Filomat 37:29 (2023), 9945–9959
https://doi.org/10.2298/FIL2329945K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Approximation properties of bivariate extension of blending type operators

Jaspreet Kaur[a], Meenu Goyal[a]

[a]*Thapar Institute of Engineering and Technology, Patiala-147004, India*

Abstract. The present article is in the continuation of our previous work [26], where we have improved the order of approximation of $\alpha-$Bernstein Păltănea operators. In the given note, we study the bivariate extension of first order modification of these operators and their approximation properties such as convergence, error of approximation in terms of complete and partial modulus of continuity and their asymptotic formula. We present numerical examples to show the convergence of functions of two variables with the help of MATLAB software. Also, we construct the GBS operators associated to the bivariate extension and present their approximation behavior.

1. Introduction

The famous advantageous proof of Weierstrass approximation theorem was proved by S. N. Bernstein, named as Bernstein operators. Due to its useful properties, many generalizations have been carried out [8, 9, 19, 30, 35]. The rate of convergence of well known Bernstein operators and its generalizations is slow, so different approaches are available to improve their order of convergence. Butzer [15] initiated it by employing linear combination of Bernstein operators. Micchelli [28] presented another procedure in which he used iterative combinations of Bernstein operators. Recently, Khosravian-Arab [27] propounded another process for improving the order of approximation by perturbing the recurrence formula satisfied by Bernstein polynomials. Using this new approach, many operators have been modified in a very short period of time as we can see [1, 3, 21, 23]. Similarly, in [26], we have modified the summation-integral operators presented by Kajla and Goyal [25] having Păltănea basis function [31] in integral depending on

2020 *Mathematics Subject Classification*. 26A15, 40A05, 41A10, 41A25, 41A35, 41A60.

Keywords. Modulus of continuity; Convergence of series and sequences; Rate of convergence; Approximation by positive operators; Asymptotic approximations.

Received: 08 January 2023; Accepted: 16 May 2023

Communicated by Snežana Č. Živković Zlatanović

Email addresses: jazzbagri3@gmail.com (Jaspreet Kaur), meenu_rani@thapar.edu (Meenu Goyal)

a parameter $\rho > 0$ given as below:

$$J^{\alpha}_{n,\rho}(g;y) = \sum_{j=0}^{n} p^{\alpha,1}_{n,j}(y) \int_0^1 \mu^{\rho}_{n,j}(r) g(r)\, dr, \qquad y \in [0,1],\ \alpha \in [0,1], \tag{1}$$

$$\text{where} \quad p^{\alpha,1}_{n,j}(y) = a(y,n)\, p^{\alpha}_{n-1,j}(y) + a(1-y,n)\, p^{\alpha}_{n-1,j-1}(y), \quad 0 \le j \le n-1,$$

$$\text{such that} \quad p^{\alpha}_{n,j}(y) = \left[\binom{n-2}{j} y(1-\alpha) + \binom{n-2}{j-2}(1-y)(1-\alpha) + \binom{n}{j}\alpha y(1-y)\right] y^{j-1}(1-y)^{n-j-1}, \quad n \ge 2,$$

$$\text{and} \quad \mu^{\rho}_{n,j}(r) = \frac{r^{j\rho}(1-r)^{(n-j)\rho}}{B(j\rho+1,(n-j)\rho+1)}. \tag{2}$$

$B(m,n)$ is the beta function and $a(y,n) = a_1(n)y + a_0(n)$. The sequences $a_0(n)$ and $a_1(n)$ are to be determined in an appropriate way. It can be easily observed that for the sequences $a_1(n) = -1, a_0(n) = 1$, operators (1) reduce to the original operators given in [25]. In order to keep the operators $J^{\alpha}_{n,\rho}(g;y)$ positive, we will assume the conditions on these sequences $a_i(n), i = 0,1$

$$2a_0(n) + a_1(n) = 1, \quad a_0(n) \ge 0 \quad \text{and} \quad a_0(n) + a_1(n) \ge 0.$$

Now onwards, we will denote $e_i = y^i$ and $e_{ij} = y^i z^j, i = 0,1,2,\cdots, j = 0,1,2,\cdots$.

Lemma 1.1. *[26] The moments of the operators (1) are given as:*

$$J^{\alpha}_{n,\rho}(e_0;y) = 1;$$

$$J^{\alpha}_{n,\rho}(e_1;y) = y + \frac{(1-2y)(\rho+1-\rho a_0(n))}{n\rho+2};$$

$$J^{\alpha}_{n,\rho}(e_2;y) = y^2 + \frac{1}{(n\rho+2)(n\rho+3)}\Big[n\rho y\{(1+\rho)(3-5y) - 2a_0(n)\rho(1-2y)\} + \rho^2(1-2y(1-y)(1+\alpha)) + 3\rho(1-2y) + 2(1-3y^2) - a_0(n)\rho(\rho(4y^2-4y+1) + 3(1-2y))\Big].$$

Lemma 1.2. *[26] For the operators (1), we have the central moments as:*

$$J^{\alpha}_{n,\rho}(r-y;y) = \frac{(1-2y)(\rho+1-\rho a_0(n))}{n\rho+2};$$

$$J^{\alpha}_{n,\rho}((r-y)^2;y) = \frac{1}{(n\rho+2)(n\rho+3)}\Big[n\rho(1+\rho)y(1-y) - y(1-y)\{6(1+2\rho) + 2\rho^2(1+\alpha)\} + (\rho+1)(\rho+2) + a_0(n)\rho(\rho+3)(4y(1-y)-1)\Big];$$

$$J^{\alpha}_{n,\rho}((r-y)^4;y) = \frac{3\rho^2(1+\rho)^2 y^2(1-y)^2 n^2}{(n\rho+2)(n\rho+3)(n\rho+4)(n\rho+5)} + O\left(\frac{1}{n^3}\right).$$

In dealing with many real life problems i.e. cost of a product, profit of a store, etc., we need more than one particular factor. Thus, Mathematical modeling of these problems requires functions of two or several variables. Moreover, these functions and their approximations are used in every field such as Economics, Continuum Mechanics, Thermo Dynamics, Fluid Dynamics etc. In order to approximate the functions of two and several variables, the initialization of new positive linear operators, defined in two as well as several dimensions in the approximation theory, is to be done by Stancu [37]. The operators defined in two variables, are popularly known as bivariate operators. In [11], Bărbosu obtained the results of the bivariate extension of Stancu generalization of $q-$Bernstein operators. Following that, there was a lot of work done on the approximation of positive linear operators in two variables [6, 17, 20, 24, 36, 38]. Agrawal and Goyal [4] presented the bivariate extensions of the different operators in their work, which contain discrete and summation integral type operators. Motivated by this literature and applications of two-dimensional operators, we provide the bivariate extension of the above defined operators $J^{\alpha}_{n,\rho}(.;y)$ in the present article. The format of the current article is as follows: we will start by defining the bivariate extension of the operators (1) followed by some preliminary findings like moments and central moments as well as some required definitions. In section 3, we present our results such as convergence of operators, Voronovskaja type asymptotic results, error of operators from the Lipschitz continuous functions and in form of approximation tools as complete and partial modulus of continuity, second order modulus of continuity. In the next section 4, we will display some graphs of certain functions to verify our above proved theoretical results. In the last section, we discuss the concept of Bögel continuity and its related terms and define the Generalized Boolean Sum operators associated with the operators $J^{\alpha}_{n,\rho}(.;y)$ (1) and study their results.

