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Further results on the EP-ness and co-EP-ness involving Mary inverses

Honglin Zou[a,*], Yuedi Zeng[b], Nan Zhou[c], Sanzhang Xu[d]

[a]College of Basic Science, Zhejiang Shuren University, Hangzhou 310015, China
Key Laboratory of Applied Mathematics (Putian University), Fujian Province University, Putian 351100, China
[b]School of Mathematics and Finance, Putian University, Putian 351100, China
[c]College of Basic Science, Zhejiang Shuren University, Hangzhou 310015, China
[d]Faculty of Mathematics and Physics, Huaiyin Institute of Technology, Huaian 223003, China

Abstract. Let R be a ring and $a, d_1, d_2 \in R$. First, we obtain several equivalent conditions for the equality $aa^{\parallel d_1} = a^{\parallel d_2}a$ to hold, under the condition $a \in R^{\parallel d_1} \cap R^{\parallel d_2}$. Then, when $a \in R^{\parallel \bullet d_1} \cap R^{\parallel \bullet d_2}$, the equality $a^m a^{\parallel d_1} = a^{\parallel d_2} a^m$ ($m \in \mathbb{N}$) is also investigated by means of Drazin inverses. Next, some characterizations for the invertibility of $aa^{\parallel d_1} - a^{\parallel d_2}a$ are obtained. Particularly, a number of examples are given to illustrate our results.

1. Introduction

Throughout this paper, R denotes an associative ring with unity 1 and $\mathbb{N}$ means the set of all positive integers. An involution $*$: $R \to R$ is an anti-isomorphism: $(a^*)^* = a$, $(a+b)^* = a^* + b^*$ and $(ab)^* = b^*a^*$ for all $a, b \in R$. We call R a $*$-ring if there exists an involution $*$ on R. First, we list several types of generalized inverses as follows.

An element $a \in R$ is said to be Moore-Penrose invertible with respect to the involution $*$ [18] if the following equations

$$(1)\ axa = a,\ \ (2)\ xax = x,\ \ (3)\ (ax)^* = ax,\ \ (4)\ (xa)^* = xa$$

have a common solution. Such solution is unique if it exists, and is denoted by $a^\dagger$.

The Drazin inverse [9] of $a \in R$ is the element $x \in R$ which satisfies

$$(1^k)\ a^k = a^{k+1}x \text{ for some } k \in \mathbb{N},\ \ (2)\ xax = x,\ \ (5)\ ax = xa.$$

The element x is unique if it exists and we will write $x = a^D$. The smallest such k is called the index of a, and denoted by ind(a). Particularly, if ind$(a) = 1$, then the Drazin inverse a^D is called the group inverse of a and it is denoted by $a^\#$.

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* Corresponding author: Honglin Zou
Email addresses: honglinzou@163.com (Honglin Zou), yuedizeng@gmail.com (Yuedi Zeng), nanzhou.math@zjsru.edu.cn (Nan Zhou), xusanzhang5222@126.com (Sanzhang Xu)

In 2010, Baksalary and Trenkler [1] introduced the core inverse and dual core inverse for complex matrices, which were extended to the $*$-ring case [19]. The core inverse of $a \in R$ is the unique element x (written $x = a^{\text{\textcircled{\#}}}$) satisfying

$$(1)\ axa = a,\ \ (2)\ xax = x,\ \ (3)\ (ax)^* = ax,\ \ (6)\ xa^2 = a,\ \ (7)\ ax^2 = x.$$

Similarly, the dual core inverse of $a \in R$ is the unique element $x \in R$ (written $x = a_{\text{\textcircled{\#}}}$) satisfying

$$(1)\ axa = a,\ \ (2)\ xax = x,\ \ (4)\ (xa)^* = xa,\ \ (6')\ a^2x = a,\ \ (7')\ x^2a = x.$$

The symbols R^{-1}, $R^{\dagger}$, R^{D}, $R^{\#}$, $R^{\text{\textcircled{\#}}}$ and $R_{\text{\textcircled{\#}}}$ stand for the sets of all invertible, Moore-Penrose invertible, Drazin invertible, group invertible, core invertible and dual core invertible elements of R, respectively.

As is well known, EP matrix $A \in \mathbb{C}^{n\times n}$ [20] means $\mathcal{R}(A) = \mathcal{R}(A^*)$, where $\mathcal{R}(A)$ denotes the column space of A, i.e., $AA^{\dagger} = A^{\dagger}A$. Then, a square matrix A is said to be co-EP [5] if $AA^{\dagger} - A^{\dagger}A$ is invertible. In a $*$-ring R, an element $a \in R$ is said to be EP (resp. co-EP) if $a \in R^{\dagger}$ and $aa^{\dagger} = a^{\dagger}a$ (resp. $aa^{\dagger} - a^{\dagger}a \in R^{-1}$). Many researchers studied the EP-ness and co-EP-ness in different settings, such as complex matrices, C^*-algebras, Banach algebras and rings [2, 4–8, 11, 15–17]. For the co-EP matrix, we have to mention the next results. Benítez and Rakočević [5] showed that the co-EP-ness of $A \in \mathbb{C}^{n\times n}$ implies the nonsingularity of $A \pm A^{\dagger}$, $A \pm A^*$, $AA^* \pm A^*A$ and $AA^{\dagger} \pm A^{\dagger}A$, which were extended to the nonsingularity [25] of $aA + bA^{\dagger} + cAA^{\dagger}$, $aA + bA^* + cAA^*$, $aAA^* + bA^*A + cA(A^*)^2A$, $aAA^{\dagger} + bA^{\dagger}A + cA(A^{\dagger})^2A$, where $a, b, c \in \mathbb{C}$ and $ab \neq 0$. Later, the authors [23] showed that if A is a co-EP matrix, then $aAA^{\dagger} + bA^{\dagger}A + cA(A^{\dagger})^2A + dA^{\dagger}A^2A^{\dagger}$ is nonsingular, where $a, b, c, d \in \mathbb{C}$ and $ab \neq cd$.

In 2011, Mary [13] defined a new generalized inverse called the inverse along an element (namely Mary inverse) in a ring or semigroup. The element $a \in R$ is said to be invertible along $d \in R$ [13] if there exists $b \in R$ such that

$$bad = d = dab,\ bR \subseteq dR \text{ and } Rb \subseteq Rd,$$

i.e.,

$$bab = b,\ bR = dR \text{ and } Rb = Rd.$$

If such b exists, then it is unique and is said to be the inverse of a along d, which will be denoted by $a^{\parallel d}$. In particular, $a^{\parallel 1} = a^{-1}$, $a^{\parallel a} = a^{\#}$ and $a^{\parallel a^*} = a^{\dagger}$. Moreover, if $aa^{\parallel d}a = a$, then we say that $a^{\parallel d}$ is an inner inverse of a along d, and a is inner invertible along d. Next, we use $R^{\parallel d}$ and $R^{\parallel \bullet d}$ to denote the sets of all invertible elements along d and inner invertible elements along d in the ring R, respectively.

