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On Markov−switching asymmetric log *GARCH* models: stationarity and estimation

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Abstract. In the present paper, we study some probabilistic and statistical properties of the Markov-switching asymmetric log *GARCH* processes, where the log −volatility follows a standard asymmetric log *GARCH* process for each regime. In these models, the coefficients of log −volatility depend on the state of a non-observed Markov chain. The main motivations of this new model can capture the asymmetries and hence leverage effect. Additionally, The volatility coefficients are not subject to positivity constraints. Therefore, some probabilistic properties of Markov-switching asymmetric log *GARCH* models have been obtained, especially, sufficient conditions ensuring the existence of stationary, causal, ergodic solution and moments properties are given. Furthermore, we show the strong consistency of the quasi-maximum likelihood estimator (*QMLE*) under mild assumptions. Finally, we provide a simulation study of the performance of the proposed estimation method and the *MS* − log *GARCH* is applied to model the exchange rate of the Algerian Dinar against the US-dollar.

1. Introduction

Autoregressive conditional heteroscedastic (*ARCH*) models introduced by Engle [14] and their various generalizations, especially the generalized *ARCH* (*GARCH*) models introduced by Bollerslev [8] have attracted considerable attention and have been widely investigated in the literature. These models belong to symmetric models, such that the volatility is formulated as a linear combination of its past values and past values of the innovations. There are many models that the (log −)volatility depends on the past values and past values of the positive and negative parts of the innovations, but an important among them is the Exponential *GARCH* (*EGARCH*) introduced and studied by Nelson [38]. The success of these models is due to the fact that they allow asymmetric in volatility. Another important reason for the growing interest is that it does not impose any positivity restrictions on the volatility coefficients. For this, we wish to address another class of *GARCH*−type models that share the same precedent characteristics. The log *GARCH* model was independently proposed by Geweke [23], Pantula [39] and Milhϕj [37] (see also Sucarrat and Escribano [40], Francq et al. [20]-[21] and Ghezal [26] for more recent works). This interest is due to the absence of positivity constraints, and asymmetric, in addition, there is no minimum value for the volatility contrary to *GARCH* models and the majority of their extensions, and it also provides persistence of large and small

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values. It appears that feature such as regime changes remain uncaptured by this class of models. This feature is best represented by regime switching models. Since the seminal paper by Hamilton [29], the use of Markov-switching (in short, *MS*) models has become a powerful tool for modeling financial time series and changes in regime into the classical time series models substantially increases their flexibility. Recently, considerable effort has been dedicated to the analysis of various aspects of Markov-switching linear and nonlinear models, including [1]-[7], [11], [13], [18], [24]-[27], [30], [35]-[36], [43] and many others. Due to the importance of both *MS* and $\log GARCH$ models, we can combine these two approaches to form a new model that can be defined as a bivariate process $(\varepsilon_t, \delta_t)$, where the process (δ_t) is a Markov chain defined on a finite state space, and (ε_t) is a $\log GARCH$ process.
The paper is organized as follows. In section 2, we introduce the class of Markov-switching asymmetric $\log GARCH$ models and give some related notations and assumptions. In this section, we also introduce the formulation of the state-space representation of the given process. Section 3 studies the existence of a strictly stationary solution to the $MS - A\log GARCH$ equation. Conditions for the existence of $\log -$moments are established. In Section 4, the strong consistency of the *QMLE* is established under mild regularity conditions. Simulation results are reported in Section 5. Section 6 applies the $MS - A\log GARCH$ specification to model the daily series of the exchange rates of the Algerian Dinar against the US-dollar. Section 7 concludes the paper.

2. Markov-switching asymmetric log *GARCH* model

Let $(e_t, t \in \mathbb{Z})$ be a sequence of independent and identically distributed (*i.i.d.*) random variables with zero mean and unit variance and let $\omega, \alpha_i, \beta_i$ and γ_j, for $1 \leq i \leq q$ and $1 \leq j \leq p$, be real coefficients. Recall that a standard asymmetric $\log GARCH$ process $(\varepsilon_t, t \in \mathbb{Z})$, $\mathbb{Z} = \{0, \pm 1, \pm 2, ...\}$ with standard $\log -$volatility process $\left(\log \sigma_t^2, t \in \mathbb{Z}\right)$ is a solution to the equations

$$\varepsilon_t = \sigma_t e_t, t \in \mathbb{Z}, \tag{2.1}$$

where $\sigma_t > 0$ and,

$$\log \sigma_t^2 = \omega + \sum\nolimits_{i=1}^{q} (\alpha_i \mathbb{I}_{\{\varepsilon_{t-i}>0\}} + \beta_i \mathbb{I}_{\{\varepsilon_{t-i}<0\}}) \log \varepsilon_{t-i}^2 + \sum\nolimits_{j=1}^{p} \gamma_j \log \sigma_{t-j}^2, \tag{2.2}$$

which has also been previously suggested by Francq et al. [19], so, in this paper, we generate a new and broader class of asymmetric $\log GARCH$ models, in which the parameters are allowed to depend on the state of a non-observed Markov chain, as a result, we will provide a Markov-switching asymmetric $\log GARCH(p, q)$ model ($MS - A\log GARCH_d$) defined by Eq. (2.1) and the $\log -$volatility process, i.e.,

