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Equivalence to uniqueness in the concept of predictability between filtrations

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Abstract. In this paper we develop the concept of dependence between filtrations given in [11], named causal predictability, which is based on the Granger's definition of causality. Then, we provide some new properties of this concept and prove a result that consider equivalence to uniqueness of the given concept. Also, a few examples that illustrate applications of the given concept are given with the main focus on stochastic differential equations (SDE) and financial mathematics.

The study of Granger's causality has been defined in the context of time series. Since continuous time models become more and more frequent in econometric practice, epidemiology, climatology, demographic, etc, we develop a concept connected to the continuous time processes.

At the same time, the modern finance theory extensively uses diffusion processes.

1. Introduction

From the seminal work of Granger [7], the theory of causality has widely developed and established itself as a fervent research area, with growing applications in the most various areas of sciences, especially in finance, econometric practice, biostatistics, optimal transport plans, neuroscience, epidemiology, demographic, etc. We consider different concepts of causality between stochastic processes and between filtrations and then relate the given concepts of causality to the to the weak solutions of SDE and financial mathematics.

The aim of the present article is to establish the result of Theorem 3.4 as the fundamental result of causal predictability which gives property of equivalence to uniqueness.

The paper is organized as follows. After Introduction, in Section 2, we provide some of the known results that we need later. Especially, we present the causality concept introduced by [15] and give some basic properties of this concept.

In Section 3 we state and prove our main results. In this part we develop the concept of dependence named causal predictability between filtrations which is based on the Granger's definition of causality and the causality concept given by Mykland [15]. More precisely, we give some equivalent definitions of causal predictability from [11] and some new characteristics of the given concept. Then, we prove a result about equivalence to uniqueness of given concept.

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Section 4 contains applications of causal predictability in continuous time to the weak uniqueness of weak solutions of SDE and financial mathematics. Finally, we mention a few ideas for some future work.

2. Preliminary notations and definitions

Suppose that $(\Omega, \mathcal{A}, \mathbf{F}, P)$ is a filtered probability space, where $(\Omega, \mathcal{A}, P)$ is a probability space and $\mathbf{F} = \{\mathcal{F}_t,\ t \in T\}$ is a "framework" filtration, that is, $\mathcal{F}_t$ is the set of all events in the model up to and including time t, a subfiltration of $\mathcal{A}$. Total information $\mathcal{F}_\infty$ carried by $\mathbf{F}$ is defined as $\mathcal{F}_\infty = \bigvee_{t \in T} \mathcal{F}_t$. We suppose that the filtration $\mathbf{F}$ satisfies usual conditions (i.e. it is right continuous and complete, see [3]). The time index set T is equal to $\mathbb{R}_+$, unless specified otherwise. Analogous notation will be used for filtrations $\mathbf{H} = \{\mathcal{H}_t\}$, $\mathbf{G} = \{\mathcal{G}_t\}$ and $\mathbf{J} = \{\mathcal{J}_t\}$.

The concept of conditional independence is widely used in probability theory and statistics. The basic properties of this concept are given in [1], [5] and [6].

In this paper, we consider different concepts of causality involving conditional independence.

Definition 2.1. (see [15], [20]) Let $\mathbf{F} = \{\mathcal{F}_t\}$, $\mathbf{G} = \{\mathcal{G}_t\}$ and $\mathbf{H} = \{\mathcal{H}_t\}$ be filtrations on the same probability space. It is said that $\mathbf{G}$ is a cause of $\mathbf{H}$ within $\mathbf{F}$ relative to P (and written as $\mathbf{H} \models \mathbf{G}; \mathbf{F}; P$) if $\mathbf{H} \subseteq \mathbf{F}$, $\mathbf{G} \subseteq \mathbf{F}$ and if $\mathcal{H}_\infty$ is conditionally independent of $\mathcal{F}_t$ given $\mathcal{G}_t$ for each t, i.e.

$$\mathcal{H}_\infty \perp \mathcal{F}_t \mid \mathcal{G}_t, \tag{1}$$

that is

$$(\forall A \in \mathcal{H}_\infty) \quad P(A|\mathcal{F}_t) = P(A|\mathcal{G}_t).$$

If there is no doubt about P, we omit "relative to P".