After introducing the notion of the inverse along an element, EP and co-EP properties were investigated by means of Mary inverses. For example, Benítez and Boasso [3] gave several equivalent characterizations for the equality $aa^{\parallel d} = a^{\parallel d}a$ (when $a \in R^{\parallel d}$), which were applied in a $*$-ring by taking $d = a^*$. Wang, Mosić and Yao [22] also studied this equality in a ring. Recently, the authors [24] showed that the invertibility of $aa^{\parallel d} - a^{\parallel d}a$ is related to the invertibility of elements expressed by certain functions of a, d and suitable elements from the center of the ring.

Motivated by the above results, in this paper we will consider more general case, that is to say when $a \in R^{\parallel d_1} \cap R^{\parallel d_2}$ or $a \in R^{\parallel \bullet d_1} \cap R^{\parallel \bullet d_2}$, the equality $aa^{\parallel d_1} = a^{\parallel d_2}a$, as well as the invertibility of $aa^{\parallel d_1} - a^{\parallel d_2}a$ is investigated, extending the special case $d_1 = d_2$. In addition, the results obtained are applied to the core and dual core inverses in a $*$-ring.

The following lemmas will be used in the sequel.

Lemma 1.1. [10, Theorem 1] *Let $a \in R$. Then $a \in R^{\#}$ if and only if $a \in a^2R \cap Ra^2$. In this case, if $a = a^2x = ya^2$, then $a^{\#} = ax^2 = y^2a = yax$.*

Lemma 1.2. [14, Theorem 2.1] *Let $a, d \in R$. Then the following statements are equivalent:*

(i) $a \in R^{\parallel d}$. (ii) *$dR \subseteq daR$ and $da \in R^{\#}$.* (iii) *$Rd \subseteq Rad$ and $ad \in R^{\#}$.*

In this case, $a^{\parallel d} = d(ad)^{\#} = (da)^{\#}d$.

Lemma 1.3. [24, Lemma 3] and [21, Corollary 1] *Let $a, d \in R$. Then the following statements are equivalent:*

(i) $a \in R^{\|\bullet d}$. (ii) $d \in R^{\|\bullet a}$. (iii) $a \in R^{\|d}$ *and* $d \in R^{\|a}$.

In this case, $aa^{\|d} = d^{\|a}d$ *and* $a^{\|d}a = dd^{\|a}$.

2. Characterizations for the equality $aa^{\|d_1} = a^{\|d_2}a$

In this section, we will mainly consider two aspects. One is the characterizations for the equality $aa^{\|d_1} = a^{\|d_2}a$, when $a \in R^{\|d_1} \cap R^{\|d_2}$. The other is the equivalent conditions of the equality $a^m a^{\|d_1} = a^{\|d_2}a^m$ ($m \in \mathbb{N}$), when $a \in R^{\|\bullet d_1} \cap R^{\|\bullet d_2}$. Both of the aspects cover the special case $d_1 = d_2$. First, we have to give the following example to illustrate that $aa^{\|d_1} = a^{\|d_2}a$ does not imply $d_1 = d_2$ or $a^{\|d_2}a = a^{\|d_1}a$ in general.

Example 2.1. *Let $R = \mathbb{C}^{2\times 2}$. Then, take* $a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $d_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ *and* $d_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. *By direct computation we see that* $a^{\|d_1} = d_1$ *and* $a^{\|d_2} = d_2$. *Clearly,* $aa^{\|d_1} = a^{\|d_2}a$. *However,* $d_1 \neq d_2$ *and* $a^{\|d_2}a \neq a^{\|d_1}a$.

Inspired by [3, Theorem 7.3], we characterize the equality $aa^{\|d_1} = a^{\|d_2}a$ under the condition $a \in R^{\|d_1} \cap R^{\|d_2}$ as follows.

Theorem 2.2. *Let $a, d_1, d_2 \in R$ be such that $a \in R^{\|d_1} \cap R^{\|d_2}$. Then the following statements are equivalent:*

(i) $aa^{\|d_1} = a^{\|d_2}a$.

(ii) $d_1 = d_1a^{\|d_2}a$ *and* $d_2 = aa^{\|d_1}d_2$.

(iii) $Rd_2a \subseteq Rd_1$ *and* $ad_1R \subseteq d_2R$.

(iv) $Rd_1 \subseteq Rd_2a$ *and* $d_2R \subseteq ad_1R$.

(v) $Rd_1 = Rd_2a$ *and* $d_2R = ad_1R$.

(vi) $Rad_1 = Rd_2a$ *and* $d_2aR = ad_1R$.

Proof. (i) ⇒ (ii), (iii) and (iv). Suppose that $aa^{\|d_1} = a^{\|d_2}a$. Then, by Lemma 1.2 we deduce

$$d_1 = d_1aa^{\|d_1} = d_1a^{\|d_2}a = d_1(d_2a)^{\#}d_2a \in Rd_2a$$

and

$$d_2 = a^{\|d_2}ad_2 = aa^{\|d_1}d_2 = ad_1(ad_1)^{\#}d_2 \in ad_1R,$$

which conclude that items (ii) and (iv) hold. In addition,

$$ad_1 = aa^{\|d_1}ad_1 = a^{\|d_2}a^2d_1 = d_2(ad_2)^{\#}a^2d_1 \in d_2R$$

and

$$d_2a = d_2aa^{\|d_2}a = d_2a^2a^{\|d_1} = d_2a^2(d_1a)^{\#}d_1 \in Rd_1.$$

So, item (iii) holds.

(ii) ⇒ (i). By item (ii), we get

$$\begin{aligned} aa^{\|d_1} &= a(d_1a)^{\#}d_1 = a(d_1a)^{\#}d_1a^{\|d_2}a = aa^{\|d_1}a^{\|d_2}a = aa^{\|d_1}d_2(ad_2)^{\#}a \\ &= d_2(ad_2)^{\#}a = a^{\|d_2}a. \end{aligned}$$

(iii) ⇒ (i). Note that $ad_1 = d_2u$ and $d_2a = vd_1$, for some $u, v \in R$. So, we claim that

$$aa^{\|d_1} = ad_1(ad_1)^{\#} = d_2u(ad_1)^{\#} = a^{\|d_2}ad_2u(ad_1)^{\#} = a^{\|d_2}aad_1(ad_1)^{\#} = a^{\|d_2}a^2a^{\|d_1}.$$

On the other hand,

$$a^{\|d_2}a = (d_2a)^{\#}d_2a = (d_2a)^{\#}vd_1 = (d_2a)^{\#}vd_1aa^{\|d_1} = (d_2a)^{\#}d_2aaa^{\|d_1} = a^{\|d_2}a^2a^{\|d_1}.$$

Therefore, $aa^{\|d_1} = a^{\|d_2}a$.

