x})*a\\ &= (\vec{x}\star\vec{y}+\vec{y}\star\vec{x})*a\\ &= 0*a\\ &= 0.\end{aligned}$$

□

Proposition 5.3. *Let $(X, *, +, 0)$ be a left zero ringoid. Then two vectors $\vec{x}$ and $\vec{y}$ $(\in X^n)$ are orthogonal if and only if $S(\vec{x},\vec{y}) = T(\vec{x},\vec{y}) = 0$.*

Proof. ($\Longrightarrow$) Assume $\vec{x}$ and $\vec{y}$ are orthogonal. Then $\vec{x}\star\vec{y}+\vec{y}\star\vec{x}=0$. Using Proposition 2.4, we obtain $\vec{x}\otimes\vec{y}=\vec{x}$. It follows that

$$S(\vec{x},\vec{y}) = (\vec{x}\otimes\vec{y})\star(\vec{y}\otimes\vec{x})+(\vec{y}\otimes\vec{x})\star(\vec{x}\otimes\vec{y}) = \vec{x}\star\vec{y}+\vec{y}\star\vec{x}=0.$$

Similarly, we obtain $T(\vec{x},\vec{y}) = 0$.

($\Longleftarrow$) Assume $S(\vec{x},\vec{y}) = 0$, for some $\vec{x},\vec{y}\in X^n$. Then by Proposition 2.4, we get $\vec{x}\otimes\vec{y}=\vec{x}$ and $\vec{y}\otimes\vec{x}=\vec{y}$. It follows that

$$0 = S(\vec{x},\vec{y}) = (\vec{x}\otimes\vec{y})\star(\vec{y}\otimes\vec{x})+(\vec{y}\otimes\vec{x})\star(\vec{x}\otimes\vec{y}) = \vec{x}\star\vec{y}+\vec{y}\star\vec{x}.$$

Therefore $\vec{x}$ and $\vec{y}$ are orthogonal. Similarly, if $T(\vec{x},\vec{y}) = 0$, then $\vec{x}$ and $\vec{y}$ are orthogonal. □

Two vectors $\vec{x}$ and $\vec{y}$ $(\in X^n)$ are said to be *incident* if

$$\vec{x}\star\vec{y}=\vec{y}\star\vec{x}. \tag{26}$$

The idea here is that in a vector-space a vector $\vec{x}$ is a position vectors from $(0,\cdots,0)$ to $(x_1,\cdots,x_n)$ and any two are naturally incident on each other from the origin. In this commutative situation $\vec{x}\star\vec{y}=\vec{y}\star\vec{x}$ as well. Given a ringoid $(K, *, +, 0)$ where K is a field, we define an $\alpha - K$*-sphere* K_α as follows:

$$K_\alpha := \{\vec{x}\in K^n \mid \vec{x}\star\vec{x}=\alpha\}.$$

Example 5.4. *Let $(\mathbf{R}, *, +, 0)$ be the ringoid discussed in Example 2.2. Given $\vec{x},\vec{y}\in\mathbf{R}^n$, we have*

$$\vec{x}\star\vec{y}=\sum_{i=1}^{n}x_i^2-\sum_{i=1}^{n}(x_i\cdot y_i) \text{ and } \vec{y}\star\vec{x}=\sum_{i=1}^{n}y_i^2-\sum_{i=1}^{n}(y_i\cdot x_i),$$

and $\vec{x}\star\vec{y}=\vec{y}\star\vec{x}$ precisely when $\sum_{i=1}^{n}x_i^2=\sum_{i=1}^{n}y_i^2$, i.e., when the vectors $\vec{x}$ and $\vec{y}$ belong to the same $\alpha - K$-sphere. This shows that two vectors $\vec{x}$ and $\vec{y}$ are incident if and only if $\vec{x}$ and $\vec{y}$ belong to the same $\alpha - \mathbf{R}$-sphere.

If take $n := 2$, $\vec{a} := (\frac{1}{2},\frac{1}{2})$ and $\vec{b} := (-\frac{1}{2},\frac{1}{2})$, then

$$\vec{a}\star\vec{b}=(\frac{1}{2},\frac{1}{2})\star(-\frac{1}{2},\frac{1}{2})=\frac{1}{2}*\frac{1}{2}+(-\frac{1}{2})*\frac{1}{2}=\frac{1}{2}\cdot(\frac{1}{2}-\frac{1}{2})+(-\frac{1}{2})\cdot(-\frac{1}{2}-\frac{1}{2})=0+\frac{1}{2}=\frac{1}{2}.$$

$$\vec{b}\star\vec{a}=(-\frac{1}{2},\frac{1}{2})\star(\frac{1}{2},\frac{1}{2})=(-\frac{1}{2})*\frac{1}{2}+\frac{1}{2}*\frac{1}{2}=(-\frac{1}{2})\cdot(-\frac{1}{2}-\frac{1}{2})+\frac{1}{2}\cdot(\frac{1}{2}-\frac{1}{2})=\frac{1}{2}+0=\frac{1}{2}.$$