$$\log \sigma_t^2 = \omega(\delta_t) + \sum\nolimits_{i=1}^{q} (\alpha_i(\delta_t) \mathbb{I}_{\{\varepsilon_{t-i}>0\}} + \beta_i(\delta_t) \mathbb{I}_{\{\varepsilon_{t-i}<0\}}) \log \varepsilon_{t-i}^2 + \sum\nolimits_{j=1}^{p} \gamma_j(\delta_t) \log \sigma_{t-j}^2, \tag{2.3}$$

where $\mathbb{I}_A$ denotes the indicator function of the set A, and $(\delta_t)_{t\in\mathbb{Z}}$ is a Markov chain with finite state space $\mathbb{S} = \{1, ..., d\}$, which is subject to the following assumption

Assumption 1. *$(\delta_t)_{t\in\mathbb{Z}}$ is a homogeneous, stationary, irreducible and aperiodic Markov chain. The stationary probabilities of $(\delta_t)_{t\in\mathbb{Z}}$ are denoted by $\pi(k) = P(\delta_0 = k) > 0$, $k \in \mathbb{S}$, the transition probability matrix is denoted by $\mathbb{P}$ and written in the following way $\mathbb{P} = \left(p_{ij}\right)_{(i,j)\in\mathbb{S}\times\mathbb{S}}$ where $p_{ij} = P(\delta_t = j|\delta_{t-1} = i)$ for $i, j \in \mathbb{S}$. In addition, we shall assume that e_t and $\{(\varepsilon_{u-1}, \delta_t), u \leq t\}$ are independent.*

Considering the assumptions made on the Markov chain we have $\pi(k) \neq 0$, $k \in \mathbb{S}$. In Eq. (2.3), for given $\delta_t = k$, ε_t satisfies a standard asymmetric $\log GARCH$ equation with coefficients $\omega(k)$, $(\alpha_i(k), \beta_i(k), 1 \leq i \leq q)$ and $\left(\gamma_j(k), 1 \leq j \leq p\right)$ for all $k \in \mathbb{S}$. The $MS - A\log GARCH_d$ is a general model including as special cases, various models such as:

- Standard asymmetric $\log GARCH$ models (i.e., $d = 1$) (see., Francq and Zakoïan [19]),

- Independent-switching $\log GARCH_d$ models: This specialization, analyzed by Wong and Li [42] in the $ARCH$ case, (δ_t) is an independent process,
- The usual symmetric $MS-\log GARCH_d$ models (i.e., $\alpha_i(.)=\beta_i(.)$ for all i) (see., Francq and Zakoïan [20]; $d=1$),
- If $p=0$, we have

$$\varepsilon_t=\sigma_t e_t,\ \log\sigma_t^2=\omega(\delta_t)+\sum\nolimits_{i=1}^{q}\left(\alpha_i(\delta_t)\mathbb{I}_{\{\varepsilon_{t-i}>0\}}+\beta_i(\delta_t)\mathbb{I}_{\{\varepsilon_{t-i}<0\}}\right)\log\varepsilon_{t-i}^2, t\in\mathbb{Z},$$

and the process is called a Markov-switching asymmetric $\log ARCH_d$ (in short, $MS-A\log ARCH_d$).

Now, we can rewrite (2.1) and (2.3) in an equivalent state-space representation in order to further simplify the study, in the following we need some notations. Define the $q-$vectors $\underline{\varepsilon}_t^{(1)\prime}:=\left(\mathbb{I}_{\{\varepsilon_t>0\}}\log\varepsilon_t^2,\ldots,\mathbb{I}_{\{\varepsilon_{t-q+1}>0\}}\log\varepsilon_{t-q+1}^2\right)$, $\underline{\varepsilon}_t^{(2)\prime}:=\left(\mathbb{I}_{\{\varepsilon_t<0\}}\log\varepsilon_t^2,\ldots,\mathbb{I}_{\{\varepsilon_{t-q+1}<0\}}\log\varepsilon_{t-q+1}^2\right)$ and the $(2q+p)-$vectors $\underline{\varepsilon}_t':=\left(\underline{\varepsilon}_t^{(1)\prime},\underline{\varepsilon}_t^{(2)\prime},\log\sigma_t^2,\ldots,\log\sigma_{t-p+1}^2\right)$, $\underline{F}_0':=\left(1,\underline{O}_{(q-1)}',1,\underline{O}_{(q+p-1)}'\right)$, $\underline{F}_1':=\left(\underline{O}_{(2q)}',1,\underline{O}_{(p-1)}'\right)$, $\underline{F}_2':=\left(1,\underline{O}_{(2q+p-1)}'\right)$ and $\underline{\Delta}_{\delta_t}(e_t):=\left(\underline{\omega}_1(\delta_t)+\underline{F}_2\log e_t^2\right)\mathbb{I}_{\{e_t>0\}}+\left(\underline{\omega}_{q+1}(\delta_t)+\left(\underline{F}_0-\underline{F}_2\right)\log e_t^2\right)\mathbb{I}_{\{e_t<0\}}+\underline{\omega}_{2q+1}(\delta_t)$ in which the $j-th$ entry of $\underline{\omega}_j(\delta_t)$ is $\omega(\delta_t)$ and all other elements are 0. Here, $O_{(k,l)}$ is the matrix of order $k\times l$ whose entries are zeros, for simplicity, we set $O_{(k)}:=O_{(k,k)}$ and $\underline{O}_{(k)}:=O_{(k,1)}$. With these notations, we obtain the following state-space representation $\log\sigma_t^2=\underline{F}_1'\underline{\varepsilon}_t$ or $\log\varepsilon_t^2=\underline{F}_0'\underline{\varepsilon}_t$ and