(iv) ⇒ (ii). Since $Rd_1 \subseteq Rd_2a$, we obtain $d_1 = xd_2a$ for some $x \in R$. Multiplying the previous equality by $a^{\|d_2}a$ from the right, we get $d_1a^{\|d_2}a = xd_2aa^{\|d_2}a = xd_2a = d_1$. Similarly, $d_2 = aa^{\|d_1}d_2$.

(i) ⇔ (v) is clear by what we have proved just now.

(v) ⇔ (vi). Note that $Rd_1 = Ra^{\|d_1}ad_1 \subseteq Rad_1$ and $Rad_1 \subseteq Rd_1$. Hence $Rd_1 = Rad_1$. Similarly, $d_2R = d_2aR$, as required. □

Let us recall the following facts in a $*$-ring [19]: (1) $a \in R^{\textcircled{\#}} \cap R_{\textcircled{\#}}$ if and only if $a \in R^{\#} \cap R^{\dagger}$. (2) If $a \in R^{\dagger}$, then $a \in R^{\|aa^*}$ if and only if $a \in R^{\textcircled{\#}}$. In this case, $a^{\|aa^*} = a^{\textcircled{\#}}$. (3) If $a \in R^{\dagger}$, then $a \in R^{\|a^*a}$ if and only if $a \in R_{\textcircled{\#}}$. In this case, $a^{\|a^*a} = a_{\textcircled{\#}}$. (4) a is EP if and only if $a \in R^{\textcircled{\#}} \cap R_{\textcircled{\#}}$ with $aa^{\textcircled{\#}} = a_{\textcircled{\#}}a$. Then, by taking $d_1 = aa^*$ and $d_2 = a^*a$ in Theorem 2.2, we directly obtained the next results, which can been seen as the new characterizations for the EP element in a $*$-ring.

Corollary 2.3. *Let R be a $*$-ring and $a \in R^{\textcircled{\#}} \cap R_{\textcircled{\#}}$. Then, the following statements are equivalent:*

(i) *a is* EP.

(ii) $a = a_{\textcircled{\#}}a^2 = a^2a^{\textcircled{\#}}$.

(iii) *$Ra^*a^2 \subseteq Raa^*$ and $a^2a^*R \subseteq a^*aR$.*

(iv) *$Raa^* \subseteq Ra^*a^2$ and $a^*aR \subseteq a^2a^*R$.*

(v) *$Raa^* = Ra^*a^2$ and $a^*aR = a^2a^*R$.*

(vi) *$Ra^2a^* = Ra^*a^2$ and $a^*a^2R = a^2a^*R$.*

Next, we show that the equality $aa^{\|d_1} = a^{\|d_2}a$ can be described by the equations.

Proposition 2.4. *Let $a, d_1, d_2 \in R$ be such that $a \in R^{\|d_1} \cap R^{\|d_2}$. Then the following statements are equivalent:*

(i) $aa^{\|d_1} = a^{\|d_2}a$.

(ii) *There exist $x, y \in R$ such that $d_1ad_1xa = d_1$, $ayd_2ad_2 = d_2$ and $ad_1xa = ayd_2a$.*

(iii) *There exist $x', y' \in R$ such that $d_1x' = d_1$, $y'd_2 = d_2$, $Rx' \subseteq Rd_2a$ and $y'R \subseteq ad_1R$.*

Proof. (i) ⇒ (ii). Let $x = (ad_1)^{\#}a^{\|d_2}$ and $y = a^{\|d_1}(d_2a)^{\#}$. Then, it is easy to check that such x, y satisfy item (ii).

(ii) ⇒ (i). Suppose that item (ii) holds. Then, we get

$$\begin{aligned} aa^{\|d_1} &= a(d_1a)^{\#}d_1 = a(d_1a)^{\#}d_1ad_1xa = aa^{\|d_1}ad_1xa = ad_1xa \\ &= ayd_2a = ayd_2aa^{\|d_2}a = ayd_2ad_2(ad_2)^{\#}a = d_2(ad_2)^{\#}a \\ &= a^{\|d_2}a. \end{aligned}$$

(i) ⇒ (iii). Let $x' = a^{\|d_2}a$ and $y' = aa^{\|d_1}$. By Theorem 2.2 (i) and (ii), we obtain $d_1x' = d_1$ and $y'd_2 = d_2$. Also, it is clear that $Rx' = R(d_2a)^{\#}d_2a \subseteq Rd_2a$ and $y'R = ad_1(ad_1)^{\#}R \subseteq ad_1R$.

(iii) ⇒ (i). Since $Rx' \subseteq Rd_2a$ and $y'R \subseteq ad_1R$, there exist $u, v \in R$ such that $x' = ud_2a$ and $y' = ad_1v$. Hence, $Rd_1 = Rd_1x' = Rd_1ud_2a \subseteq Rd_2a$ and $d_2R = y'd_2R = ad_1vd_2R \subseteq ad_1R$. Using Theorem 2.2 (i) and (iv), we have $aa^{\|d_1} = a^{\|d_2}a$. □

In the following theorem, we consider the relationship between $ad_1 = d_2a$ and $aa^{\|d_1} = a^{\|d_2}a$.

Theorem 2.5. *Let $a, d_1, d_2 \in R$ be such that $a \in R^{\|d_1} \cap R^{\|d_2}$. Then the following statements are equivalent:*

(i) $ad_1 = d_2a$.

(ii) $aa^{\|d_1} = a^{\|d_2}a$ *and* $d_1a^{\|d_2} = a^{\|d_1}d_2$.

(iii) *There exists* $x \in R$ *such that* $d_1ad_1x = d_1$, $xd_2ad_2 = d_2$ *and* $ad_1x = xd_2a$.

Proof. (i) ⇒ (ii) and (iii). Suppose that $ad_1 = d_2a$. Then we have

$$\begin{aligned} aa^{\|d_1} &= ad_1(ad_1)^{\#} = d_2a(ad_1)^{\#} = a^{\|d_2}ad_2a(ad_1)^{\#} = a^{\|d_2}aad_1(ad_1)^{\#} \\ &= a^{\|d_2}aaa^{\|d_1} = (d_2a)^{\#}d_2aaa^{\|d_1} = (d_2a)^{\#}ad_1aa^{\|d_1} \\ &= (d_2a)^{\#}ad_1 = (d_2a)^{\#}d_2a \\ &= a^{\|d_2}a \end{aligned}$$

and

$$\begin{aligned} d_1a^{\|d_2} &= d_1(d_2a)^{\#}d_2 = d_1d_2a\left((d_2a)^{\#}\right)^2 d_2 = d_1ad_1\left((d_2a)^{\#}\right)^2 d_2 \\ &= d_1(ad_1)^{\#}(ad_1)^2\left((d_2a)^{\#}\right)^2 d_2 = a^{\|d_1}(d_2a)^2\left((d_2a)^{\#}\right)^2 d_2 \\ &= a^{\|d_1}d_2a(d_2a)^{\#}d_2 = a^{\|d_1}d_2aa^{\|d_2} \\ &= a^{\|d_1}d_2. \end{aligned}$$

Hence, item (ii) holds.