2. Construction of the operators

Let $I^2 = [0,1]\times[0,1]$, and $C(I^2)$ be the space of all continuous functions on I^2, with the norm defined as:

$$\|g\| = \sup_{(y,z)\in I^2} \mid g(y,z) \mid .$$

The bivariate extension of the operators (1) is defined as:

$$J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;y,z) = \sum_{i=0}^{n}\sum_{j=0}^{m} p^{\alpha_1,1}_{n,i}(y)\, p^{\alpha_2,1}_{m,j}(z) \int_0^1\int_0^1 \mu^{\rho_1}_{n,i}(r)\,\mu^{\rho_2}_{m,j}(s)\, g(r,s)\, ds\, dr, \tag{3}$$

where $\mu^{\rho_1}_{n,i}(r)$, $\mu^{\rho_2}_{m,j}(s)$ are same as in (2) and

$$p^{\alpha_1,1}_{n,i}(y) = a(y,n)\,p^{\alpha_1}_{n-1,i}(y) + a(1-y,n)\,p^{\alpha_1}_{n-1,i-1}(y),$$

$$p^{\alpha_2,1}_{m,j}(z) = b(z,m)\,p^{\alpha_2}_{m-1,j}(z) + b(1-z,m)\,p^{\alpha_2}_{m-1,j-1}(z).$$

For any $f(y), g(z) \in C(I)$, our operators (3) satisfy the following relationship:

$$J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(f(y).g(z);y,z) = J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(f(y);y,z).J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g(z);y,z).$$

Lemma 2.1. *For the operators (3) and test functions e_{ij}, we have the following:*

$$\begin{aligned}
J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(e_{00};y,z) &= 1;\\
J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(e_{10};y,z) &= y + \frac{(1-2y)(\rho_1+1-\rho_1 a_0(n))}{n\rho_1+2};\\
J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(e_{01};y,z) &= z + \frac{(1-2z)(\rho_2+1-\rho_2 b_0(m))}{m\rho_2+2};\\
J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(e_{20};y,z) &= y^2 + \frac{1}{(n\rho_1+2)(n\rho_1+3)}\left[n\rho_1 y\{(1+\rho_1)(3-5y) - 2a_0(n)\rho_1(1-2y)\}\right.\\
&\quad + {\rho_1}^2(1-2y(1-y)(1+\alpha_1)) + 3\rho_1(1-2y) + 2(1-3y^2)\\
&\quad \left. -a_0(n)\rho_1(\rho_1(4y^2-4y+1)+3(1-2y))\right];\\
J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(e_{02};y,z) &= z^2 + \frac{1}{(m\rho_2+2)(m\rho_2+3)}\left[m\rho_2 z\{(1+\rho_2)(3-5z) - 2b_0(m)\rho_2(1-2z)\}\right.\\
&\quad + {\rho_2}^2(1-2z(1-z)(1+\alpha_2)) + 3\rho_2(1-2z) + 2(1-3z^2)\\
&\quad \left. -b_0(m)\rho_2(\rho_2(4z^2-4z+1)+3(1-2z))\right].
\end{aligned}$$

Lemma 2.2. *For the operators (3), we have the following central moments:*

$$\begin{aligned}
J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(r-y;y,z) &= \frac{(1-2y)(\rho_1+1-\rho_1 a_0(n))}{n\rho_1+2};\\
J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(s-z;y,z) &= \frac{(1-2z)(\rho_2+1-\rho_2 b_0(m))}{m\rho_2+2};\\
J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((r-y)^2;y,z) &= \frac{1}{(n\rho_1+2)(n\rho_1+3)}\Big[n\rho_1(1+\rho_1)y(1-y)-y(1-y)\{6(1+2\rho_1)+2{\rho_1}^2(1+\alpha_1)\}\\
&\quad +(\rho_1+1)(\rho_1+2)+a_0(n)\rho_1(\rho_1+3)(4y(1-y)-1)\Big];\\
J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((s-z)^2;y,z) &= \frac{1}{(m\rho_2+2)(m\rho_2+3)}\Big[m\rho_2(1+\rho_2)z(1-z)-z(1-z)\{6(1+2\rho_2)+2{\rho_2}^2(1+\alpha_2)\}\\
&\quad +(\rho_2+1)(\rho_2+2)+b_0(m)\rho_2(\rho_2+3)(4z(1-z)-1)\Big];\\
J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((r-y)^4;y,z) &= \frac{3{\rho_1}^2(1+\rho_1)^2y^2(1-y)^2n^2}{(n\rho_1+2)(n\rho_1+3)(n\rho_1+4)(n\rho_1+5)}+O\left(\frac{1}{n^3}\right);\\
J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((s-z)^4;y,z) &= \frac{3{\rho_2}^2(1+\rho_2)^2z^2(1-z)^2m^2}{(m\rho_2+2)(m\rho_2+3)(m\rho_2+4)(m\rho_2+5)}+O\left(\frac{1}{m^3}\right).
\end{aligned}$$

Corollary 2.3. *From the Lemma 2.2, we can easily get:*

$$\begin{aligned}
J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((r-y)^2;y,z) &\le \frac{1+\rho_1}{n\rho_1+2}\left[y(1-y)+\frac{2+\rho_1}{n\rho_1+2}\right],\\
J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((s-z)^2;y,z) &\le \frac{1+\rho_2}{m\rho_2+2}\left[z(1-z)+\frac{2+\rho_2}{m\rho_2+2}\right].
\end{aligned}$$

Now, we define some definitions related to bivariate functions:

Definition 2.4. ***Complete Modulus of Continuity:*** *([7], p.80) For any $g \in C(I^2)$, we have:*

$$\begin{aligned}
\omega(g;\delta_1,\delta_2) &= \sup\{\mid g(v,w)-g(y,z)\mid; (v,w),(y,z)\in I^2 \;\&\; \mid v-y\mid\le\delta_1, \mid w-z\mid\le\delta_2\},\\
\text{or}\quad \omega(g;\delta) &= \sup\{\mid g(v,w)-g(y,z)\mid; \sqrt{(v-y)^2+(w-z)^2}\le\delta,\ (v,w),(y,z)\in I^2\},
\end{aligned}$$

with the following properties:

i) $\omega(g;\delta_1,\delta_2)\to 0$ as $\delta_1,\delta_2\to 0$.

ii) $\mid g(v,w)-g(y,z)\mid\le \omega(g;\delta_1,\delta_2)\left(1+\frac{\mid v-y\mid}{\delta_1}\right)\left(1+\frac{\mid w-z\mid}{\delta_2}\right)$.

Definition 2.5. ***Partial Modulus of Continuity:*** *([7], p.81) For any $g\in C(I^2)$, the partial modulus of continuity is defined by:*

$$\begin{aligned}
\omega^1(g;\delta_1) &= \sup\{\mid g(y_1,z)-g(y_2,z)\mid; z\in I, \mid y_1-y_2\mid\le\delta_1\},\\
\text{and}\quad \omega^2(g;\delta_2) &= \sup\{\mid g(y,z_1)-g(y,z_2)\mid; y\in I, \mid z_1-z_2\mid\le\delta_2\}.
\end{aligned}$$

Definition 2.6. ***Lipschitz Condition:*** *([5], p.377) A function g satisfies Lipschitz condition i.e. $g\in Lip_M(\zeta,\eta)$ is defined as*

$$\mid g(r,s)-g(y,z)\mid < M\mid r-y\mid^{\zeta}\mid s-z\mid^{\eta},$$

where $\zeta,\eta\in(0,1]$.