$$\underline{\varepsilon}_t=\Gamma_{\delta_t}(e_t)\underline{\varepsilon}_{t-1}+\underline{\Delta}_{\delta_t}(e_t), t\in\mathbb{Z} \tag{2.4}$$

with $\Gamma_{\delta_t}(e_t):=\Gamma_1(\delta_t)\mathbb{I}_{\{e_t>0\}}+\Gamma_2(\delta_t)\mathbb{I}_{\{e_t<0\}}+\Gamma_3(\delta_t)$ where

$$\Gamma_1(\delta_t)=\begin{pmatrix}\alpha_1(\delta_t) & \cdots & \alpha_q(\delta_t) & \beta_1(\delta_t) & \cdots & \beta_q(\delta_t) & \gamma_1(\delta_t) & \cdots & \gamma_p(\delta_t)\\ & & & O_{(2q+p-1,2q+p)} & & & & & \end{pmatrix},$$

$$\Gamma_2(\delta_t)=\begin{pmatrix} & & & O_{(q,2q+p)} & & & & & \\ \alpha_1(\delta_t) & \cdots & \alpha_q(\delta_t) & \beta_1(\delta_t) & \cdots & \beta_q(\delta_t) & \gamma_1(\delta_t) & \cdots & \gamma_p(\delta_t)\\ & & & O_{(q+p-1,2q+p)} & & & & & \end{pmatrix},$$

$$\Gamma_3(\delta_t)=\begin{pmatrix}0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\ I_{(q-1)} & & \underline{O}_{(q-1)} & O_{(q-1)} & & \underline{O}_{(q-1)} & O_{(q-1,p-1)} & & \underline{O}_{(p-1)}\\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\ O_{(q-1)} & & \underline{O}_{(q-1)} & I_{(q-1)} & & \underline{O}_{(q-1)} & O_{(q-1,p-1)} & & \underline{O}_{(p-1)}\\ \alpha_1(\delta_t) & \cdots & \alpha_q(\delta_t) & \beta_1(\delta_t) & \cdots & \beta_q(\delta_t) & \gamma_1(\delta_t) & \cdots & \gamma_p(\delta_t)\\ O_{(p-1,q)} & & & O_{(p-1,q)} & & & I_{(p-1)} & & \underline{O}_{(p-1)}\end{pmatrix}.$$

We are now in a position to present the first major finding of this paper

3. Stationarity and log −moments of the $MS-A\log GARCH_d(p,q)$

In this section, we begin by studying the existence of solutions to model (2.1) and (2.3). Note that $(\Gamma_k(e_t),k)$ is a sequence of $i.i.d.$ random matrices independent of $\left\{\underline{\varepsilon}_u, u<t\right\}$ and $\left(\underline{\Delta}_k\left(\underline{e}_t\right),k\right)$ is a sequence of $i.i.d.$ random vectors, for all $k\in\mathbb{S}$.

So, the existence of the strict stationary and ergodic solution to (2.1) and (2.3) is equivalent to the existence of the strict stationary solution to (2.4). Processes similar to $\underline{\varepsilon}_t$ of (2.4) has been examined by many authors., e.g., Bibi and Ghezal [7] (see also Bougerol and Picard [9]) who established that the series

$$\underline{\varepsilon}_t=\sum\nolimits_{l\geq1}\left\{\prod\nolimits_{j=0}^{l-1}\Gamma_{\delta_{t-j}}\left(e_{t-j}\right)\right\}\underline{\Delta}_{\delta_{t-l}}(e_{t-l})+\underline{\Delta}_{\delta_t}(e_t), \tag{3.1}$$

constitute the unique, strictly stationary and ergodic solution of Eq. (2.4) if, the top-Lyapunov exponent $\gamma(\Gamma)$ associated with the strictly stationary and ergodic sequence of random matrices $\Gamma = (\Gamma_{\delta_t}(e_t), t \in \mathbb{Z})$ defined by

$$\gamma(\Gamma) = \inf_{t>0}\left\{\frac{1}{t}E\left\{\log\left\|\prod_{j=0}^{t-1}\Gamma_{\delta_{t-j}}\left(e_{t-j}\right)\right\|\right\}\right\} \overset{a.s.}{=} \lim_{t\to+\infty}\left\{\frac{1}{t}\log\left\|\prod_{j=0}^{t-1}\Gamma_{\delta_{t-j}}\left(e_{t-j}\right)\right\|\right\} \tag{3.2}$$