Let $x = (ad_1)^{\#} = (d_2a)^{\#}$. Then, we get $d_1ad_1x = d_1ad_1(ad_1)^{\#} = d_1aa^{\|d_1} = d_1$ and $xd_2ad_2 = d_2$ goes similarly. In addition, $ad_1x = ad_1(ad_1)^{\#} = aa^{\|d_1} = a^{\|d_2}a = (d_2a)^{\#}d_2a = xd_2a$, which means item (iii) holds.

(ii) ⇒ (i). Since $aa^{\|d_1} = a^{\|d_2}a$ and $d_1a^{\|d_2} = a^{\|d_1}d_2$, we have

$$ad_1 = ad_1aa^{\|d_1} = ad_1a^{\|d_2}a = aa^{\|d_1}d_2a = a^{\|d_2}ad_2a = d_2a.$$

(iii) ⇒ (i). Suppose that (iii) holds. Then,

$$ad_1 = ad_1ad_1x = ad_1xd_2a = xd_2ad_2a = d_2a. \quad \square$$

Now, we focus on the equivalent conditions for $a^ma^{\|d_1} = a^{\|d_2}a^m$ to hold, when $a \in R^{\|\bullet d_1} \cap R^{\|\bullet d_2}$.

Theorem 2.6. *Let* $a, d_1, d_2 \in R$ *be such that* $a \in R^{\|\bullet d_1} \cap R^{\|\bullet d_2}$ *and* $m \in \mathbb{N}$. *Then the following statements are equivalent:*

(i) $a^ma^{\|d_1} = a^{\|d_2}a^m$.

(ii) $Ra^m \subseteq Rd_1$ *and* $a^mR \subseteq d_2R$.

(iii) *There exist* $x \in Rd_1$ *and* $y \in d_2R$ *such that* $a^m = a^{m+1}x = ya^{m+1}$.

Proof. (i) ⇒ (iii). Let $x = a^{\|d_1}$ and $y = a^{\|d_2}$. Clearly, $x \in Rd_1$ and $y \in d_2R$. Also, we see that $a^m = aya^m = a^{m+1}x$ and $a^m = a^mxa = ya^{m+1}$.

(iii) ⇒ (ii) is obvious.

(ii) ⇒ (i). Let us write $a^m = ud_1 = d_2v$, for $u, v \in R$. Then,

$$a^{m+1}a^{\|d_1} = a^maa^{\|d_1} = ud_1aa^{\|d_1} = ud_1 = a^m$$

and

$$a^{\|d_2}a^{m+1} = a^{\|d_2}aa^m = a^{\|d_2}ad_2v = d_2v = a^m.$$

Hence, $a^ma^{\|d_1} = a^{\|d_2}a^{m+1}a^{\|d_1} = a^{\|d_2}a^m$. $\square$

Let $m = 1$ in Theorem 2.6, we have

Corollary 2.7. *Let* $a, d_1, d_2 \in R$ *be such that* $a \in R^{\|\bullet d_1} \cap R^{\|\bullet d_2}$. *Then the following statements are equivalent:*

(i) $aa^{\|d_1} = a^{\|d_2}a$.

(ii) *$Ra \subseteq Rd_1$ and $aR \subseteq d_2R$.*

(iii) *$a \in R^{\#}$ and $a^{\#} = a^{\|d_2}aa^{\|d_1}$.*

Proof. (i) ⇔ (ii) and (i) ⇒ (iii) are trivial by Theorem 2.6 and Lemma 1.1.

(iii) ⇒ (i). From item (iii), we deduce that

$$aa^{\|d_1} = aa^{\|d_2}aa^{\|d_1} = aa^{\#} = a^{\#}a = a^{\|d_2}aa^{\|d_1}a = a^{\|d_2}a. \quad \square$$

Applying Corollary 2.7 (i)(ii) and Lemma 1.3, we deduce the following result.

Corollary 2.8. *Let $a, b, d \in R$ be such that $a, b \in R^{\|\bullet d}$. Then, the following statements are equivalent:*

(i) *$aa^{\|d} = b^{\|d}b$.*

(ii) *$dR \subseteq aR$ and $Rd \subseteq Rb$.*

By Theorem 2.6 (iii) and [9, Theorem 4], we see that if $a \in R^{\|\bullet d_1} \cap R^{\|\bullet d_2}$ and $a^m a^{\|d_1} = a^{\|d_2}a^m$, then $a \in R^D$. So, we will characterize the equality $a^m a^{\|d_1} = a^{\|d_2}a^m$ by using Drazin inverses.

Theorem 2.9. *Let $a, d_1, d_2 \in R$ be such that $a \in R^{\|\bullet d_1} \cap R^{\|\bullet d_2} \cap R^D$ and let $m, n \geq \mathrm{ind}(a)$, $i, j, l \in \mathbb{N}$. Then the following statements are equivalent:*

(i) *$a^m a^{\|d_1} = a^{\|d_2}a^m$.*

(ii) *$a^n a^{\|d_1} = a^{\|d_2}a^n$.*

(iii) *$R(a^D)^i \subseteq Rd_1$ and $(a^D)^i R \subseteq d_2R$.*

(iv) *$(a^D)^j a^{\|d_1} = a^{\|d_2}(a^D)^j$.*

(v) *$a^l a^D a^{\|d_1} = a^{\|d_2}a^D a^l$.*

Proof. (i) ⇔ (ii). Obviously, we only need to show that (i) ⇒ (ii). Suppose that $a^m a^{\|d_1} = a^{\|d_2}a^m$.

Case 1. If $n > m$, then we get

$$a^n a^{\|d_1} = a^{n-m}(a^m a^{\|d_1}) = a^{n-m}a^{\|d_2}a^m = a^{n-m-1}(aa^{\|d_2}a)a^{m-1} = a^{n-1}.$$

Similarly, we have $a^{\|d_2}a^n = a^{n-1}$. Hence, $a^n a^{\|d_1} = a^{\|d_2}a^n$.

Case 2: If $n < m$, then by the hypotheses we conclude that

$$\begin{aligned} a^n a^{\|d_1} &= (a^D)^{m-n}(a^m a^{\|d_1}) = (a^D)^{m-n}a^{\|d_2}a^m = (a^D)^{m-n+1}(aa^{\|d_2}a)a^{m-1} \\ &= (a^D)^{m-n+1}a^m = a^D a^n. \end{aligned}$$

Similarly, we have $a^{\|d_2}a^n = a^n a^D$. So, $a^n a^{\|d_1} = a^{\|d_2}a^n$.

(i) ⇔ (iii). Since $a \in R^D$ and $m \geq \mathrm{ind}(a)$, we get

$$Ra^m = Ra^D = R(a^D)^i \text{ and } a^m R = a^D R = (a^D)^i R.$$

Then, by Theorem 2.6 we obtain the equivalence of (i) and (iii).