It is easy to see that (1) may be formulated as

$$(\forall u \in T), (\forall t \in T) \quad \mathcal{H}_u \perp \mathcal{F}_t \mid \mathcal{G}_t.$$

Intuitively, (1) means that, for arbitrary t, information about $\mathcal{H}_\infty$ provided by $\mathcal{F}_t$ is not "bigger" than that provided by $\mathcal{G}_t$.

If $\mathbf{G}$ and $\mathbf{F}$ are such that $\mathbf{G} \models \mathbf{G}; \mathbf{F}$, we shall say that $\mathbf{G}$ is its own cause within $\mathbf{F}$. It should be mentioned that the notion of subordination (as introduced by [17]) is equivalent to the notion of being one's own cause, as defined here.

If $\mathbf{G}$ and $\mathbf{F}$ are such that $\mathbf{G} \models \mathbf{G}; \mathbf{G} \bigvee \mathbf{F}$ (where $\mathbf{G} \bigvee \mathbf{F}$ is a family determined by $(\mathcal{G} \bigvee \mathcal{F})_t = \mathcal{G}_t \bigvee \mathcal{F}_t$), we shall say that $\mathbf{F}$ does not cause $\mathbf{G}$. It is clear that the interpretation of Granger's causality is now that $\mathbf{F}$ does not cause $\mathbf{G}$ if $\mathbf{G} \models \mathbf{G}; \mathbf{G} \bigvee \mathbf{F}$ (see [15]).

It should be mentioned that Definition 2.1 of causality is equivalent to the definition of strong global noncausality, given by [4].

A family of σ-algebras induced by a stochastic process $\mathbf{X} = \{X_t, t \in T\}$ is given by $\mathbf{F}^X = \{\mathcal{F}_t^X, t \in T\}$, where

$$\mathcal{F}_t^X = \sigma\{X_u, u \in T, u \leq t\},$$

being the smallest σ-algebra with respect to which the random variables $X_u, u \leq t$ are measurable.

The process $\mathbf{X} = \{X_t\}$ is adapted to $\mathbf{F} = \{\mathcal{F}_t\}$ or $\mathbf{F}$-adapted if $\mathcal{F}_t^X \subseteq \mathcal{F}_t$ for each t .

The notation $(X_t, \mathcal{F}_t)$ means that proces $\mathbf{X} = \{X_t\}$ is $\mathbf{F}$-adapted.

Definition 2.1 can be applied to stochastic processes. It will be said that stochastic processes are in a certain relationship if and only if the corresponding induced filtrations are in that relationship.

3. Equivalence to uniqueness for causal predictability

In this section, we consider some properties of causal predictability relation for filtration given in [11] and prove the equivalence to uniqueness for this relation. Finally, we give the alternative definition of causal predictability given in Definition 3.1.

A probability space $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{P})$ is an extension of probabilty space $(\Omega, \mathcal{A}, P)$ if there is a measurable surjective function $f : (\hat{\Omega}, \hat{\mathcal{A}}) \longrightarrow (\Omega, \mathcal{A})$ such that $\hat{P} f^{-1} = P$ on $\mathcal{A}$ (see [8]). Both spaces must have the same time axis T. For filtrations $\mathbf{G} = \{\mathcal{G}_t\}$, $\mathbf{H} = \{\mathcal{H}_t\}$ and $\mathbf{J} = \{\mathcal{J}_t\}$ on the same probability space $(\Omega, \mathcal{A}, P)$, we have

$$\hat{\mathcal{G}}_t = \{f^{-1}(A) : A \in \mathcal{G}_t\}, \;\; \hat{\mathcal{H}}_t = \{f^{-1}(A) : A \in \mathcal{H}_t\}, \;\; \hat{\mathcal{J}}_t = \{f^{-1}(A) : A \in \mathcal{J}_t\}.$$

Note that there may be filtrations on the probability space $(\hat{\Omega}, \hat{\mathcal{A}})$ which are not defined as an inverse image of some filtration on the original probability space $(\Omega, \mathcal{A})$. Also, note that $\hat{\mathcal{A}} \neq f^{-1}(\mathcal{A})$.

Let us recall the following definition from [11].