is such that $\gamma(\Gamma) < 0$, where, as usual, empty products are set to be equal to $I_{(2q+p)}$. The choice of the norm is unimportant for the value of the top Lyapunov exponent. However, in the sequel, the matrix norm will be assumed to be multiplicative. Moreover, the existence of $\gamma(\Gamma)$ is guaranteed by the fact that $E\left\{\log^+\left\|\Gamma_{\delta_t}(e_t)\right\|\right\} \le E\left\{\left\|\Gamma_{\delta_t}(e_t)\right\|\right\} < \infty$ and $E\left\{\log^+\left\|\underline{\Delta}_{\delta_t}(e_t)\right\|\right\} \le E\left\{\left\|\underline{\Delta}_{\delta_t}(e_t)\right\|\right\} < \infty$ where $\log^+(x) = \max(\log x, 0)$ and the right-hand member in (3.2) can be justified using Kingman's subadditive ergodic theorem. Francq et al. [16] showed that if an equation in the form of (2.4) with positive coefficients, (δ_t, e_t) is a strictly stationary ergodic process and if $E\left\{\log^+\left\|\Gamma_{\delta_t}(e_t)\right\|\right\}$ and $E\left\{\log^+\left\|\underline{\Delta}_{\delta_t}(e_t)\right\|\right\}$ are finite, $\gamma(\Gamma) < 0$ is the necessary and sufficient condition for the existence of a stationary solution to (2.4). Bibi and Ghezal [4] showed that, for the Markov-Switching BiLinear ($MS - BL$) model, there exists a representation of the form (2.4), and for which the necessary and sufficient condition for the existence of a stationary $MS - BL$ model is $\gamma(\Gamma) < 0$. The result can be extended to more general classes of $MS - BLGARCH$ models (see., e.g. Bibi and Ghezal [7]). The problem is more delicate with the $MS - A\log GARCH$ because the coefficients of (2.4) are not constrained to be positive. The following theorem gives us the main result for stochastic difference Equation (2.4).

Theorem 3.1. *Suppose that* $E\left\{\log^+\left|\log e_0^2\right|\right\} < \infty$ *and if* $\gamma(\Gamma)$ *corresponding to a* $MS - A\log GARCH_d(p,q)$ *models is strictly negative, then*

1. *Eq.* (2.4) *has a unique, strictly stationary, causal and ergodic solution given by the series* (3.1) *which converges absolutely almost surely.*
2. *Eq.* (2.3) *and hence* (2.1) *admits a unique, strictly stationary, causal and ergodic solution given by* $\varepsilon_t = \exp\left\{\frac{1}{2}\underline{F}_0'\underline{\varepsilon}_t\right\}$ *or* $\varepsilon_t = e_t\exp\left\{\frac{1}{2}\underline{F}_1'\underline{\varepsilon}_t\right\}$ *where* $\underline{\varepsilon}_t$ *is given by the series* (3.1).

Proof. Through the $E\left\{\log^+\left\|\underline{\Delta}_{\delta_t}(e_t)\right\|\right\} < \infty$ and $E\left\{\log^+\left\|\Gamma_{\delta_t}(e_t)\right\|\right\} < \infty$ and by the subadditive ergodic theorem (see., Kingman [33]), almost surely,

$$\limsup_{n\to+\infty}\left\|\left\{\prod_{j=0}^{n-1}\Gamma_{\delta_{t-j}}\left(e_{t-j}\right)\right\}\underline{\Delta}_{\delta_{t-n}}(e_{t-n})\right\|^{1/n} \le \exp\{\gamma(\Gamma)\},$$

when $\gamma(\Gamma) < 0$, Cauchy's root test shows that, the series (3.1) converges absolutely almost surely. The rest of the proof follows essentially the same arguments as in Bibi and Ghezal [4] or Bougerol and Picard [9] (see., Theorem 1.1) and Brandt [10] (see., Theorem 1). □

Remark 3.2. *Under the first assumption of Theorem* 3.1 *(i.e.,* $E\left\{\log^+\left|\log e_0^2\right|\right\} < \infty$*),* $P(e_0 = 0) = 0$ *for all t. Thus, the observed process satisfies* $\varepsilon_t^2 \ne 0$ *a.s.*

Remark 3.3. *The top-Lyapunov exponent* $\gamma(\Gamma)$ *is independent of the intercepts coefficients* $(\omega(k), k \in \mathbb{S})$.

Though the condition $\gamma(\Gamma) < 0$ could be used as a test for the strict stationarity, it is of little use in practice since this condition involves the limit of products of infinitely many random matrices. On the other hand, some simple sufficient conditions ensuring the negativity of $\gamma(\Gamma)$ can be given

Theorem 3.4. *Consider the* $MS - A\log GARCH_d(p,q)$ *model* (2.1) *and* (2.3) *with state-space representation* (2.4)*. Then* $\gamma(\Gamma) < 0$*, if one of the following conditions holds true.*