(i) ⇒ (iv). By the condition $a^m a^{\|d_1} = a^{\|d_2}a^m$, we have

$$(a^D)^j a^{\|d_1} = (a^D)^{m+j}a^m a^{\|d_1} = (a^D)^{m+j}a^{\|d_2}a^m = (a^D)^{m+j+1}aa^{\|d_2}a^m = (a^D)^{j+1}.$$

Similarly, we get $a^{\|d_2}(a^D)^j = (a^D)^{j+1}$. Hence, $(a^D)^j a^{\|d_1} = a^{\|d_2}(a^D)^j$.

(iv) ⇒ (v). Suppose that item (iv) holds. Then, we get

$$a^l a^D a^{\|d_1} = a^{l+j-1}(a^D)^j a^{\|d_1} = a^{l+j-1}a^{\|d_2}(a^D)^j = a^{l+j-1}a^{\|d_2}a(a^D)^{j+1} = a^{l-1}a^D.$$

Similarly, $a^{\|d_2}a^Da^l = a^Da^{l-1}$. Hence, $a^la^Da^{\|d_1} = a^{\|d_2}a^Da^l$.

(v) ⇒ (i). By the hypotheses, we conclude that

$$a^m a^{\|d_1} = a^{m+1}a^Da^{\|d_1} = a^m(aa^D)^l a^{\|d_1} = a^m(a^D)^{l-1}a^la^Da^{\|d_1} = a^m(a^D)^{l-1}a^{\|d_2}a^Da^l = a^ma^D.$$

Analogously, we get $a^{\|d_2}a^m = a^Da^m$. So, $a^ma^{\|d_1} = a^{\|d_2}a^m$. □

As a consequence of Theorem 2.9 (i) and (ii), we get the following.

Corollary 2.10. *Let R be a $*$-ring and $a \in R^{\#} \cap R_{\#}$ and $m, n \in \mathbb{N}$. Then, the following statements are equivalent:*

(i) $a^ma^{\#} = a_{\#}a^m$.

(ii) $a^na^{\#} = a_{\#}a^n$.

3. Characterizations for the invertibility of $aa^{\|d_1} - a^{\|d_2}a$

In this section, for given $a, d_1, d_2 \in R$, when $a \in R^{\|d_1} \cap R^{\|d_2}$, we investigate several equivalent conditions for the invertibility of $aa^{\|d_1} - a^{\|d_2}a$, extending related results in [24]. In the beginning, we need to give an example to show that $aa^{\|d_1} - a^{\|d_2}a \in R^{-1}$ does not imply $d_1 \neq d_2$ or $a^{\|d_2}a \neq a^{\|d_1}a$ in general.

Example 3.1. *Setting $R = M_2(\mathbb{Z}_2)$. Let $a = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $d_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $d_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then, we can check that $a^{\|d_1} = d_1$, $a^{\|d_2} = d_2$ and $aa^{\|d_1} - a^{\|d_2}a \in R^{-1}$. But, $d_1 \neq d_2$ and $a^{\|d_2}a \neq a^{\|d_1}a$.*

The following lemmas are necessary to prove our main theorems.

Lemma 3.2. [12, Theorem 3.2] and [4, Theorem 1] *Let $f, g \in R$ be idempotents. Then the following statements are equivalent:*

(i) $f - g \in R^{-1}$.

(ii) $fR \oplus gR = R$ *and* $Rf \oplus Rg = R$.

(iii) *There exist idempotents $h, k \in R$ such that $fh = h$, $hf = f$, $g(1-h) = 1-h$, $(1-h)g = g$, $kf = k$, $fk = f$, $(1-k)g = 1-k$ and $g(1-k) = g$.*

By Lemma 3.2 and the definition of the inverse along an element, we directly obtain

Lemma 3.3. *Let $a, d_1, d_2 \in R$ be such that $a \in R^{\|d_1} \cap R^{\|d_2}$. If $aa^{\|d_1} - a^{\|d_2}a \in R^{-1}$, then there exist idempotents $h, k \in R$ satisfying*

$$\begin{aligned} &aa^{\|d_1}h = h, had_1 = ad_1, a^{\|d_2}a(1-h) = 1-h, hd_2 = 0, \\ &\qquad\qquad\qquad\text{and} \\ &kaa^{\|d_1} = k, d_1k = d_1, (1-k)a^{\|d_2}a = 1-k, d_2ak = 0. \end{aligned} \tag{$*1$}$$

Denote by $C(R)$ the center of R, that is the set of such elements that commute with all elements of R. The right annihilator of $a \in R$ is defined by $a^0 = \{x \in R \mid ax = 0\}$. Now, we are ready to establish the following result concerning the invertibility of $aa^{\|d_1} - a^{\|d_2}a$.

Theorem 3.4. *Let $a, d_1, d_2 \in R$ be such that $a \in R^{\|d_1} \cap R^{\|d_2}$. Then, the following statements are equivalent:*

(i) $aa^{\|d_1} - a^{\|d_2}a \in R^{-1}$.

(ii) $r = \lambda_1(ad_1)^m + \lambda_2(d_2a)^n + \lambda_3(ad_1)^m(d_2a)^n + \lambda_4(d_2a)^n(ad_1)^m \in R^{-1}$, $\lambda_1d_1r^{-1}(ad_1)^m = d_1$, $\lambda_2(d_2a)^nr^{-1}d_2 = d_2$, $\lambda_1\lambda_2(d_2a)^nr^{-1}(ad_1)^m = -\lambda_4(d_2a)^n(ad_1)^m$ *and* $\lambda_1\lambda_2d_1r^{-1}d_2 = -\lambda_3d_1d_2$, *where* $\lambda_i \in C(R)$ $(i \in \overline{1,4})$, $\lambda_1\lambda_2 \in R^{-1}$, $\lambda_3\lambda_4 \in a^0$ *and* $m, n \in \mathbb{N}$.

Proof. (i) ⇒ (ii). Suppose that $aa^{\|d_1} - a^{\|d_2}a \in R^{-1}$. In view of Lemma 3.3, there exist idempotents $h, k \in R$ satisfying (∗1). Now, let

$$r' = \lambda_1(1-k)\left((d_2a)^{\#}\right)^n(1-h) + \lambda_2 k\left((ad_1)^{\#}\right)^m h - \lambda_3 k(1-h) - \lambda_4(1-k)h. \qquad (*2)$$