Definition 3.1. ([11]) Let $\mathbf{G} = \{\mathcal{G}_t\}$, $\mathbf{H} = \{\mathcal{H}_t\}$ and $\mathbf{J} = \{\mathcal{J}_t\}$ be filtrations on the same probability space $(\Omega, \mathcal{A}, P)$ with a common time axis T. It is said that **J** is causally predictable by **H** relative to **G** (or that **H** causally predicts **J** relative to **G**) if

$$\mathbf{H} \subseteq \mathbf{G}$$

and

$$\mathbf{H} \mid\!< \mathbf{G}; \mathbf{G}'; P,$$

where

$$\mathbf{G}' = \mathbf{G} \vee \mathbf{J} \tag{2}$$

and if

any extension $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{P})$ of the probability space $(\Omega, \mathcal{G}'_\infty, P)$ (with $\hat{P} f^{-1} = P$) containing filtrations $\hat{\mathbf{F}}, \hat{\mathbf{G}}', \hat{\mathbf{G}}, \hat{\mathbf{H}}$ and $\hat{\mathbf{J}}$ satisfying

a)

$$\hat{\mathbf{G}}' \subseteq \hat{\mathbf{F}}; \tag{3}$$

b) for every $A \in H_\infty$ and $t \in T$

$$\left.\begin{array}{l} g(A) = P(A \mid \mathcal{G}'_t)\,(P\text{-a.s.}) \\ g \text{ is } \mathbf{G}'\text{-measurable} \end{array}\right\} \Rightarrow g \circ f(A) = \hat{P}(f^{-1}(A) \mid \hat{\mathcal{G}}'_t)\,(\hat{P}\text{-a.s.}); \tag{4}$$

c)

$$\hat{\mathbf{H}} \mid\!< \hat{\mathbf{G}}; \hat{\mathbf{F}}; \hat{P}; \tag{5}$$

must also satisfy

$$\hat{\mathbf{J}} \mid\!< \hat{\mathbf{G}}'; \hat{\mathbf{F}}; \hat{P}. \tag{6}$$

If $\mathbf{G} = \mathbf{H}$ in the above definition, we say that **J** is causally predictable by **H**.

It should be noted that causal predictability is a notion of dependence, but one which is weaker than adaptedness with respect to a filtration.

First we give some new properties of causal predictability. Actually, we give some alternative definitions of causal predictability.

Lemma 3.1. *(see [11]) Condition c) of the definition of causal predictability can be replaced by*

$$c') \qquad \hat{\mathbf{H}} \mathrel{|\!\!<} \hat{\mathbf{G}}'; \hat{\mathbf{F}}; \hat{P}.$$

Lemma 3.2. *(see [12]) Conditions b) and c) of the definition of causal predictability can be replaced by the following condition*
d) for every $A \in \mathcal{H}_\infty$, $t \in T$

$$\left.\begin{array}{l} g(A) = P(A \mid \mathcal{G}'_t)\,(P\text{-a.s.}) \\ g \text{ is } \mathbf{G}'\text{-measurable} \end{array}\right\} \Rightarrow g \circ f(A) = \hat{P}(f^{-1}(A) \mid \hat{\mathcal{F}}_t)\ (\hat{P}\text{-a.s.});$$

Proof. From Lemma 3.1 it follows that condition *c*) from definition of causal predictability could be replaced by *c*′), which reads

$$\hat{P}(f^{-1}(A)|\hat{\mathcal{G}}'_t) = \hat{P}(f^{-1}(A)|\hat{\mathcal{F}}_t), \text{ for every } A \in \mathcal{H}_\infty \text{ and } t \in T \tag{7}$$

thus *b*) and *d*) are equivalent in the presence of *c*′). Condition *d*) implies that $\hat{P}(f^{-1}(A)|\hat{\mathcal{F}}_t)$ can be chosen $\hat{\mathbf{G}}'$−measurable for $A \in \mathcal{H}_\infty$, which is (7).

The following result gives equivalent conditions for causal predictability between filtrations.

Proposition 3.3. *[13] Let* $(\Omega, \mathcal{A}, \mathbf{F}, P)$ *be a filtered probability space and let* $\mathbf{H}, \mathbf{J}$ *and* $\mathbf{G}$ *be filtration in* $\mathcal{A}$. *Let* $\mathbf{H} \subset \mathbf{G}, \mathbf{J} \subset \mathbf{G}$ *and*

$$H = \{(M_t) : M_t \text{ is a } \mathbf{G}\text{-martingale and } \exists A \in \mathcal{H}_\infty \ M_t = P(A|\mathcal{G}_t), \forall t \in T\},$$

$$J = \{(M_t) : M_t \text{ is a } \mathbf{G}\text{-martingale and } \exists A \in \mathcal{H}_\infty \ M_t = P(A|\mathcal{G}_t), \forall t \in T\}.$$