1. $E\left\{\log\left\|\prod_{j=0}^{t-1}\Gamma_{\delta_{t-j}}\left(e_{t-j}\right)\right\|\right\}<0$ *or* $E\left\{\left\|\prod_{j=0}^{t-1}\Gamma_{\delta_{t-j}}\left(e_{t-j}\right)\right\|\right\}<1$ *for some* $t\geq 1$,
2. $\rho\left(|\Gamma|\right)<1$, *where* $\rho\left(M\right)$ *is the spectral radius of squared matrix* M *and* $|\Gamma|:=E\left\{\left|\Gamma_{\delta_t}\left(e_t\right)\right|\right\}$,
3. $\rho\left(\mathbb{P}\left(|\Gamma_\nu|\right)\right)<1$, *for some* $\nu\in\,]0,1]$ *with* $E\left\{|e_1|^{2\nu}\right\}<\infty$, *where* $|\Gamma_\nu|:=\left(E\left\{|\Gamma_k\left(e_1\right)|^\nu\right\},k\in\mathbb{S}\right)$ *for some* $\nu\in\,]0,1]$, *and let* $|M|^\nu:=\left(\left|m_{ij}\right|^\nu\right)$, *then it is easy to see that the operator* $|.|^\nu$ *is submultiplicative, i.e.,* $|M_1M_2|^\nu\leq|M_1|^\nu|M_2|^\nu$, $\left|M\underline{x}\right|^\nu\leq|M|^\nu\left|\underline{x}\right|^\nu$ *for any appropriate vector* $\underline{x}$ *and for any function* $M:\mathbb{S}\rightarrow\mathcal{M}\left(n\times n\right)$, *we shall note*

$$\mathbb{P}(M)=\begin{pmatrix} p_{11}M(1) & \dots & p_{d1}M(1)\\ \vdots & \dots & \vdots\\ p_{1d}M(d) & \dots & p_{dd}M(d)\end{pmatrix}.$$

Proof. Because the top−Lyapunov exponent is independent of the norm, by choosing an absolute norm, i.e., a norm $\|.\|$ such that $\|.\|^\nu\leq\||.|\|^\nu$ (e.g. $\|M\|=\sum\limits_{i,j}\left|m_{ij}\right|$), then from the definition of $\gamma\left(\Gamma\right)$ and according to Kesten and Spitzer [32], we have almost surely $\lim\limits_{t\to\infty}\frac{1}{t}\log\left\|\prod_{j=0}^{t-1}\Gamma_{\delta_{t-j}}\left(e_{t-j}\right)\right\|\leq\log\rho\left(|\Gamma|\right)$. On the other hand, by Jensen's inequality we get almost surely $\gamma\left(\Gamma\right)\leq\dfrac{1}{t}E\left\{\log\left\|\prod_{j=0}^{t-1}\Gamma_{\delta_{t-j}}\left(e_{t-j}\right)\right\|\right\}\leq\dfrac{1}{t}\log E\left\{\left\|\prod_{j=0}^{t-1}\Gamma_{\delta_{t-j}}\left(e_{t-j}\right)\right\|\right\}\leq\dfrac{1}{t}\log E\left\{\left\|\prod_{j=0}^{t-1}\left|\Gamma_{\delta_{t-j}}\left(e_{t-j}\right)\right|\right\|\right\}\leq\log\rho\left(|\Gamma|\right)$, so the result follows. Moreover, since $\rho\left(\mathbb{P}\left(|\Gamma_\nu|\right)\right)<1$, there exists $\mu\in\,]0,1[$ such that $\limsup_t\left\|\mathbb{P}^t\left(|\Gamma_\nu|\right)\right\|^{1/t}<\mu$. By Jensen inequality and submultiplicativity of the operator $|.|^\nu$ we obtain

$$\begin{aligned}\gamma\left(\Gamma\right)\nu &= \lim_t\frac{1}{t}E\left\{\log\left\|\prod_{j=0}^{t-1}\Gamma_{\delta_{t-j}}\left(e_{t-j}\right)\right\|^\nu\right\}\\ &\leq \lim_t\frac{1}{t}\log E\left\{\left\|\prod_{j=0}^{t-1}\Gamma_{\delta_{t-j}}\left(e_{t-j}\right)\right\|^\nu\right\}\\ &\leq \lim_t\frac{1}{t}\log E\left\{\left\|\prod_{j=0}^{t-1}\left|\Gamma_{\delta_{t-j}}\left(e_{t-j}\right)\right|^\nu\right\|\right\}\\ &\leq \limsup_t\log\left\|\mathbb{P}^t\left(|\Gamma_\nu|\right)\right\|^{1/t}<0.\end{aligned}$$

□

Example 3.5. *In the following table, we summarize the condition* $\gamma\left(\Gamma\right)<0$ *for some particular cases*

Specification	*Condition* $\gamma\left(\Gamma\right)<0$
Standard $A\log GARCH_1\left(1,1\right)^{(a)}$	$s\log\left\|\alpha_1\left(1\right)+\gamma_1\left(1\right)\right\|<\left(s-1\right)\log\left\|\beta_1\left(1\right)+\gamma_1\left(1\right)\right\|$
Symmetric $MS-\log GARCH_d\left(1,1\right)$	$\sum_{k=1}^d\pi\left(k\right)\log\left\|\alpha_1\left(k\right)+\gamma_1\left(k\right)\right\|<0$
$MS-A\log GARCH_d\left(1,1\right)$	$\sum_{k=1}^d\pi\left(k\right)s\log\left\|\alpha_1\left(k\right)+\gamma_1\left(k\right)\right\|<\sum_{k=1}^d\pi\left(k\right)\left(s-1\right)\log\left\|\beta_1\left(k\right)+\gamma_1\left(k\right)\right\|$
$MS-A\log ARCH_d\left(1\right)$	$\prod_{k=1}^d\left\|\alpha_1\left(k\right)\right\|^{s\pi(k)}\left\|\beta_1\left(k\right)\right\|^{(1-s)\pi(k)}<1$
$^{(a)}\ s=P\left(e_0>0\right)>0$	