Let $C^2(I^2)$ be the space of all functions $g \in C(I^2)$ such that $\dfrac{\partial^i g}{\partial y^i}, \dfrac{\partial^i g}{\partial z^i} \in C(I^2), i = 1,2$.
The norm on the space $C^2(I^2)$ is defined as:

$$\|g\|_{C^2(I^2)} = \|g\| + \sum_{i=1}^{2}\left(\left\|\frac{\partial^i g}{\partial y^i}\right\| + \left\|\frac{\partial^i g}{\partial z^i}\right\|\right).$$

Definition 2.7. $K-$**functional:** *Peetre [32] introduced the K–functional for $g \in C(I^2)$:*

$$K(g;\delta) = \inf\{\|g - f\| + \delta\|f\|_{C^2(I^2)}; f \in C^2(I^2)\}.$$

Definition 2.8. Second order modulus of smoothness: *([2], p.5558) For $g \in C(I^2)$:*

$$\omega_2(g;\delta) = \sup\{|g(y+2v, z+2w) - 2g(y+v, z+w) + g(y,z)|; (y,z), (y+2v, z+2w) \in I^2, |\, v \,|\leq \delta^2, |\, w \,|\leq \delta^2\}.$$

Also, K-functional and second order modulus of continuity are related by the following relation[16]:

$$K(g;\delta) \leq C\left[\omega_2(g;\sqrt{\delta}) + min(1,\delta)\|g\|\right]. \tag{4}$$

3. Approximation Results

Theorem 3.1. *Let $g \in C(I^2)$, the operators $J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;y,z)$ converge uniformly to $g(y,z)$.*

Proof. By Lemma 2.1, we can calculate:

$$\begin{aligned}
\lim_{n,m\to\infty} J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(e_{10};y,z) &= y;\\
\lim_{n,m\to\infty} J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(e_{01};y,z) &= z;\\
\lim_{n,m\to\infty} J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(e_{20}+e_{02};y,z) &= y^2+z^2.
\end{aligned}$$

Now, by using Volkov's result for bivariate functions, we get that the operators $J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;y,z)$ converge uniformly to $g(y,z)$. □

Theorem 3.2. *(Voronovskaja type result) For any $g \in C^2(I^2)$ and $(y,z) \in I^2$ we obtain:*

$$\begin{aligned}
\lim_{n\to\infty} n(J^{\alpha_1,\alpha_1}_{n,n,\rho_1,\rho_1}(g;y,z) - g(y,z)) &= \frac{(\rho_1 + 1 - \rho_1 a_0(n))}{\rho_1}\Big[(1-2y)g_y(y,z) + (1-2z)g_z(y,z)\Big]\\
&\quad + \frac{(1+\rho_1)}{2\rho_1}\Big[y(1-y)g_{yy}(y,z) + z(1-z)g_{zz}(y,z)\Big].
\end{aligned}$$

Proof. Using the Taylor's formula for a fixed point $(y,z) \in I^2$:

$$\begin{aligned}
g(r,s) &= g(y,z) + g_y(y,z)(r-y) + g_z(y,z)(s-z) + \frac{1}{2}\Big[g_{yy}(y,z)(r-y)^2\\
&\quad + 2g_{yz}(y,z)(r-y)(s-z) + g_{zz}(y,z)(s-z)^2\Big] + \Theta(r,s)\{(r-y)^2 + (s-z)^2\},
\end{aligned}$$

where $\lim\limits_{(r,s)\to(y,z)} \Theta(r,s) = 0$ and $(y,z) \in I^2$.
Applying the operators and using its linearity property:

$$\begin{aligned} n\left(J^{\alpha_1,\alpha_1}_{n,n,\rho_1,\rho_1}(g;y,z) - g(y,z)\right) &= n\Big(g_y(y,z)J^{\alpha_1,\alpha_1}_{n,n,\rho_1,\rho_1}(r-y;y,z) + g_z(y,z)J^{\alpha_1,\alpha_1}_{n,n,\rho_1,\rho_1}(s-z;y,z) \\ &+\frac{1}{2}\Big\{g_{yy}(y,z)J^{\alpha_1,\alpha_1}_{n,n,\rho_1,\rho_1}((r-y)^2;y,z) \\ &+ 2g_{yz}(y,z)J^{\alpha_1,\alpha_1}_{n,n,\rho_1,\rho_1}((r-y)(s-z);y,z) \\ &+ g_{zz}(y,z)J^{\alpha_1,\alpha_1}_{n,n,\rho_1,\rho_1}((s-z)^2;y,z)\Big\} \\ &+J^{\alpha_1,\alpha_1}_{n,n,\rho_1,\rho_1}(\Theta(r,s)\{(r-y)^2+(s-z)^2\};y,z)\Big). \end{aligned} \tag{5}$$

Apply Holder's inequality to the last term of right hand side of (5):
$nJ^{\alpha_1,\alpha_1}_{n,n,\rho_1,\rho_1}(\Theta(r,s)\{(r-y)^2+(s-z)^2\};y,z)$

$$\begin{aligned} &\leq n\left[J^{\alpha_1,\alpha_1}_{n,n,\rho_1,\rho_1}(\Theta^2(r,s);y,z)\right]^{\frac{1}{2}}.\left[J^{\alpha_1,\alpha_1}_{n,n,\rho_1,\rho_1}(((r-y)^2+(s-z)^2)^2;y,z)\right]^{\frac{1}{2}} \\ &\leq \left[J^{\alpha_1,\alpha_1}_{n,n,\rho_1,\rho_1}(\Theta^2(r,s);y,z)\right]^{\frac{1}{2}} \\ &\quad\times\sqrt{2}n\left[J^{\alpha_1,\alpha_1}_{n,n,\rho_1,\rho_1}((r-y)^4;y,z) + J^{\alpha_1,\alpha_1}_{n,n,\rho_1,\rho_1}((s-z)^4;y,z)\right]^{\frac{1}{2}}. \end{aligned} \tag{6}$$

$[\because (a+b)^2 \leq 2(a^2+b^2)]$

By using Theorem 3.1, we get:

$$\lim_{n\to\infty} J^{\alpha_1,\alpha_1}_{n,n,\rho_1,\rho_1}(\Theta^2(r,s);y,z) = 0.$$

With the help of central moments of order 4 in Lemma 2.2, we get:

$$\lim_{n\to\infty} n\,J^{\alpha_1,\alpha_1}_{n,n,\rho_1,\rho_1}(\Theta(r,s)\{(r-y)^2+(s-z)^2\};y,z) = 0.$$

Using Lemma 2.2, (5) and (6), we get our required result. □

Theorem 3.3. *For any $g \in C(I^2)$, the operators $J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;y,z)$ satisfy the following relation:*

$$\mid J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;y,z) - g(y,z) \mid \;\leq\; 4\,\omega(g;\delta_1(y),\delta_2(z)),$$

where $\delta_1(y) := \sqrt{J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((r-y)^2;y,z)}$ and $\delta_2(z) := \sqrt{J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((s-z)^2;y,z)}$.