Hence $\vec{a}$ and $\vec{b}$ are incident, but not orthogonal, since

$$\vec{a}\star\vec{b}+\vec{b}\star\vec{a}=\frac{1}{2}+\frac{1}{2}=1\neq 0.$$

If take $\vec{x} = (\alpha, \beta)$*, then* $\vec{x} \star \vec{x} = \alpha * \alpha + \beta * \beta = 0$*. Thus every element* $\vec{x} \in \mathbf{R}^2$ *is orthogonal to itself* (*see Example 5.1*(a))*. Furthermore,* $\vec{0} = (0,0)$ *is neither orthogonal nor incident with* $\vec{x} = (x_1, x_2) \neq \vec{0}$*, since*

$$\vec{0} \star \vec{x} = (0,0) \star (x_1, x_2) = 0 * x_1 + 0 * x_2 = 0 \cdot (0 - x_1) + 0 \cdot (0 - x_2) = 0 + 0 = 0.$$

$$\vec{x} \star \vec{0} = (x_1, x_2) \star (0,0) = x_1 * 0 + x_2 * = x_1 \cdot (x_1 - 0) + x_2 \cdot (x_2 - 0) = x_1^2 + x_2^2 \neq 0.$$

Proposition 5.5. *Let* $(X, *, +, 0)$ *be a ringoid. The column vector* $\begin{bmatrix} x_l \\ y_l \end{bmatrix}$ *is incident with column vector* $\begin{bmatrix} y_k \\ x_k \end{bmatrix}$ $(l < k)$ *if and only if* $TRD \begin{bmatrix} x_l & x_k \\ y_l & y_k \end{bmatrix} + TRD \begin{bmatrix} y_l & y_k \\ x_l & x_k \end{bmatrix} = 0$*.*

Proof. Assume a column vector $\begin{bmatrix} x_l \\ y_l \end{bmatrix}$ is incident with a column vector $\begin{bmatrix} y_k \\ x_k \end{bmatrix}$ $(l < k)$. Then

$$(x_l, y_l) \star (y_k, x_k) = (y_k, x_k) \star (x_l, y_l), \text{ i.e., } x_l * y_k + y_l * x_k = x_k * y_l + y_k * x_l.$$

It follows that

$$TRD \begin{bmatrix} x_l & x_k \\ y_l & y_k \end{bmatrix} + TRD \begin{bmatrix} y_l & y_k \\ x_l & x_k \end{bmatrix} = x_l * y_k - x_k * y_l + y_l * x_k - y_k * x_l = 0.$$

The converse is trivial, and we omit it. □

A ringoid $(X, *, +, 0)$ is said to have a *strong orthogonality condition* if it satisfies the following:
(SO_n): for all $\vec{x}, \vec{y} \in X^n$, we have

$$\vec{x} \star (\vec{x} \ \nabla \ \vec{y}) = (\vec{x} \ \nabla \ \vec{y}) \star \vec{x} = \vec{y} \star (\vec{x} \ \nabla \ \vec{y}) = (\vec{x} \ \nabla \ \vec{y}) \star \vec{y} = 0. \tag{27}$$

Proposition 5.6. *If* $(X, *, +, 0)$ *is a commutative ringoid, then* (SO_3) *holds.*

Proof. Assume $(X, *, +, 0)$ is a commutative ringoid. Given $\vec{x}, \vec{y} \in X^3$, we have

$$\begin{aligned}
\vec{x} \star (\vec{x} \ \nabla \ \vec{y}) &= (x_1, x_2, x_3) \star (x_2 * y_3 - x_3 * y_2, y_1 * x_3 - y_3 * x_1, x_1 * y_2 - x_2 * y_1) \\
&= x_1 * (x_2 * y_3 - x_3 * y_2) + x_2 * (y_1 * x_3 - y_3 * x_1) + x_3 * (x_1 * y_2 - x_2 * y_1) \\
&= x_1 * x_2 * y_3 - x_1 * x_3 * y_2 + x_2 * y_1 * x_3 - x_2 * y_3 * x_1 + x_3 * x_1 * y_2 - x_3 * x_2 * y_1 \\
&= 0.
\end{aligned}$$

and

$$\begin{aligned}
(\vec{x} \ \nabla \ \vec{y}) \star \vec{x} &= (x_2 * y_3 - x_3 * y_2, y_1 * x_3 - y_3 * x_1, x_1 * y_2 - x_2 * y_1) \star (x_1, x_2, x_3) \\
&= (x_2 * y_3 - x_3 * y_2) * x_1 + (y_1 * x_3 - y_3 * x_1) * x_2 + (x_1 * y_2 - x_2 * y_1) * x_3 \\
&= x_2 * y_3 * x_1 - x_3 * y_2 * x_1 + y_1 * x_3 * x_2 - y_3 * x_1 * x_2 + x_1 * y_2 * x_3 - x_2 * y_1 * x_3 \\
&= 0.
\end{aligned}$$