Table 1. Condition $\gamma\left(\Gamma\right)<0$ *for some specifications.*

For the model $MS-A\log GARCH_d\left(1,1\right)$, *the existence of explosive regimes, (i.e., for some* $k\in\mathbb{S}$, *such that,* $\left|\alpha_1\left(k\right)+\gamma_1\left(k\right)\right|^{\pi(k)s}\left|\beta_1\left(k\right)+\gamma_1\left(k\right)\right|^{\pi(k)(1-s)}>1$) *does not preclude the existence of strictly stationary solution. So, local stationarity is not necessary for global stationarity and global stationarity is not sufficiently for local stationarity. For* $d=2$ *and* $\gamma_1\left(.\right)=0$, $\alpha_1\left(1\right)=2\beta_1\left(2\right)=a$, $\beta_1(1)=\alpha_1(2)=b$, $\pi\left(2\right)=0.25$ *and with* $e_t\sim\mathcal{N}(0,1)$, *the regions of* *strictly stationary are shown in Fig*1. *below*

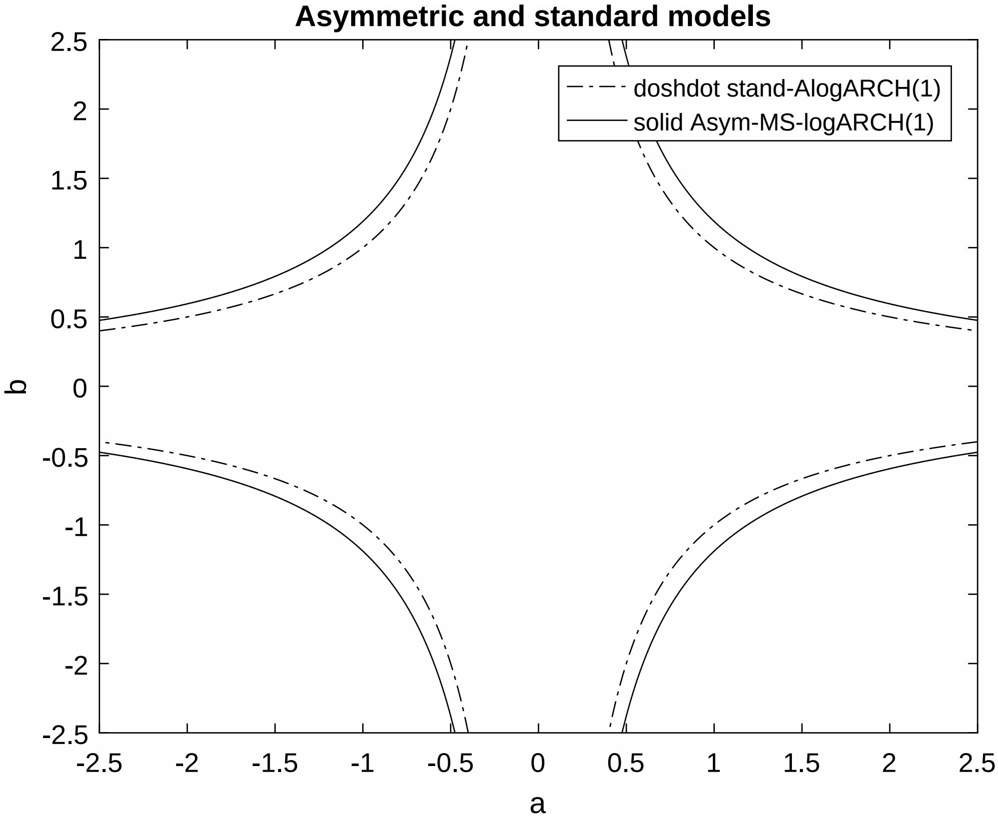

Fig1.Region of stationarity for Example 3.5.

Remark 3.6. *The process* (2.4) *is not a Markov process in general. But the extended process* $\left(\widetilde{\underline{\varepsilon}}_t = \left(\underline{\varepsilon}_t', \delta_t\right)', t \in \mathbb{Z}\right)$ *is a Markov chain with a state space* $\mathbb{R}^{2q+p} \times \mathbb{S}$ *whose* $n-$*step transition probability is given by* $P^{(n)}\left(\underline{z}, A\right) = P\left(\widetilde{\underline{\varepsilon}}_n \in A \left| \widetilde{\underline{\varepsilon}}_0 = z\right.\right)$ *for any Borel set* $A \in \mathcal{B}_{\mathbb{R}^{2q+p}\times\mathbb{S}}$, *where* $\widetilde{\underline{\varepsilon}}_0$ *be an arbitrarily specified random vector in* $\mathbb{R}^{2q+p+1}$ *and independent of* $\left(\Gamma_k\left(e_t\right), \underline{\Delta}_k\left(e_t\right), t \in \mathbb{Z}\right)$, *for all* $k \in \mathbb{S}$.