Proof. By using linearity of the operators (3), we have:

$$\mid J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;y,z) - g(y,z) \mid \;\leq\; J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\mid g(r,s) - g(y,z) \mid;y,z).$$

With the help of the relation (ii) of complete modulus of continuity:

$$\begin{aligned} \mid J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;y,z) - g(y,z) \mid &\leq \omega(g;\delta_1(y),\delta_2(z))\left(1 + \frac{J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\mid r-y \mid;y,z)}{\delta_1(y)}\right) \\ &\quad.\left(1 + \frac{J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\mid s-z \mid;y,z)}{\delta_2(z)}\right). \end{aligned} \tag{7}$$

By applying the Cauchy-Schwarz's inequality, we obtain:

$$J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\mid r-y \mid;y,z) \;\leq\; \left[J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((r-y)^2;y,z)\right]^{\frac{1}{2}}.1 := \delta_1(y),$$

and

$$J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(| s-z |; y,z) \leq \left[J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((s-z)^2; y,z)\right]^{\frac{1}{2}}.1 := \delta_2(z).$$

Now, substituting the above two inequalities in (7), we obtain the result. □

Theorem 3.4. *For $g \in C(I^2)$, the operators (3) have the following inequality:*

$$| J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g; y,z) - g(y,z) | \leq 2\left[\omega^1(g;\delta_1(y)) + \omega^2(g;\delta_2(z))\right],$$

where $\delta_1(y)$ and $\delta_2(z)$ are defined as in Thm 3.3.

Proof. With the help of linearity and partial modulus of continuity, we get:

$$\begin{aligned} | J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g; y,z) - g(y,z) | &\leq J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(| g(r,s) - g(y,z) |; y,z) \\ &\leq J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(| g(r,z) - g(y,z) |; y,z) + J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(| g(r,s) - g(r,z) |; y,z) \\ &\leq J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\omega^1(g; | r-y |); y,z) + J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\omega^2(g; | s-z |); y,z). \end{aligned} \tag{8}$$

As we know the relation of modulus of continuity $\omega(\lambda\delta) \leq (1+\lambda)\,\omega(\delta)$, for $\lambda > 0$, then, using this relation in (8), we obtain:

$$\begin{aligned} | J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g; y,z) - g(y,z) | &\leq J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}\left(\left(1 + \frac{| r-y |}{\delta_1(y)}\right)\omega^1\left(g;\delta_1(y)\right); y,z\right) \\ &\quad + J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}\left(\left(1 + \frac{| s-z |}{\delta_2(z)}\right)\omega^2\left(g;\delta_2(z)\right); y,z\right). \end{aligned}$$

With the help of Cauchy-Schwarz inequality the above term reduces to:

$$\begin{aligned} | J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g; y,z) - g(y,z) | &\leq \omega^1(g;\delta_1(y))\left\{1 + \frac{\sqrt{J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((r-y)^2; y,z)}}{\delta_1(y)}\right\} \\ &\quad + \omega^2(g;\delta_2(z))\left\{1 + \frac{\sqrt{J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((s-z)^2; y,z)}}{\delta_2(z)}\right\}. \end{aligned}$$

On choosing $\sqrt{J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((r-y)^2; y,z)} := \delta_1(y)$ and $\sqrt{J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((s-z)^2; y,z)} := \delta_2(z)$, the proof is done. □

Theorem 3.5. *Let $g \in Lip_M(\zeta,\eta)$, then we have the following inequality:*

$$| J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g; y,z) - g(y,z) | \leq M(\delta_1(y))^{\zeta}.(\delta_2(z))^{\eta},$$

where M is a positive constant and $\delta_1(y)$ and $\delta_2(z)$ are same as defined in Thm 3.3.

Proof. As $g \in Lip_M(\zeta,\eta)$, then it gives:

$$| g(r,s) - g(y,z) | \leq M| r-y |^{\zeta} | s-z |^{\eta}.$$

Now,

$$\begin{aligned} | J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g; y,z) - g(y,z) | &\leq J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(| g(r,s) - g(y,z) |; y,z) \\ &\leq J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(M| r-y |^{\zeta} | s-z |^{\eta}; y,z) \\ &= MJ^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(| r-y |^{\zeta}; y,z).J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(| s-z |^{\eta}; y,z). \end{aligned} \tag{9}$$

Using Holder's inequality

$$J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\mid r-y\mid^{\zeta};y,z)\leq\left[J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((r-y)^2;y,z)\right]^{\frac{\zeta}{2}}=(\delta_1(y))^{\zeta}.$$

Similarly,

$$J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\mid s-z\mid^{\eta};y,z)\leq\left[J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((s-z)^2;y,z)\right]^{\frac{\eta}{2}}=(\delta_2(z))^{\eta}.$$

Now by using these inequalities in (9), we attain the desired outcome. □

Theorem 3.6. *For $g\in C(I^2)$, we get the error estimation in terms of first and second order modulus of continuity:*

$$\begin{aligned}\mid J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;y,z)-g(y,z)\mid &\leq 4C\left[\omega_2\left(g;\frac{1}{2}\sqrt{V_{n,m}(y,z)}\right)+\min\left(1,\frac{1}{4}V_{n,m}(y,z)\right)\right]\\ &\quad+\omega(g;\mu_{n,m}(y,z)),\\ \textit{where}\quad V_{n,m}(y,z)&=\frac{1}{2}\left[(\delta_1(y))^2+\left(\frac{(1-2y)(\rho_1+1-\rho_1a_0(n))}{n\rho_1+2}\right)^2\right.\\ &\quad\left.+(\delta_2(z))^2+\left(\frac{(1-2z)(\rho_2+1-\rho_2b_0(m))}{m\rho_2+2}\right)^2\right],\\ \textit{and}\quad \mu_{n,m}(y,z)&=\left(\left(\frac{(1-2y)(\rho_1+1-\rho_1a_0(n))}{n\rho_1+2}\right)^2+\left(\frac{(1-2z)(\rho_2+1-\rho_2b_0(m))}{m\rho_2+2}\right)^2\right)^{\frac{1}{2}},\end{aligned}$$

where $C>0$ is a constant and $\delta_1(y)$ and $\delta_2(z)$ are same as defined in Thm 3.3.

Proof. Firstly, we define an auxiliary operators for $(y,z)\in I^2$:

$$\begin{aligned}\tilde{J}^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;y,z)&=J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;y,z)+g(y,z)\\ &\quad-g\left(y+\frac{(1-2y)(\rho_1+1-\rho_1a_0(n))}{n\rho_1+2},z+\frac{(1-2z)(\rho_2+1-\rho_2b_0(m))}{m\rho_2+2}\right).\end{aligned}$$

By using this definition, we get:

$$\tilde{J}^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(r-y;y,z)=0;\qquad \tilde{J}^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(s-z;y,z)=0.$$

For $h\in C^2(I^2)$, we consider
$\mid J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;y,z)-g(y,z)\mid$

$$\begin{aligned}&\leq \left|J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;y,z)-\tilde{J}^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;y,z)\right|+\left|\tilde{J}^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;y,z)-\tilde{J}^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(h;y,z)\right|\\ &\quad+\left|\tilde{J}^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(h;y,z)-h(y,z)\right|+\left|h(y,z)-g(y,z)\right|\\ &=\left|g\left(y+\frac{(1-2y)(\rho_1+1-\rho_1a_0(n))}{n\rho_1+2},z+\frac{(1-2z)(\rho_2+1-\rho_2b_0(m))}{m\rho_2+2}\right)-g(y,z)\right|\\ &\quad+\left|\tilde{J}^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g-h;y,z)\right|+\left|\tilde{J}^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(h;y,z)-h(y,z)\right|+\left|h(y,z)-g(y,z)\right|.\end{aligned}\tag{10}$$