Since $\lambda_i \in C(R)$ ($i \in \overline{1,4}$) and $\lambda_3\lambda_4 \in a^0$, combining what we have shown yields that

$$\begin{aligned} rr' &= (\lambda_1(ad_1)^m + \lambda_2(d_2a)^n + \lambda_3(ad_1)^m(d_2a)^n + \lambda_4(d_2a)^n(ad_1)^m)\cdot \\ &\quad \left(\lambda_1(1-k)\left((d_2a)^{\#}\right)^n(1-h) + \lambda_2 k\left((ad_1)^{\#}\right)^m h - \lambda_3 k(1-h) - \lambda_4(1-k)h\right) \\ &= \lambda_1\lambda_2 ad_1(ad_1)^{\#}h - \lambda_1\lambda_3(ad_1)^m(1-h) + \lambda_1\lambda_2 d_2a(d_2a)^{\#}(1-h) - \lambda_2\lambda_4(d_2a)^n h \\ &\quad +\lambda_1\lambda_3(ad_1)^m d_2a(d_2a)^{\#}(1-h) + \lambda_2\lambda_4(d_2a)^n ad_1(ad_1)^{\#}h \\ &= \lambda_1\lambda_2 aa^{\|d_1}h - \lambda_1\lambda_3(ad_1)^m(1-h) + \lambda_1\lambda_2 a^{\|d_2}a(1-h) - \lambda_2\lambda_4(d_2a)^n h \\ &\quad +\lambda_1\lambda_3(ad_1)^m a^{\|d_2}a(1-h) + \lambda_2\lambda_4(d_2a)^n aa^{\|d_1}h \\ &= \lambda_1\lambda_2 h - \lambda_1\lambda_3(ad_1)^m(1-h) + \lambda_1\lambda_2(1-h) - \lambda_2\lambda_4(d_2a)^n h \\ &\quad +\lambda_1\lambda_3(ad_1)^m(1-h) + \lambda_2\lambda_4(d_2a)^n h \\ &= \lambda_1\lambda_2. \end{aligned}$$

On the other hand, one can check that $r'r = \lambda_1\lambda_2$. Owing to $\lambda_1\lambda_2 \in R^{-1}$, then we get $r \in R^{-1}$ and $r^{-1} = (\lambda_1\lambda_2)^{-1}r'$, which leads to the equality $\lambda_1 d_1 r^{-1}(ad_1)^m = \lambda_2^{-1}d_1 r'(ad_1)^m$. Now, substituting (∗2) into the previous equality, we conclude $\lambda_1 d_1 r^{-1}(ad_1)^m = d_1$. In addition, $\lambda_2(d_2a)^n r^{-1}d_2 = d_2$, $\lambda_1\lambda_2(d_2a)^n r^{-1}(ad_1)^m = -\lambda_4(d_2a)^n(ad_1)^m$ and $\lambda_1\lambda_2 d_1 r^{-1}d_2 = -\lambda_3 d_1 d_2$ go similarly.

(ii) ⇒ (i). First we show that there exist $h, k \in R$ such that $had_1 = ad_1$, $hd_2 = 0$, $d_1k = d_1$ and $d_2ak = 0$. In order to verify this, we need to define $h = (\lambda_1(ad_1)^m + \lambda_3(ad_1)^m(d_2a)^n)\, r^{-1}$ and $k = r^{-1}(\lambda_1(ad_1)^m + \lambda_4(d_2a)^n(ad_1)^m)$. By item (ii), we obtain

$$\begin{aligned} h(ad_1)^m &= (\lambda_1(ad_1)^m + \lambda_3(ad_1)^m(d_2a)^n)\, r^{-1}(ad_1)^m \\ &= (ad_1)^{m-1}a\left(\lambda_1 d_1 r^{-1}(ad_1)^m\right) - (\lambda_1\lambda_2)^{-1}(\lambda_3\lambda_4)(ad_1)^m(d_2a)^n(ad_1)^m \\ &= (ad_1)^m, \end{aligned}$$

which implies $had_1 = h(ad_1)^m\left((ad_1)^{\#}\right)^{m-1} = (ad_1)^m\left((ad_1)^{\#}\right)^{m-1} = ad_1$. Also, we get

$$\begin{aligned} hd_2 &= (r - \lambda_2(d_2a)^n - \lambda_4(d_2a)^n(ad_1)^m)\, r^{-1}d_2 \\ &= d_2 - \lambda_2(d_2a)^n r^{-1}d_2 - \lambda_4(d_2a)^n(ad_1)^{m-1}a(d_1r^{-1}d_2) \\ &= d_2 - d_2 + (\lambda_1\lambda_2)^{-1}(d_2a)^n(\lambda_3\lambda_4)(ad_1)^m d_2 \\ &= 0. \end{aligned}$$

Analogously, we have $d_1k = d_1$ and $d_2ak = 0$.

Next, our aim is to see that $aa^{\|d_1} - a^{\|d_2}a \in R^{-1}$. By Lemma 3.2, we only need to infer $aa^{\|d_1}R \oplus a^{\|d_2}aR = R$ and $Raa^{\|d_1} \oplus Ra^{\|d_2}a = R$, which is clearly equivalent to $ad_1R \oplus d_2aR = R$ and $Rad_1 \oplus Rd_2a = R$. From the invertibility of r, we get $ad_1R + d_2aR = R$. Let $x \in ad_1R \cap d_2aR$. So, $x = ad_1w_1 = d_2aw_2$, for suitable $w_1, w_2 \in R$. Hence, $x = had_1w_1 = hd_2aw_2 = 0$, which means $ad_1R \cap d_2aR = \{0\}$. Therefore, $ad_1R \oplus d_2aR = R$. Similarly, $Rad_1 \oplus Rd_2a = R$, as announced above. □

In particular, when $a \in R^{\|\bullet d_1} \cap R^{\|\bullet d_2}$, we further characterize the invertibility of $aa^{\|d_1} - a^{\|d_2}a$ as follows.

Theorem 3.5. *Let $a, d_1, d_2 \in R$ be such that $a \in R^{\|\bullet d_1} \cap R^{\|\bullet d_2}$. Then, the following statements are equivalent:*

(i) $aa^{\|d_1} - a^{\|d_2}a \in R^{-1}$.

(ii) $s = \mu_1 a + \mu_2 ad_1 + \mu_3 d_2a + \mu_4 ad_1d_2a \in R^{-1}$, *$as^{-1}a = 0$ and $\mu_2 ad_1 s^{-1}a = \mu_3 as^{-1}d_2a = a$, where $\mu_i \in C(R)$ ($i \in \overline{1,4}$) and $\mu_2\mu_3 \in R^{-1}$.*

Proof. (i) ⇒ (ii). Now, we know that there exist idempotents $h, k \in R$ satisfying (∗1). Furthermore, we find that $ha = a$ and $ak = 0$, because $ha = haa^{\parallel d_1}a = had_1(ad_1)^{\#}a = ad_1(ad_1)^{\#}a = aa^{\parallel d_1}a = a$ and $ak = aa^{\parallel d_2}ak = a(d_2a)^{\#}d_2ak = 0$. Write

$$s' = -\mu_1 k(ad_1)^{\#}(ad_2)^{\#}a(1-h) + \mu_2(1-k)(d_2a)^{\#}(1-h) + \mu_3 k(ad_1)^{\#}h - \mu_4 k(1-h).$$

Note that $a(d_2a)^{\#} = (ad_2)^{\#}a$. Then, one can check that

$$\begin{aligned} ss' &= \mu_2\mu_3 - \mu_1\mu_2 ad_1(ad_1)^{\#}(ad_2)^{\#}a(1-h) + \mu_1\mu_2 a(d_2a)^{\#}(1-h) \\ &= \mu_2\mu_3 - \mu_1\mu_2(aa^{\parallel d_1}a)d_2((ad_2)^{\#})^2a(1-h) + \mu_1\mu_2(ad_2)^{\#}a(1-h) \\ &= \mu_2\mu_3 - \mu_1\mu_2 ad_2((ad_2)^{\#})^2a(1-h) + \mu_1\mu_2(ad_2)^{\#}a(1-h) \\ &= \mu_2\mu_3. \end{aligned}$$

A symmetric argument shows that it is true for $s's = \mu_2\mu_3$. So, $s \in R^{-1}$ and $s^{-1} = (\mu_2\mu_3)^{-1}s'$. Using the expression of s^{-1}, we conclude that the equalities in item (ii) hold.