The following statements are equivalent:

i) $\mathbf{J}$ *is causally predictable by* $\mathbf{H}$ *relative to* $\mathbf{G}$.

ii) Every extension $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{P})$ *of probabilty space* $(\Omega, \mathcal{A}, P)$ *with filtrations* $\hat{\mathbf{F}}$ *and* $\hat{\mathbf{G}}$ *with the same time axis as* $\mathbf{G}$ *satisfying* $\hat{\mathbf{G}} \subseteq \hat{\mathbf{F}}$ *and* $\hat{P}f^{-1} \ll P$ *on* $\mathcal{G}_\infty$, $\frac{d\hat{P}f^{-1}}{dP} \in L^1(\Omega, \mathcal{G}_\infty, P)$
and
$\hat{H} = \{(\hat{M}_t) : \hat{M}_t(\hat{\omega}) = M_t(f(\hat{\omega})) : (M_t) \in H\}$ *is collection of* $\hat{\mathbf{F}}$*-martingale under* $\hat{P}$,
must also satisfy that
$\hat{J} = \{(\hat{M}_t) : \hat{M}_t(\hat{\omega}) = M_t(f(\hat{\omega})) : (M_t) \in J\}$ *is a collection of* $\hat{\mathbf{F}}$*-martingale under* $\hat{P}$.

We will now prove the main result of causal predictability which gives property of equivalence to uniqueness.

Theorem 3.4. *Let* $\mathbf{G} = \{\mathcal{G}_t\}$, $\mathbf{H} = \{\mathcal{H}_t\}$ *and* $\mathbf{J} = \{\mathcal{J}_t\}$ *be filtrations on* $(\Omega, \mathcal{A}, P)$ *with a common time axis* $T = [0, t_0]$ *such that* $\mathbf{H} \subseteq \mathbf{G}$ *and* $\mathbf{J} \subseteq \mathbf{G}$.
The following statements are equivalent:

i) $\mathbf{J}$ *is causally predictable by* $\mathbf{H}$ *relative to* $\mathbf{G}$ *in the sense of Definition 3.1.*

ii) For any probability measure Q *on* $\mathcal{G}_\infty$ *satisfying*

$$Q \ll P \ (\text{on } \mathcal{G}_\infty) \tag{8}$$

and

for every $A \in \mathcal{H}_\infty$ *and* $t \in T$

$$\left.\begin{array}{l} g = P(A \mid \mathcal{G}_t) \ (P\text{-a.s.}) \\ g \text{ is } \mathbf{G}\text{-measurable} \end{array}\right\} \Rightarrow g = Q(A \mid \mathcal{G}_t) \ (Q\text{-a.s.}); \tag{9}$$

the following holds

for every $A \in \mathcal{J}_\infty$ *and* $t \in T$

$$\left.\begin{array}{l} g = P(A \mid \mathcal{G}_t) \ (P\text{-a.s.}) \\ g \text{ is } \mathbf{G}\text{-measurable} \end{array}\right\} \Rightarrow g = Q(A \mid \mathcal{G}_t) \ (Q\text{-a.s.}). \tag{10}$$

Proof. $(i) \Rightarrow (ii)$ This is proved as Proposition 2 in [13].

$(ii) \Rightarrow (i)$ It is enough to show that statement *ii*) from Proposition 3.3 holds. Suppose that $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{P})$ is an extension of probability space $(\Omega, \mathcal{A}, P)$ defined with $f : (\hat{\Omega}, \hat{\mathcal{A}}) \longrightarrow (\Omega, \mathcal{A})$ which satisfies conditions of *ii*) from Proposition 3.3. We want to show that $\hat{J} = \{(\hat{M}_t) : \hat{M}_t(\hat{\omega}) = M_t(f(\hat{\omega})) : (M_t) \in J\}$ is a class of $\hat{\mathbf{F}}$-martingales under $\hat{P}$. Suppose that elements of $\hat{J}$ are not necessarily $\hat{\mathbf{F}}$-martingales under $\hat{P}$, actually that there is $(M_t) \in J$, $u \in T$ and $\hat{A} \in \hat{\mathcal{F}}_u$, such that, if $\hat{M}_t(\hat{\omega}) = M_t(f(\hat{\omega}))$ it follows that

$$\int_{\hat{A}} \hat{M}_\infty d\hat{P} \neq \int_{\hat{A}} \hat{M}_u d\hat{P}. \tag{11}$$