Now, by Taylor's polynomial for $h(r,s)\in C^2(I^2)$, we have:

$$h(r,s)=h(y,z)+(r-y)\frac{\partial h(y,z)}{\partial y}+\int_y^r(r-v)\frac{\partial^2h(v,z)}{\partial v^2}\,dv+(s-z)\frac{\partial h(y,z)}{\partial z}+\int_z^s(s-w)\frac{\partial^2h(y,w)}{\partial w^2}\,dw.$$

Applying the operators $\tilde{J}^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(.;y,z)$:

$$\left|\tilde{J}^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(h;y,z)-h(y,z)\right|$$

$$\begin{aligned}
&\leq \left|\tilde{J}^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}\left(\int_y^r (r-v)\frac{\partial^2 h(v,z)}{\partial v^2}\,dv;y,z\right)\right| + \left|\tilde{J}^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}\left(\int_z^s (s-w)\frac{\partial^2 h(y,w)}{\partial w^2}\,dw;y,z\right)\right| \\
&\leq J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}\left(\left|\int_y^r |r-v|\left|\frac{\partial^2 h(v,z)}{\partial v^2}\right| dv\right|;y,z\right) \\
&\quad + \left|\int_y^{y+\frac{(1-2y)(\rho_1+1-\rho_1 a_0(n))}{n\rho_1+2}} \left|y+\frac{(1-2y)(\rho_1+1-\rho_1 a_0(n))}{n\rho_1+2}-v\right|\left|\frac{\partial^2 h(v,z)}{\partial v^2}\right| dv\right| \\
&\quad + J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}\left(\left|\int_z^s |s-w|\left|\frac{\partial^2 h(y,w)}{\partial w^2}\right| dw\right|;y,z\right) \\
&\quad + \left|\int_z^{z+\frac{(1-2z)(\rho_2+1-\rho_2 b_0(m))}{m\rho_2+2}} \left|z+\frac{(1-2z)(\rho_2+1-\rho_2 b_0(m))}{m\rho_2+2}-w\right|\left|\frac{\partial^2 h(y,w)}{\partial w^2}\right| dw\right| \\
&\leq \left[J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((r-y)^2;y,z)+\left(\frac{(1-2y)(\rho_1+1-\rho_1 a_0(n))}{n\rho_1+2}\right)^2\right. \\
&\quad \left. + J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((s-z)^2;y,z)+\left(\frac{(1-2z)(\rho_2+1-\rho_2 b_0(m))}{m\rho_2+2}\right)^2\right]\|h\|_{C^2(I^2)}.
\end{aligned}$$

With the values of $\delta_1(y)$ and $\delta_2(z)$, we have:

$$\left|\tilde{J}^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(h;y,z)-h(y,z)\right| \leq V_{n,m}(y,z)\|h\|_{C^2(I^2)}.$$

By using the definition of operators $\tilde{J}^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;y,z)$, we obtain:

$$|\ \tilde{J}^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;y,z)\ | \leq 3\|g\|_{C(I^2)}.$$

Thus, the equation (10) becomes:

$$\begin{aligned}
|\ J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;y,z)-g(y,z)\ | &\leq 4\|g-h\|_{C(I^2)}+V_{n,m}(y,z)\|h\|_{C^2(I^2)} \\
&\quad +\omega\left(g;\mu_{n,m}(y,z)\right) \\
&= 4\left\{\|g-h\|_{C(I^2)}+\frac{1}{4}V_{n,m}(y,z)\|h\|_{C^2(I^2)}\right\}+\omega(g;\mu_{n,m}(y,z)).
\end{aligned}$$

Taking the infimum over $h\in C^2(I^2)$ and using the relation (4), we get our desired result. □

4. Numerical Verification

In this section, we provide an example with various values of parameters and sequences $a_i(n), i=0,1$ to support our previously proven theoretical findings. For this purpose, we take into the account the function $g(y,z)=y^4z^2-2y^2z^3+yz^2$.

Firstly, we present the convergence of the operators $J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;y,z)$ with specific values of parameters to the function $g(y,z)$ in Fig. 1 having sequences $a_1(n)=\dfrac{1}{n}, a_0(n)=\dfrac{n-1}{2n}, b_1(m)=\dfrac{1}{m}, b_0(m)=\dfrac{m-1}{2m}$.

Also, we give their error estimation, which is $E^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;y,z)=|\ J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;y,z)-g(y,z)\ |$ for the particular values of parameters in Fig. 2.

We can see in both images that when the values of n,m increase, the operators converge quicker to the specified function and the error term decrease rapidly.

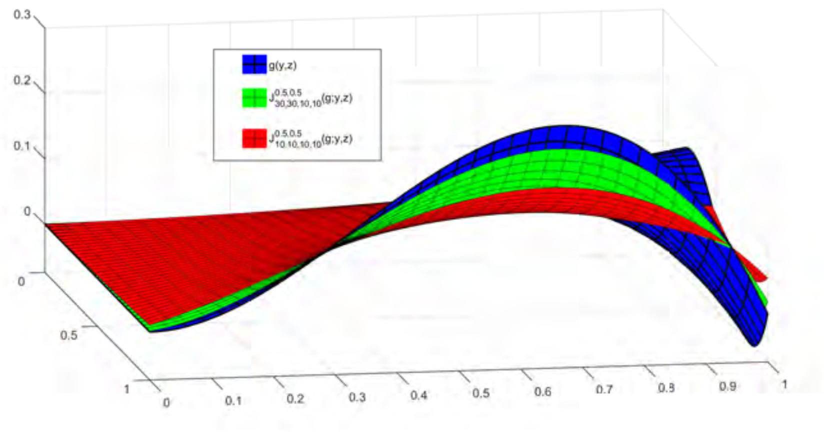

Figure 1: Approximation Process

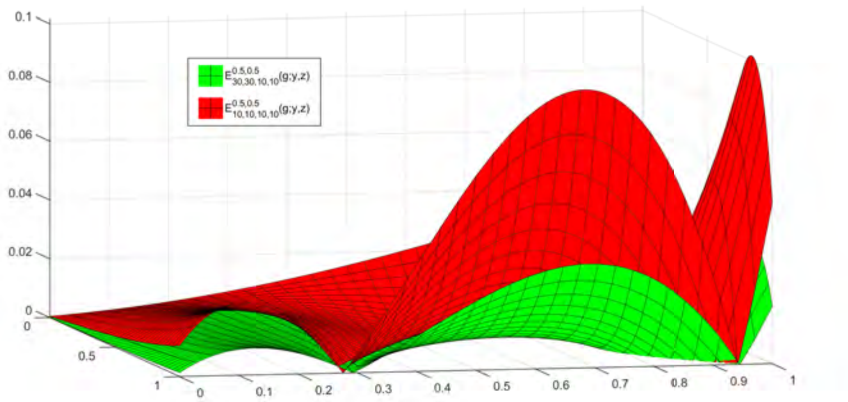

Figure 2: Error estimation

In Fig. 3, we show the effect of the different sequences $a_i(n), i = 0, 1$ in the convergence of the operators with specific parameters $n = m = 30, \alpha_1 = \alpha_2 = 0.3, \rho_1 = \rho_2 = 4$. For the operators $J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g; y, z)$, we have chosen the different sets of sequences with different rate of convergence:

$$a_1(n) = \frac{n-1}{n}, a_0(n) = \frac{1}{2n}, b_1(m) = \frac{m-1}{m}, b_0(m) = \frac{1}{2m};$$

$$a_1(n) = \frac{1-n^2}{n^2}, a_0(n) = \frac{2n^2-1}{2n^2}, b_1(m) = \frac{1-m^2}{m^2}, b_0(m) = \frac{2m^2-1}{2m^2};$$

$$a_1(n) = -1, a_0(n) = 1, b_1(m) = -1, b_0(m) = 1.$$

Also, we have presented the error of approximation of the given function from the operators in Fig. 4.