(ii) ⇒ (i). Suppose that item (ii) holds. Set $h = (\mu_1 a + \mu_2 ad_1 + \mu_4 ad_1d_2a)s^{-1}$ and $k = \mu_2 s^{-1}ad_1$. Since $\mu_3 as^{-1}d_2a = a$, we deduce that $\mu_3 d_2as^{-1}d_2 = d_2(\mu_3 as^{-1}d_2a)a^{\parallel d_2} = d_2aa^{\parallel d_2} = d_2$. Also, from $\mu_2 ad_1s^{-1}a = a$, it follows that $\mu_2 d_1s^{-1}ad_1 = a^{\parallel d_1}(\mu_2 ad_1s^{-1}a)d_1 = a^{\parallel d_1}ad_1 = d_1$. Then, it is straightforward to check that $ha = a$, $hd_2 = 0$, $d_1k = d_1$ and $ak = 0$.

Now, we have to claim that $aR \oplus d_2R = R$. Since $m = \mu_1 a + \mu_2 ad_1 + \mu_3 d_2a + \mu_4 ad_1d_2a \in R^{-1}$, we get $aR + d_2R = R$. Let $y \in aR \cap d_2R$. Then, $y = aw_1 = d_2w_2$, for some $w_1, w_2 \in R$. Thereby, $y = haw_1 = hd_2w_2 = 0$. This implies $aR \cap d_2R = \{0\}$. Consequently, $aR \oplus d_2R = R$. Observe that $aR = aa^{\parallel d_1}aR = aa^{\parallel d_1}R$ and $d_2R = a^{\parallel d_2}aR$. Hence, $aa^{\parallel d_1}R \oplus a^{\parallel d_2}aR = R$. Dually, $Raa^{\parallel d_1} \oplus Ra^{\parallel d_2}a = R$. Therefore, $aa^{\parallel d_1} - a^{\parallel d_2}a \in R^{-1}$. □

From Lemma 1.3, it follows that Theorem 3.5 becomes to the next result.

Corollary 3.6. *Let $a, b, d \in R$ be such that $a, b \in R^{\parallel \bullet d}$. Then, the following statements are equivalent:*

(i) $aa^{\parallel d} - b^{\parallel d}b \in R^{-1}$.

(ii) $t = \xi_1 d + \xi_2 ad + \xi_3 db + \xi_4 dbad \in R^{-1}$, $dt^{-1}d = 0$ *and* $\xi_2 dt^{-1}ad = \xi_3 dbt^{-1}d = d$, *where* $\xi_i \in C(R)$ $(i \in \overline{1,4})$ *and* $\xi_2\xi_3 \in R^{-1}$.

Let us recall [16, Theorem 4.3]: if $a \in R^{\circledast} \cap R_{\circledast}$, then a is co-EP if and only if $aa^{\circledast} - a_{\circledast}a \in R^{-1}$. Motivated by this, we get the following result, which is a new property of the co-EP element.

Corollary 3.7. *Let R be a ∗-ring and $a \in R^{\circledast} \cap R_{\circledast}$. Then, the following statements are equivalent:*

(i) *a is* co-EP.

(ii) $r = \tau_1 a^2a^* + \tau_2 a^*a^2 + \tau_3 a^2(a^*)^2a^2 + \tau_4 a^*a^4a^* \in R^{-1}$, $\tau_1\tau_2 ar^{-1}a = -\tau_4 a^2$, $\tau_1 a^*r^{-1}a = a_{\circledast}$, $\tau_2 ar^{-1}a^* = a^{\circledast}$, $\tau_1\tau_2 a^*r^{-1}a^* = -\tau_3(a^*)^2$, *where* $\tau_i \in C(R)$ $(i \in \overline{1,4})$, $\tau_1\tau_2 \in R^{-1}$ *and* $\tau_3\tau_4 \in a^0$.

(iii) $s = \nu_1 a + \nu_2 a^2a^* + \nu_3 a^*a^2 + \nu_4 a^2(a^*)^2a^2 \in R^{-1}$, $as^{-1}a = 0$, $\nu_2 a^*s^{-1}a = a_{\circledast}$, $\nu_3 as^{-1}a^* = a^{\circledast}$, *where* $\nu_i \in C(R)$ $(i \in \overline{1,4})$ *and* $\nu_2\nu_3 \in R^{-1}$.

Proof. Note that $a^{\circledast} = a^{\parallel aa^*}$ and $a_{\circledast} = a^{\parallel a^*a}$, when $a \in R^{\dagger}$. Then, by taking $d_1 = aa^*$ and $d_2 = a^*a$ in Theorem 3.4 and Theorem 3.5, we conclude that Corollary 3.7 holds. Indeed, since $a \in R^{\circledast} \cap R_{\circledast}$, we have $a \in R^{\dagger} \cap R^{\#}$, $a^{\circledast} = a^{\#}aa^{\dagger}$ and $a_{\circledast} = a^{\dagger}aa^{\#}$. Combining that a is ∗-cancellable, we get

$$\tau_1\tau_2 a^*a^2r^{-1}a^2a^* = -\tau_4 a^*a^4a^* \Leftrightarrow \tau_1\tau_2 a^2r^{-1}a^2 = -\tau_4 a^4 \Leftrightarrow \tau_1\tau_2 ar^{-1}a = -\tau_4 a^2$$

and

$$\tau_1 aa^*r^{-1}a^2a^* = aa^* \Leftrightarrow \tau_1 aa^*r^{-1}a^2 = a \Leftrightarrow \tau_1 a^{\dagger}aa^*r^{-1}a^2a^{\#} = a^{\dagger}aa^{\#} \Leftrightarrow \tau_1 a^*r^{-1}a = a_{\circledast},$$

as required. □

If we add the condition $d_1 \in d_2R$ and $d_2 \in Rd_1$ in Theorem 3.5, then we obtain

Theorem 3.8. *Let $a, d_1, d_2 \in R$ be such that $a \in R^{\|\bullet d_1} \cap R^{\|\bullet d_2}$, $d_1 \in d_2R$ and $d_2 \in Rd_1$. Then, the following statements are equivalent:*

(i) $aa^{\|d_1} - a^{\|d_2}a \in R^{-1}$.