Remark 3.7. [*Non-necessity conditions of Theorem* 3.1]*Assume for instance that* $p = q = 1$ *and* $\alpha_1(.) = \beta_1(.)$. *In that case* $\gamma(\Gamma) < 0$ *is equivalent to* $\prod_{k=1}^{d}\left|\alpha_1(k) + \gamma_1(k)\right|^{\pi(k)} < 1$. *In addition, assume that* $e_t^2 = 1$ *a.s. Then, when* $\prod_{k=1}^{d}\left|\alpha_1(k) + \gamma_1(k)\right|^{\pi(k)} \neq 1$ *with* $\left|\alpha_1(k) + \gamma_1(k)\right|^{\pi(k)} \neq 1$ $\forall k \in \mathbb{S}$, *there exists a stationary solution to* (2.1) *and* (2.3) *for each regime* $k \in \mathbb{S}$ *defined by* $\varepsilon_t = e_t \exp\left(\frac{1}{2}\lambda(k)\right)$, *with* $\lambda(k) = \alpha_0(k)\left(1 - \alpha_1(k) - \gamma_1(k)\right)^{-1}, k \in \mathbb{S}$.

For the *GARCH* model, in a general, and the *MS* − *BL* model, in particular, the strict stationarity condition requires the existence of a moment of certain order ν for $|\varepsilon_t|$. Therefore, we will reinforce this important consequence of the proposed *MS* − *A* log *GARCH* models. The following theorem gives us the existence of fractional log −moments

Theorem 3.8. *Consider the* $MS - A\log GARCH_d(p, q)$ *model with associated state-space representation* (2.4) *and assume that* $\gamma(\Gamma) < 0$ *and that* $E\left\{\left|\log e_t^2\right|^{\tau}\right\} < \infty$ *for some* $\tau > 0$. *Then there exists* $\nu > 0$ *such that* $E\left\{\left|\log \sigma_t^2\right|^{\nu}\right\} < \infty$ *and* $E\left\{\left|\log \varepsilon_t^2\right|^{\nu}\right\} < \infty$.

Proof. Since $\gamma(\Gamma) < 0$, there is a positive integer t such that $E\left\{\log\left\|\prod_{i=0}^{t-1}\Gamma_{\delta_{t-i}}(e_{t-i})\right\|\right\} < 0$ and we get

$$E\left\{\left\|\prod_{i=0}^{t-1}\Gamma_{\delta_{t-i}}(e_{t-i})\right\|^{\nu}\right\} \leq \left\|\mathbb{I}'_{(d(p+2q))}\mathbb{P}^{t}(|\Gamma_{\nu}|)\underline{\Pi}\left(I_{(p+2q)}\right)\right\| < \infty,$$

for all t, where $\mathbb{I}'_{(mn)}$ means the $m-$block matrix $\left(I_{(n)}\vdots\ldots\vdots I_{(n)}\right)$, $I_{(n)}$ is the $n\times n$ identity matrix and $\underline{\Pi}'(I_{(n)}) = \left(\pi(1)I_{(n)},\cdots,\pi(d)I_{(n)}\right)$. Otherwise, we use Lemma 4.2 of Bibi and Ghezal [4], to get $E\left\{\left\|\prod_{i=0}^{t_0-1}\Gamma_{\delta_{t_0-i}}(e_{t_0-i})\right\|^{\nu}\right\} < 1$ for some (ν, t_0). The rest of the proof is the same as the proof of Lemma 4.2 in Bibi and Ghezal [4]. □

Remark 3.9. *Many references assumed independent and identically distributed standardized innovations (i.e., the strong case). Theorem* 3.8 *is no longer valid when the i.i.d. assumption on* $(e_t, t \in \mathbb{Z})$ *is breached (Hamadeh and Zakoïan* [28] *for further discussions). While Escanciano* ([15], 2009)*, and Francq and Thieu* ([22], 2019) *established statistical inference of QMLE for semi-strong standard GARCH and semi-strong standard APARCH − X models respectively, i.e., when* $(e_t, t \in \mathbb{Z})$ *is a quasi-strong assumption of type* $E\left\{e_t^2|\mathcal{F}_{t-1}\right\} = 1$ *for each t, thus, it would be beneficial to generalize the current results of MS − AlogGARCH when* $(e_t, t \in \mathbb{Z})$ *is semi-strong.*

By the previous theorem, it will be important to know if the strictly stationary solution has log −moments of higher-order. For this, we shall be interested in conditions ensuring the existence of higher-order log −moments for strictly stationary $\left(\left|\log\varepsilon_t^2\right|\right)$ in the $MS - A\log GARCH$ model having state-space representation (2.4).