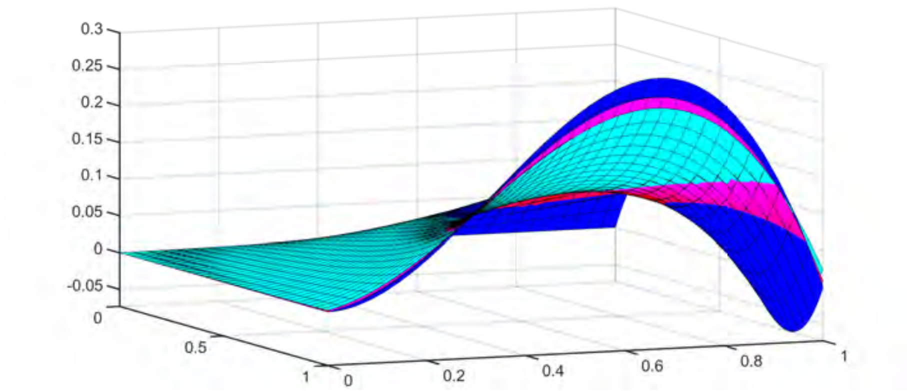

Figure 3: Approximation Process

$g(y,z)(Blue)$, $[a_1(n)=1, a_0(n)=-1, b_1(m)=-1, b_0(m)=-1](Red)$,
$\left[a_1(n)=\frac{n-1}{n}, a_0(n)=\frac{1}{2n}, b_1(m)=\frac{m-1}{m}, b_0(m)=\frac{1}{2m}\right](Cyan)$
$\left[a_1(n)=\frac{1-n^2}{n^2}, a_0(n)=\frac{2n^2-1}{2n^2}, b_1(m)=\frac{1-m^2}{m^2}, b_0(m)=\frac{2m^2-1}{2m^2}\right](Meganta)$

5. Generalized Boolean Sum (GBS) operators

Bögel [13, 14] proposed the concept of B-continuous and B-differentiable functions in 1934 and 1935. The approximation results concerning these functions were firstly introduced by Dobrescu and Matei [18]. Badea and Cottin [10] established the Korovkin type theorem for B-continuous functions, which is also famous as test function theorem. In 2013, Miclăuş [29] studied the approximation results of GBS of Bernstein-Stancu operators. Agrawal and İspir [5] studied the degree of approximation for bivariate Chlodowsky-Szasz-Charlier type operators. Similarly, Bŏrbosu et al. [12] proposed the GBS Durrmeyer operators based on $q-$integers. The authors studied convergence and degree of approximation of these variants. A lot of research on these operators is going on as we can see [22, 33, 34]. Inspired by these papers, we define the GBS operators associated with the operators (3).
Firstly, we present some definitions that are required in subsequent work defined as in [13, 14].
Let Y and Z are compact subsets of real numbers.

Definition 5.1. *A function $f: Y\times Z \to \mathbb{R}$ is said to be B-continuous or Bögel continuous at a point $(y,z)\in Y\times Z$ if*

$$\lim_{(r,s)\to(y,z)} \Delta f[(r,s),(y,z)] = 0,$$

where $\Delta f[(r,s),(y,z)] = f(r,s) - f(y,s) - f(r,z) + f(y,z)$.

The space of all B-continuous functions is denoted by $C_b(Y\times Z)$.

Definition 5.2. *A function $f: Y\times Z \to \mathbb{R}$ is called B-differentiable at $(y,z)\in Y\times Z$ if*

$$\lim_{(r,s)\to(y,z)} \frac{\Delta f[(r,s),(y,z)]}{(r-y)(s-z)}$$

exists and finite. It is denoted by $D_B f(y,z)$.

The space of all B-differentiable functions is denoted by $D_b(Y\times Z)$.

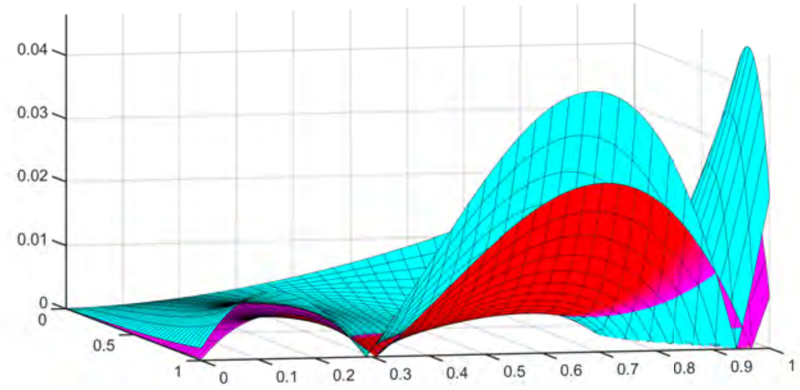

Figure 4: Error estimation

$$[a_1(n) = 1, a_0(n) = -1, b_1(m) = -1, b_0(m) = -1](Red),$$
$$\left[a_1(n) = \frac{n-1}{n}, a_0(n) = \frac{1}{2n}, b_1(m) = \frac{m-1}{m}, b_0(m) = \frac{1}{2m}\right](Cyan)$$
$$\left[a_1(n) = \frac{1-n^2}{n^2}, a_0(n) = \frac{2n^2-1}{2n^2}, b_1(m) = \frac{1-m^2}{m^2}, b_0(m) = \frac{2m^2-1}{2m^2}\right](Meganta)$$

Definition 5.3. *A function $f : Y \times Z \to \mathbb{R}$ is called B-bounded on $Y \times Z$ if there exists a constant $M > 0$ such that*

$$\mid \Delta f[(r,s),(y,z)] \mid \leq M,$$

for every $(r,s),(y,z) \in Y \times Z$.

Definition 5.4. *For any $f \in C_b(I^2)$, the mixed modulus of continuity $\omega_{mixed} : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is defined as:*

$$\omega_{mixed}(f;\sigma_1,\sigma_2) = \sup\{\mid \Delta f[(r,s),(y,z)] \mid ; \mid r-y \mid < \sigma_1, \mid s-z \mid < \sigma_2\}$$

for all $(y,z),(r,s) \in I^2$.