(ii) $u = \eta_1 d_1 + \eta_2 ad_1 + \eta_3 d_2a + \eta_4 d_2a^2d_1 \in R^{-1}$, $d_1u^{-1}d_2 = 0$ *and* $\eta_2 ad_1u^{-1}a = \eta_3 au^{-1}d_2a = a$, *where* $\eta_i \in C(R)$ $(i \in \overline{1,4})$ *and* $\eta_2\eta_3 \in R^{-1}$.

(iii) $v = \delta_1 d_2 + \delta_2 ad_1 + \delta_3 d_2a + \delta_4 d_2a^2d_1 \in R^{-1}$, $d_1v^{-1}d_2 = 0$ *and* $\delta_2 ad_1v^{-1}a = \delta_3 av^{-1}d_2a = a$, *where* $\delta_i \in C(R)$ $(i \in \overline{1,4})$ *and* $\delta_2\delta_3 \in R^{-1}$.

Proof. To begin with, we show that $aa^{\|d_1} - a^{\|d_2}a \in R^{-1}$ imply $u, v \in R^{-1}$. Note that $d_1 \in d_2R$ and $d_2 \in Rd_1$. So, we obtain $d_1 = d_2z_1$ and $d_2 = z_2d_1$, for $z_1, z_2 \in R$. Hence, we get

$$\begin{aligned} aa^{\|d_1} &= aa^{\|d_2}aa^{\|d_1} = a(d_2a)^{\#}d_2aa^{\|d_1} = a(d_2a)^{\#}z_2d_1aa^{\|d_1} \\ &= a(d_2a)^{\#}z_2d_1 = a(d_2a)^{\#}d_2 \\ &= aa^{\|d_2}. \end{aligned}$$

On the other hand, it is clear that

$$\begin{aligned} a^{\|d_1}a &= d_1(ad_1)^{\#}a = d_2z_1(ad_1)^{\#}a = a^{\|d_2}ad_2z_1(ad_1)^{\#}a \\ &= a^{\|d_2}ad_1(ad_1)^{\#}a = a^{\|d_2}aa^{\|d_1}a \\ &= a^{\|d_2}a. \end{aligned}$$

Hence, $a^{\|d_2}ad_1 = a^{\|d_1}ad_1 = d_1$ and $a^{\|d_2} = a^{\|d_2}aa^{\|d_2} = a^{\|d_1}aa^{\|d_2} \in d_1R$. Dually, $d_1aa^{\|d_2} = d_1$ and $a^{\|d_2} \in Rd_1$. Then, by the definition of the inverse along an element we claim $a^{\|d_1} = a^{\|d_2}$, which implies that $d_1(ad_1)^{\#} = (d_1a)^{\#}d_1 = d_2(ad_2)^{\#} = (d_2a)^{\#}d_2$. When item (i) holds, it has been known to us that there exist idempotents $h, k \in R$ satisfying (∗1), and we further have $ha = a$, $ak = 0$, $hd_1 = hd_2z_1 = 0$, $d_2k = z_2d_1k = z_2d_1 = d_2$. Now, let

$$u' = -\eta_1(1-k)(d_2a)^{\#}(d_1a)^{\#}d_1h + \eta_2(1-k)(d_2a)^{\#}(1-h) + \eta_3k(ad_1)^{\#}h - \eta_4(1-k)h$$

and

$$v' = -\delta_1(1-k)d_2(ad_2)^{\#}(ad_1)^{\#}h + \delta_2(1-k)(d_2a)^{\#}(1-h) + \delta_3k(ad_1)^{\#}h - \delta_4(1-k)h.$$

Then, it is easily verified that $uu' = u'u = \eta_2\eta_3$ and $vv' = v'v = \delta_2\delta_3$ by what we have shown already, as desired.

Next, the remaining part of this theorem can be inferred by applying the same strategy as the proof of Theorem 3.5. □

Remark that, the condition $d_1 \in d_2R$ and $d_2 \in Rd_1$ of Theorem 3.8 in general can not be deleted, which can been seen from the following example.

Example 3.9. *In $R = \mathbb{C}^{2\times 2}$, let us choose $a = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $d_1 = \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}$ and $d_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$. Then, we can check that $a \in R^{\|\bullet d_1} \cap R^{\|\bullet d_2}$, $a^{\|d_1} = \begin{pmatrix} 0 & 0 \\ 1 & \frac{1}{2} \end{pmatrix}$ and $a^{\|d_2} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$. Clearly, $aa^{\|d_1} - a^{\|d_2}a = \begin{pmatrix} 1 & \frac{1}{2} \\ -1 & -1 \end{pmatrix}$ is invertible. But, $d_2 + ad_1 - d_2a = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$ is not invertible.*

Although by the proof of Theorem 3.8 we see that the condition $a \in R^{\|\bullet d_1} \cap R^{\|\bullet d_2}$, $d_1 \in d_2R$, $d_2 \in Rd_1$ and $aa^{\|d_1} - a^{\|d_2}a \in R^{-1}$ yields $a^{\|d_1} = a^{\|d_2}$. However, such condition does not imply $d_1 = d_2$ or $d_1a = d_2a$ in general, as we will see in the next example.

Example 3.10. *Let $R = \mathbb{C}^{2\times 2}$. Setting $a = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $d_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{pmatrix}$ and $d_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. Observe that $d_1 \in d_2R$, $d_2 \in Rd_1$ and $a^{\|d_1} = a^{\|d_2} = d_1$. Hence, $a \in R^{\|\bullet d_1} \cap R^{\|\bullet d_2}$and $aa^{\|d_1} - a^{\|d_2}a \in R^{-1}$. However, $d_1 \neq d_2$ and $d_1a \neq d_2a$.*

Apply Theorem 3.8 and Lemma 1.3, we directly have

Corollary 3.11. *Let $a, b, d \in R$ be such that $a, b \in R^{\|\bullet d}$, $a \in Rb$ and $b \in aR$. Then, the following statements are equivalent:*

(i) $aa^{\|d} - b^{\|d}b \in R^{-1}$.

(ii) $p = \beta_1 a + \beta_2 ad + \beta_3 db + \beta_4 ad^2 b \in R^{-1}$, $bp^{-1}a = 0$ *and* $\beta_2 dp^{-1}ad = \beta_3 dbp^{-1}d = d$, *where* $\beta_i \in C(R)$ $(i \in \overline{1,4})$ *and* $\beta_2\beta_3 \in R^{-1}$.

(iii) $q = \gamma_1 b + \gamma_2 ad + \gamma_3 db + \gamma_4 ad^2 b \in R^{-1}$, $bq^{-1}a = 0$ *and* $\gamma_2 dq^{-1}ad = \gamma_3 dbq^{-1}d = d$, *where* $\gamma_i \in C(R)$ $(i \in \overline{1,4})$ *and* $\gamma_2\gamma_3 \in R^{-1}$.

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