This shows that $\vec{x} \star (\vec{x} \ \nabla \ \vec{y}) = (\vec{x} \ \nabla \ \vec{y}) \star \vec{x} = 0$. Similarly, we obtain $(\vec{x} \ \nabla \ \vec{y}) \star \vec{y} = \vec{y} \star (\vec{x} \ \nabla \ \vec{y}) = 0$. □

Corollary 5.7. *Let* $(X, *, +, 0)$ *be a commutative ringoid. Then* $\vec{x} \nabla \vec{y}$ *is orthogonal with both* $\vec{x}$ *and* $\vec{y}$*.*

Proof. It follows immediately from Proposition 5.6. □

The following example shows that the commutative law in Proposition 5.6, is necessary.

Example 5.8. *Let* $(\mathbf{R}, *, +, 0)$ *be the non commutative ringoid discussed in Example 4.9. Then*

$$\begin{aligned}\vec{x} \star (\vec{x} \nabla \vec{y}) &= (-1,2,4) \star (-6,16,33)\\ &= (-1) * (-6) + 2 * 16 + 4 * 33\\ &= (-1)\cdot(-1-(-6)) + 2\cdot(2-16) + 4\cdot(4-33)\\ &= -5 - 28 - 116\\ &= -149\\ &\neq 0.\end{aligned}$$

Similarly, we see that $(\vec{x} \nabla \vec{y})\star \vec{x} = 1211$, $\vec{y} \star (\vec{x} \nabla \vec{y}) = -23$ *and* $(\vec{x} \nabla \vec{y})\star \vec{y} = 1308$.

6. Future work

We suggest three definitions related to orthogonal conditions which will be future research topics in this area.

(i) The *orthogonality* condition can be written:
(O_n): $\vec{x}, \vec{y} \in X^n$ implies:

$$\vec{x} \star(\vec{x} \nabla \vec{y}) + (\vec{x} \nabla \vec{y})\star \vec{x} = \vec{y} \star(\vec{x} \nabla \vec{y}) + (\vec{x} \nabla \vec{y})\star \vec{y} = 0. \tag{28}$$

(ii) The *weak orthogonality* condition becomes:
(WO_n): $\vec{x}, \vec{y} \in X^n$ implies:

$$\vec{x} \star(\vec{x} \nabla \vec{y}) + (\vec{x} \nabla \vec{y})\star \vec{x} + \vec{y} \star(\vec{x} \nabla \vec{y}) + (\vec{x} \nabla \vec{y})\star \vec{y} = 0. \tag{29}$$

(iii) The *very weak orthogonality* condition becomes:
(VWO_n): $\vec{x}, \vec{y} \in X^n$ implies:

$$\begin{aligned}&\vec{x} \star(\vec{x} \nabla \vec{y}) + \vec{x} \star(\vec{y} \nabla \vec{x}) + (\vec{x} \nabla \vec{y})\star \vec{x} + (\vec{y} \nabla \vec{x})\star \vec{x} + \vec{y} \star(\vec{x} \nabla \vec{y}) + \vec{y} \star(\vec{y} \nabla \vec{x})\\ &\quad + (\vec{x} \nabla \vec{y})\star \vec{y} + (\vec{y} \nabla \vec{x})\star \vec{y}\\ &\quad = 0.\end{aligned} \tag{30}$$

We observe that there is a chain of implications:

$$(SO_n) \implies (O_n) \implies (WO_n) \implies (VWO_n), \text{ for all } n \geq 3. \tag{31}$$

Open problem. Under what condition/conditions the converse of implications in (31) are valid?

7. Conclusions

In this paper, it is shown that there are many right distributive ringoids over a field which are linear groupoids, but not rings in general. Beside, we investigated (strongly, (very-) weak) orthogonality conditions of vectors in ringoids. Moreover, we introduced a new notion as top-row-determinate (resp., down-row-determinate) in a ringoid and investigated some of its properties. As a direction of research, one could extend these results to other algebraic structures to get more results. Also, we will define among various ideals in a ringoid, and investigate the relationship between them. Another direction of research, one could define linear-quadratic ringoids, and quadratic-quadratic ringoids. Since hyper algebraic structures are a generalization of algebraic structures, one could define hyper ringoids, and discuss the relationship between ordered semihyperrings (see, [6, 9, 10]).

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