It satisfies the property:

$$\omega_{mixed}(g;\lambda_1\sigma_1,\lambda_2\sigma_2) \leq (1+\lambda_1)(1+\lambda_2)\,\omega_{mixed}(g;\sigma_1,\sigma_2), \tag{11}$$

where $\lambda_1,\lambda_2 > 0$.
$B_b(Y \times Z)$ denotes the space of all B-bounded functions having the norm

$$\|f\|_B = \sup_{(r,s),(y,z)\in Y\times Z} \mid \Delta f[(r,s),(y,z)] \mid .$$

5.1. Construction of GBS operators

Let $g \in C_b(I^2)$. The GBS operators $K^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;y,z)$ associated to the operators $J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;y,z)$ is defined as follows:

$$K^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;y,z) = J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}[g(r,z)+g(y,s)-g(r,s);y,z], \quad (y,z) \in I^2. \tag{12}$$

Theorem 5.5. *For every $g \in C_b(I^2)$, the operators (12) satisfy the following inequality:*

$$\mid K^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;y,z) - g(y,z) \mid \leq M\omega_{mixed}\left(g;\frac{1}{\sqrt{n\rho_1+2}},\frac{1}{\sqrt{m\rho_2+2}}\right),$$

where $M > 0$ is a constant depending only on ρ_1 and ρ_2.

Proof. As we know:

$$\begin{aligned} \Delta g[(r,s),(y,z)] &= g(y,z) - g(y,s) - g(r,z) + g(r,s), \\ \text{then} \quad g(y,s) + g(r,z) - g(r,s) &= g(y,z) - \Delta g[(r,s),(y,z)]. \end{aligned}$$

Applying the operators (3) and using eq. (11), we have:

$$\begin{aligned} \mid K^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;y,z) - g(y,z) \mid &\leq J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\mid \Delta g[(r,s),(y,z)] \mid; y,z) \\ &\leq J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}\left(\left(1 + \frac{\mid r-y \mid}{\sigma_1}\right)\left(1 + \frac{\mid s-z \mid}{\sigma_2}\right)\omega_{mixed}(g;\sigma_1,\sigma_2); y,z\right) \\ &\leq \left\{1 + \sigma_1^{-1}\left[J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((r-y)^2;y,z)\right]^{\frac{1}{2}} + \sigma_2^{-1}\left[J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((s-z)^2;y,z)\right]^{\frac{1}{2}} \right. \\ &\quad \left. + \sigma_1^{-1}\sigma_2^{-1}\left[J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((r-y)^2;y,z).J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((s-z)^2;y,z)\right]^{\frac{1}{2}}\right\}\omega_{mixed}(g;\sigma_1,\sigma_2). \end{aligned}$$

By using Corollary 2.3, we can get:

$$J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((r-y)^2;y,z) \leq \frac{2(1+\rho_1)}{n\rho_1+2} \quad \& \quad J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((s-z)^2;y,z) \leq \frac{2(1+\rho_2)}{m\rho_2+2}.$$

Also, by choosing $\sigma_1 = \frac{1}{\sqrt{n\rho_1+2}}$ and $\sigma_2 = \frac{1}{\sqrt{m\rho_2+2}}$, the proof is completed. □

Theorem 5.6. *Let $g \in D_b(I^2)$ with $D_B(g) \in B(I^2)$. Then for every $(y,z) \in I^2$, we get:*

$$\mid K^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;y,z) - g(y,z) \mid \leq \frac{M}{\sqrt{n\rho_1+2}.\sqrt{m\rho_2+2}}\left[\|D_B g\| + \omega_{mixed}\left(D_B g; \frac{1}{\sqrt{n\rho_1+2}}, \frac{1}{\sqrt{m\rho_2+2}}\right)\right].$$

Proof. For $g \in D_b(I^2)$,

$$\begin{aligned} \Delta g[(r,s),(y,z)] &= (r-y)(s-z)D_B g(\zeta,\eta) \quad \text{with} \quad y < \zeta < r,\ z < \eta < s, \\ \text{where} \quad D_B g(\zeta,\eta) &= \Delta D_B g[(\zeta,\eta),(y,z)] + D_B g(\zeta,z) + D_B g(y,\eta) - D_B g(y,z). \end{aligned}$$

By using this relation and $D_B g \in B(I^2)$, we get:

$$\begin{aligned} \mid K^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;y,z) - g(y,z) \mid &= \mid J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\Delta g[(r,s),(y,z)];y,z) \mid \\ &= \mid J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((r-y)(s-z)D_B g(\zeta,\eta);y,z) \mid \\ &= \mid J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((r-y)(s-z)\{\Delta D_B g[(\zeta,\eta),(y,z)] + D_B g(\zeta,z) + D_B g(y,\eta) \\ &\quad - D_B g(y,z)\};y,z) \mid \\ &\leq J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\mid r-y \mid\mid s-z \mid\mid \Delta D_B g[(\zeta,\eta),(y,z)] \mid;y,z) \\ &\quad + J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\mid r-y \mid\mid s-z \mid \{\mid D_B g(\zeta,z) \mid + \mid D_B g(y,\eta) \mid \\ &\quad + \mid D_B g(y,z) \mid\};y,z) \\ &\leq J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\mid r-y \mid\mid s-z \mid \omega_{mixed}(D_B g; \mid r-y \mid\mid s-z \mid);y,z) \\ &\quad + 3\|D_B g\| J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\mid r-y \mid\mid s-z \mid;y,z) \\ &\leq \left[J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\mid r-y \mid\mid s-z \mid;y,z) \right. \\ &\quad + \sigma_1{}^{-1} J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((r-y)^2 \mid s-z \mid;y,z) \\ &\quad + \sigma_2{}^{-1} J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\mid r-y \mid (s-z)^2;y,z) \\ &\quad \left. + \sigma_1{}^{-1}\sigma_2{}^{-1} J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((r-y)^2(s-z)^2;y,z)\right]\omega_{mixed}(D_B g;\sigma_1,\sigma_2) \\ &\quad + 3\|D_B g\| J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\mid r-y \mid\mid s-z \mid;y,z). \end{aligned}$$

With the help of Cauchy-Schwarz inequality, it turns out to be:

$$\begin{aligned}
\mid K^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;y,z)-g(y,z)\mid \;\leq\; & \Bigg[\sqrt{J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((r-y)^2(s-z)^2;y,z)} \\
& +\sigma_1{}^{-1}\sqrt{J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((r-y)^4(s-z)^2;y,z)} \\
& +\sigma_2{}^{-1}\sqrt{J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((r-y)^2(s-z)^4;y,z)} \\
& +\sigma_1{}^{-1}\sigma_2{}^{-1}J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((r-y)^2(s-z)^2;y,z)\Bigg]\omega_{mixed}(D_Bg;\sigma_1,\sigma_2) \\
& +3\|D_Bf\|\sqrt{J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((r-y)^2(s-z)^2;y,z)}.
\end{aligned}$$

Using Corollary 2.3, we obtain:

$$J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((r-y)^2;y,z)\leq\frac{2(1+\rho_1)}{n\rho_1+2}\text{ and }J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}((s-z)^2;y,z)\leq\frac{2(1+\rho_2)}{m\rho_2+2}.$$

Now, by choosing $\sigma_1=\dfrac{1}{\sqrt{n\rho_1+2}}$ and $\sigma_2=\dfrac{1}{\sqrt{m\rho_2+2}}$ and Lemma 2.2, we get our result. □