Following the proof of Theorem 3 from [13], for $\eta \in L^\infty(\Omega, \mathcal{G}_\infty, P)$, we have that the functional

$$\Phi(\eta) = \hat{E}[I_{\hat{A}} \hat{E}(\eta \circ f | \hat{\mathcal{G}}_u) + I_{\hat{A}^C}(\eta \circ f)]$$

is well defined, it is linear and bounded. As $\Phi(\eta) \geq 0$ for $\eta \geq 0$ and as $\Phi(1) = 1$, Φ is the expectation operator of a measure Q on $\mathcal{G}_\infty$,

$$\forall C \in \mathcal{G}_\infty \quad Q(C) = \Phi(I_C).$$

Now, we have that $Q \ll \nu$ holds and that by Riesz' Representation Theorem $\frac{dQ}{d\nu} \in L^\infty(\Omega, \mathcal{G}_\infty, \nu)$ holds.

Similar techniques from the proof of Theorem 3 from [13], taking function X to be previously introduced function f, lead to the conclusion that H is a class of $\mathbf{G}$-martingales under Q and hence J is also a class of $\mathbf{G}$-martingales under Q. So, we have

$$\int M_\infty dQ = \int M_u dQ = \hat{E}(I_{\hat{A}} \hat{M}_u) + \hat{E}(I_{\hat{A}^c} \hat{M}_u).$$

According to Proposition 15.1 in [18] we have that (M_t) is a $\mathbf{G}$−martingale under $\hat{P}f^{-1}$, so $\hat{E}(\hat{M}_\infty | \hat{\mathcal{G}}_u) = \hat{M}_u$. Hence,

$$\int M_\infty dQ = \hat{E}(I_{\hat{A}} \hat{M}_u) + \hat{E}(I_{\hat{A}^c} \hat{M}_\infty)$$

which would contradict (11) and prove the result.

The previous result remains valid if (8) is replaced by $Q \sim P$ on $\mathcal{G}_\infty$. In this case, (9) is equivalent to

$$\forall A \in \mathcal{H}_\infty \ \forall t \in T \quad P(A|\mathcal{G}_t) = Q(A|\mathcal{G}_t) \ \ (P - \text{a.s. and } Q - \text{a.s.})$$

and similarly (10) is equivalent to

$$\forall A \in \mathcal{J}_\infty \ \forall t \in T \quad P(A|\mathcal{G}_t) = Q(A|\mathcal{G}_t) \ \ (P - \text{a.s. and } Q - \text{a.s.}).$$

In [11] the following result is shown, from which we will obtain the equivalent definition of causal predictability to Definition 3.1.

Proposition 3.5. *(see [11]) Let* $\mathbf{G}$, $\tilde{\mathbf{G}}$, $\mathbf{H}$, $\tilde{\mathbf{H}}$, $\mathbf{J}$ *and* $\tilde{\mathbf{J}}$ *be filtrations in a probability space* $(\Omega, \mathcal{A}, P)$ *such that*

$$\mathbf{G} = \tilde{\mathbf{G}},\ \mathbf{H} = \tilde{\mathbf{H}},\ \mathbf{J} = \tilde{\mathbf{J}}\ a.s.$$

and

$$\tilde{\mathbf{H}} \subseteq \tilde{\mathbf{G}}.$$

If $\mathbf{J}$ *is causally predictable by* $\mathbf{H}$ *relative to* $\mathbf{G}$*, then* $\tilde{\mathbf{J}}$ *is causally predictable by* $\tilde{\mathbf{H}}$ *relative to* $\tilde{\mathbf{G}}$*.*

The invariance under stochastic equivalence from this proposition is crucially dependent on a relation of absolute continuity. By setting $A = \emptyset$, the condition (4) from Definition 3.1 obviously implies that

$$\hat{P}f^{-1} \ll P \ \text{ on } \ \mathcal{G}'_\infty. \tag{12}$$

As a consequence of (12), considering the statement

b') for every $A \in \mathcal{H}_\infty, t \in T$ there exists g which is $\mathbf{G}'$-measurable such that

$$g(A) = P(A \mid \mathcal{G}'_t)\,(P\text{-a.s.}) \ \ \& \ \ g \circ f(A) = \widehat{P}(f^{-1}(A) \mid \widehat{\mathcal{G}'_t})\,(\widehat{P}\text{-a.s.})$$

it is easy to obtain the following result.