6. Declarations

Ethical Approval: NA
Competing interests: Both the authors declare no competing interests.
Author's Contributions: M. G. and J. K. both done analysis, interpretation of results and draft manuscript. J. K. prepared the figures. Both authors reviewed the results and approved the final version of the manuscript.
Funding: The first author is thankful to the "University Grants Commission (UGC) (1001/(CSIR-UGC NET DEC. 2017))" India and DST-FIST (grant SR/FST/MS-1/2017/13) for financial support to carry out the above research work.
Availability of data and materials: NA

References

[1] T. Acar, A. M. Acu, and N. Manav, *Approximation of functions by genuine Bernstein-Durrmeyer type operators*, J. Math. Inequalities. **12** (4) (2018), 975–987.
[2] A. M. Acu, T. Acar, C. V. Muraru, and V. A. Radu, *Some approximation properties by a class of bivariate operators*, Math. Meth. Appl. Sci. **42** (2019), 5551-5565.
[3] A. M. Acu, V. Gupta, and G. Tachev, *Better numerical approximation by Durrmeyer type operators*, Results Math. **74** (3) (2019), 1–24.
[4] P. N. Agrawal and M. Goyal, *Bivariate extension of linear positive operators*, Mathematical Analysis, Approximation Theory and Their Applications, Springer **111** (2016), 15–62.
[5] P. N. Agrawal and N. İspir, *Degree of approximation for bivariate Chlodowsky-Szasz-Charlier type operators*, Results Math. **69** (2016), 369–385.
[6] F. Altomare and M. Campiti, *Korovkin-type approximation theory and its applications*, de Gruyter Studies in Mathematics, 17 Walter de Grutyer and Co. Berlin (1994).
[7] G. A. Anastassiou and S. Gal, *Approximation Theory: Moduli of Continuity and Global Smoothness Preservation*, Springer Science & Business Media (2000).
[8] K. Ansari, S. Karakiliç, and F. Özger, *Bivariate Bernstein-Kantorovich operators with a summability method and related GBS operators*, Filomat **36**(19) (2022), 6751–6765.
[9] A. Aral and V. Gupta, *On the q−analogue of Stancu-Beta operators*, Appl. Math. Lett. **25** (2012), 67–71.
[10] C. Badea and C. Cottin, *Korovkin-type theorems for generalized boolean sum operators, approximation theory*, In Colloq. Math. Soc. János Bolyai **58** (1990), 51–68.
[11] D. Bărbosu, *Some generalized bivariate Bernstein operators*, Math. Notes (Miskolc) **1**(1) (2000), 3–10.
[12] D. Bărbosu, A. M. Acu, and C. V. Muraru, *On certain GBS-Durrmeyer operators based on q-integers*, Turk. J. Math. **41** (2017), 368–380.
[13] K. Bögel, *Mehrdimensionale Differentiation von Funtionen mehrerer veränderlicher*, J. Reine Angew. Math. **170** (1934), 197–217.
[14] K. Bögel, *Über die mehrdimensionale differentiation, integration und beschränkte variation*, J. Reine Angew. Math. **173** (1935), 5–30.
[15] P. L. Butzer, *Linear combinations of Bernstein polynomials*, Canad. J. Math. **5** (2) (1953), 559–567.

[16] P. L. Butzer and H. Berens, *Semi-Groups of Operators and Approximation*, New York, Springer (1967).
[17] W. Chen, *On the modified Bernstein-Durrmeyer operator, Report of the Fifth Chinese Conference on Approximation Theory*, Zhen Zhou, China (1987).
[18] E. Dobrescu and I. Matei, *The approximation by Bernstein type polynomials of bidimensionally continuous functions*, An. Univ. Timişoara Ser. Sti. Mat. Fiz. **4** (1966), 85–90.
[19] N. K. Govil, V. Gupta, and D. Soybaş, *Certain new classes of Durrmeyer type operators*, Appl. Math. Comput. **225** (2013), 195–203.
[20] M. Goyal, A. Kajla, P. N. Agrawal, and S. Araci, *Approximation by Bivariate Bernstein-Durrmeyer operators on a triangle*, Appl. Math. Inf. Sci. **11** (3) (2017), 693–702.
[21] V. Gupta, G. Tachev, and A. M. Acu, *Modified Kantorovich operators with better approximation properties*, Numer. Algor. **81** (2019), 125–149.
[22] N. İspir, P. N. Agrawal, and A. Kajla, *GBS operators of Lupas-Durrmeyer type based on Polya distribution*, Results Math. **3** (4) (2016), 397–418.
[23] A. Kajla and T. Acar, *Modified α-Bernstein operators with better approximation properties*, Ann. Funct. Anal. **10** (2019), 570–582.
[24] A. Kajla and M. Goyal, *Modified Bernstein-Kantorovich operators for functions of one and two varables*, Rend. Circ. Mat. Palermo II Ser **67** (2018), 379–395.
[25] A. Kajla and M. Goyal, *Generalized Bernstein-Durrmeyer operators of blending type*, Afrika Mat. **30** (2019), 1103–1118.
[26] J. Kaur and M. Goyal, *Order improvement for the sequence of $\alpha-$Bernstein-Păltănea operators*, Int. J. Nonlinear Anal. Appl. DOI:10.22075/IJNAA.2023.28762.3982.
[27] H. Khosravian-Arab, M. Dehghan, and M. R. Eslahchi, *A new approach to improve the order of approximation of the Bernstein operators: theory and applications*, Numer. Algor. **77** (1) (2018), 111–150.
[28] C. A. Micchelli, *Saturation classes and iterates of operators*, Ph.D. Thesis. Standford, CA: Standford University (1969).
[29] D. Miclăuş, *On the GBS Bernstein-Stancu's type operators*, Creat. Math. Inform. **22** (2013), 73–80.
[30] G. V. Milovanović, V. Gupta, and N. Malik, *(p, q)-Beta functions and applications in aproximation*, Bol. Soc. Mat. Mex. **24**(1) (2018), 219–237.
[31] R. Păltănea, *A class of Durrmeyer type operators preserving linear functions*, J. Ann. Tiberiu popoviciu Semin. Funct. Equ. Approx. Convexity (Cluj-Napoca) **5** (2007), 109–118.
[32] J. Peetre, *A Theory of Interpolation of Normed Spaces. Noteas de Mathematica, Instituto de Mathemática Pura e Applicada*, Conselho Nacional de Pesquidas, Rio de Janeiro **39** (1968).
[33] O. T. Pop, *Approximation of B-differentiable functions by GBS operators*, Anal. Univ. Oradea Fasc. Mat. **14** (2007), 15–31.
[34] O. T. Pop, *The approximation of bivariate functions by bivariate operators and GBS operators*, Rev. Anal. Numér. Théorie. Approximation. **40** (2011), 64–79.
[35] N. Rao and A. Wafi, *Bivariate-Schurer-Stancu operators based on (p, q) integers*, Filomat **32**(4) (2018), 1251–1258.
[36] M. Skorupka, *Approximation of functions of two variables by some linear positive operators*, Mathematiche (Catania) **50** (2) (1995), 323–336.
[37] D. D. Stancu, *A new class of uniform approximating polynomial operators in two and several variables*, In proceedings of the conference on the constructive theory of the functions (Approximation Theory), Akadémiai Kiadó Budhapest (1969).
[38] A. Wafi and S. Khatoon, *Approximation by generalized Baskakov operators for functions of one and two variables in exponential and polynomial weight space*, Thai. J. Math. **2** (2004), 53–66.