Proposition 3.6. *The equivalent definition of causal predictability is obtained if condition* b) *in Definition 3.1 is replaced by* $\hat{P}f^{-1} \ll P$ *on* $\mathcal{G}'_\infty$ *and* b')*.*

4. Some applications

In [11] we showed how the theory of causal predictability can be applied to the weak uniqueness of weak solution of the stochastic differential equation

$$dX_t = a_t(X)dt + b_t(X)dW_t, \ \ X_0 = \eta, \tag{13}$$

where X is a continuous d-dimensional process and W a d-dimensional Wiener process, which is well defined when the following elements are given: the dimension d (of X and W), functionals a_t (d-dimensional vector) and b_t($d \times d$ matrix) and d-dimensional distribution F_η function (see [10], [16]).

Theorem 4.1. *([11]) Assume that SDE (13) has a weak solution. If for every weak solution* $(\Omega, \mathcal{A}, \mathbf{F}, P, W, X)$ *of (13),* $\mathbf{F}^X$ *is causally predictable by* $\mathbf{F}^W$ *relative to* $\mathbf{F}^{X,W}$*, then the solution is weakly unique.*

Also, in [11] we gave another example of the application of causal predictability considering SDE

$$dX_t = u_t(X)dZ_t, \ \ X_0 = \eta, \tag{14}$$

where $Z = \{Z_t, t \in T\}$ is an $m-$dimenisonal semimartingale and coefficient u_t is an $(n \times m)-$dimensional predictable functional (see [8] and [9] for more details). This SDE generalizes the diffusion equation (13). We proved the following result which gives conditions for a solution of stochastic differential equation (14) to be weakly unique in terms of causal predictability.

Theorem 4.2. *([11]) Assume that SDE (14) has a weak solution. If for every weak solution* $(\Omega, \mathcal{A}, \mathbf{F}, P, Z, X)$ *of (14),* $\mathbf{F}^{X,Z}$ *is causally predictable by* $\mathbf{F}^X$ *relative to* $\mathbf{F}^{X,Z}$*, then the solution is weakly unique.*

In this paper, we will discuss an application in financial mathematics. Actually, we will give an example in the view of modeling of default risk which is well represented in the literature (see, for example, [2]). There are two main approaches: either the default time τ is a stopping time in the asset's filtration (structural approach), or it is a stopping time in a larger filtration (intensity-based approach). We shall focus our attention on the intensity-based valuation. A default occurs at a random time τ (i.e. a non-negative random variable). In the defaultable world, the payment of a contingent claim depends whether or not the default has appeared before the maturity. We associate with this random time the counting process $D = \{D_t\}$ defined as $D_t = I_{\{\tau<t\}}$. The process D is an increasing process, cádlág, equal to 0 before the default and equal to 1 after default. Essentially, the intensity of τ is defined as the nonnegative adapted process λ such that

$$M_t := D_t - \int_0^{t\wedge\tau} \lambda_u du$$

is a martingale.

Definition 4.1. The increasing function $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$ given by the formula

$$\Gamma(t) = -ln(1 - F(t))),$$

where $F(t) = P(\tau \leq t)$, is called the hazard function of τ.

Definition 4.2. A function $\Lambda : \mathbb{R}_+ \to \mathbb{R}_+$ is called a martingale hazard function of a random time τ with respect to the natural filtration $\mathbf{F}^D$ if and only if the process $D_t - \Lambda(t \wedge \tau)$ is a $\mathbf{F}^D$-martingale.

Assume that B is a $\mathbf{G}$-adapted continuous process of finite variation given by the formula

$$B_r = e^{\int_0^t r_u du},$$

for some $\mathbf{G}$-adapted integrable process r. The process B is referred to as the savings account. Let us fix $K > 0$ and set

$$S_t := B_t E\left(\int_{]t,K]} B_u^{-1} Z_u dD_u + B_K^{-1} X I_{\{K<\tau\}} \mid \mathcal{F}_t\right), \tag{15}$$

where $\mathbf{Z}$ is a $\mathbf{F}$-predictable process and X is a $\mathcal{F}_K$ -measurable random variable.

Let us define the process V by setting

$$V_t = \tilde{B}_t E\left(\int_t^K \tilde{B}_u^{-1} Z_u \lambda_u du + \tilde{B}_K^{-1} X \mid \mathcal{F}_t\right), \tag{16}$$

where $\tilde{B}$ is the 'savings account' corresponding to the default-adjusted short-term rate $R_t = r_t + \lambda_t$, that is,

$$\tilde{B}_t = \exp\left(\int_0^t (r_u + \lambda_u)\, du\right).$$

Now, we are ready to illustrate an application of the concept of causal predictability from Definition 3.1 in this case.

Example 4.1. Let $\mathbf{F}$ be causally predictable by $\mathbf{G}$ and let τ admit an absolutely continuous $\mathbf{G}$-martingale hazard function Λ. Let $\mathbf{Z}$ be a $\mathbf{G}$-predictable process and X be a $\mathcal{G}_K$-measurable random variable. If $\Gamma = \Lambda$ it follows that $S_t = I_{\{\tau>t\}} V_t$ for $t \leq K$, where processes S and V are given by (15) and (16), respectively.

Another example of application was given in [11], where we have mentioned the strong connection between the theory of optimal transport in continuous time and stochastic analysis.

Finally, we give some ideas for the future work.

Remark 1. It might be interesting to see how the theory of causal predictability can be applied to stochastic filtering and control theory (see, for example, [19]).

Also, it might be interesting to deal with the case of progressive causal predictability, i.e. with the case when $\{\mathcal{I}_t, t \in \tau \cap (-\infty, u)\}$ is causally predictable by $\{\mathcal{H}_t, t \in \tau \cap (-\infty, u)\}$ for stopping time τ and all $u \in \mathbb{R}$.

References

[1] C. Dellacherie, P.A. Meyer, Probability and Potentials B, Theory of Martingales. Blaisdell Publishing Company, Waltham, Mass, 1982.

[2] D. Duffie, M. Schroder and C. Skiadas, Recursive valuation of defaultable securities and the timing of resolution of uncertainty. Annals of Applied Probability 6. (1997) 1075-1090.

[3] R.J. Elliot, Stochastic calculus and applications. Springer-Verlag, New York, 1982.

[4] J.P. Florens, D. Fougères, Noncausality in continuous time. Econometrica, 64(5) (1996) 1195–1212.

[5] J.P. Florens, M. Mouchart, A Note on Noncausality. Econometrica, 50(3), (1982) 583–591.

[6] J.P. Florens, M. Mouchart, J.M. Rolin, Elements of Bayesian statistics. Pure and Applied Mathematics, A Series of Monographs and Textbooks, Marcel Dekker inc, New York and Basel, 1990.

[7] C.W.J. Granger, Investigating Causal Relations by Econometric Models and Cross Spectral Methods. Econometrica 37 (1969) 424–438.

[8] J. Jacod, Weak and strong solutions of stochastic differential equations. Stochastics 3. (1980) 171-191.

[9] J. Jacod, J. Memin, Existence of weak solutions for stochastic differential equation with driving semimartingales. Stochastics 4. (1981) 317-337.

[10] R.S. Liptser, A.N. Shiryayev, Statistics of random processes I. Springer-Verlag, New York, 1979.

[11] A. Merkle, Predictability and uniqueness of weak solutions of the stochastic differential equations, Analele Stiintifice ale Universitatii Ovidius Constanta, 31 (1) (2023), 207-219.

[12] A. Merkle, Causal predictability and weak solutions of the stochastic differential equations with driving semimartingales, Statistics and Probability Letters, (2023).

[13] A. Merkle, Causal predictability between stochastic processes and filtrations, Stochastics: An International Journal of Probability and Stochastic Processes, accepted, 2023.

[14] M. Mouchart, J.M. Rolin, . A note on conditional independence with statistical applications. Statistica 44(4). (1985) 557–584.

[15] P.A. Mykland, Statistical Causality. Report No.14. University of Bergen, (1986) 1–26.

[16] B. Øksendal, Stochastic Differential Equations: An Introduction with Applications, Springer, 2013.

[17] Yu.A. Rozanov, Theory of Innovation Processes, Monographs in Probabolity Theory and Mathematical Statistics. Izdat. Nauka, Moscow, 1974.

[18] H.L. Royden, Real Analysis, McMillan Publishing Co, New York, 1968.

[19] J.H. Schuppen, Control and System Theory of Discrete-Time Stochastic Systems, Springer, 2021.

[20] D Valjarević, A. Merkle, Statistical causality and measurable separability of σ-algebras. Statistics and Probability Letters, Elsevier. vol. 177(C) (2021).