Theorem 3.10. *Consider the* $MS - A\log GARCH_d(p,q)$ *model with associated state-space representation* (2.4) *and assume that* $\gamma(\Gamma) < 0$ *and that* $E\left\{\left|\log e_t^2\right|^m\right\} < \infty$ *for any integer* $m \geq 1$*. If* $\rho\left(\mathbb{P}\left(\left|\Gamma^{(m)}\right|\right)\right) < 1$*, where* $\left|\Gamma^{(m)}\right| = \left(E\left\{\left|\Gamma_{\delta_t=k}^{\otimes m}(e_1)\right|\right\}, k\in\mathbb{S}\right)$ *with* $\otimes$ *is the usual Kronecker product of matrices and* $\Gamma^{\otimes r} = \Gamma\otimes\Gamma\otimes\ldots\otimes\Gamma$*, r−times. Then* $E\left\{\left|\log\sigma_t^2\right|^m\right\} < \infty$ *and* $E\left\{\left|\log\varepsilon_t^2\right|^m\right\} < \infty$*.*

Proof. It is easily seen that $E\left\{\left|\underline{\varepsilon}_t^{\otimes m}(n)\right|\right\} \leq \mathbb{I}'_{((p+2q)d)}\mathbb{P}^k\left(\left|\Gamma^{(m)}\right|\right)\underline{\Pi}\left(\left|\underline{\Delta}^{(m)}\right|\right)$, where $\underline{\varepsilon}_t(n) := \left\{\prod_{j=0}^{n-1}\Gamma_{\delta_{t-j}}\left(e_{t-j}\right)\right\}\underline{\Delta}_{\delta_{t-n}}\left(\underline{e}_{t-n}\right)$ and $\left|\underline{\Delta}^{(m)}\right| := \left(E\left\{\left|\underline{\Delta}_{\delta_t=k}^{\otimes m}(e_1)\right|\right\}', k\in\mathbb{S}\right)'$. Hence $\left\|\underline{\varepsilon}_t(k)\right\|_m \leq \left\|\left|\Gamma^{(m)}\right|^k\right\|^{1/m}\left\|\left|\underline{\Delta}^{(m)}\right|\right\|^{1/m}$. So, by Jordan decomposition, $\left\|\left|\Gamma^{(m)}\right|^k\right\|$ converge to zero at an exponential rate as $k\to\infty$, the rest of statements are immediate. □

Next, we present the log −moment conditions around a smaller size of another sequence of matrices. For this, we introduce the $p\vee q-$vectors $\underline{\sigma}'_t := \left(\log\sigma_t^2,\ldots,\log\sigma_{t-p\vee q+1}^2\right)$ and $\underline{F}'_3 := \left(1, \underline{O}'_{(p\vee q-1)}\right)$. Now, it is shown how these notations lead to obtain the following state-space representation $\log\sigma_t^2 = \underline{F}'_3\underline{\sigma}_t$ and

$$\underline{\sigma}_t = \Lambda_{\delta_t}(e_t)\underline{\sigma}_{t-1} + \eta_{\delta_t}(e_t)\underline{F}_3, t\in\mathbb{Z}, \tag{3.3}$$

where

$$\Lambda_{\delta_t}(e_t) := \begin{pmatrix} \lambda_{1,\delta_t}(Le_t) & \ldots & \lambda_{p\vee q,\delta_t}(L^{p\vee q}e_t) \\ I_{(p\vee q-1)} & & \underline{O}_{(p\vee q-1)} \end{pmatrix}$$

is an $(p\vee q)\times(p\vee q)$ matrix, L denotes the lag operator, $\lambda_{i,\delta_t}(x) = \alpha_i(\delta_t)\mathbb{I}_{\{x>0\}} + \beta_i(\delta_t)\mathbb{I}_{\{x<0\}} + \gamma_i(\delta_t)$, $i = 1,\ldots,p\vee q$ with the convention $\gamma_i(.) = 0$ for $i > q$, $\alpha_i(.) = \beta_i(.) = 0$ for $i > p$ and $\eta_{\delta_t}(e_t) = \omega(\delta_t) + \sum_{i=1}^{q}\left(\alpha_i(\delta_t)\mathbb{I}_{\{L^ie_t>0\}} + \beta_i(\delta_t)\mathbb{I}_{\{L^ie_t<0\}}\right)L^i\log e_t^2$. Note here that $(\Lambda_{\delta_t=k}(e_t))$ is a sequence of dependent random matrices and $\left(\eta_{\delta_t=k}(e_t)\underline{F}_3\right)$ is a sequence of dependent vectors for all $k\in\mathbb{S}$.

Theorem 3.11. *Let $(\varepsilon_t, t \in \mathbb{Z})$ be a strict stationary solution of Eq.* (2.1) *and* (2.3), *with associated state-space representation* (3.3) *and assume that* $E\left\{\left|\log e_t^2\right|\right\} < \infty$. *If* $\rho\left(\mathbb{P}\left(|\Lambda|\right)\right) < 1$, *where* $|\Lambda| := \left(\left|\Lambda\left(\delta_t = k\right)\right|, k \in \mathbb{S}\right)$, *then* $E\left\{\left|\log \sigma_t^2\right|\right\} < \infty$ *and* $E\left\{\left|\log \varepsilon_t^2\right|\right\} < \infty$.

Proof. From (3.3), we then have

$$\left|\underline{\sigma}_t\right| \leq \sum_{i \geq 0}\left\{\prod_{j=0}^{i-1}\left|\Lambda_{\delta_{t-j}}\left(e_{t-j}\right)\right|\right\}\left|\eta_{\delta_{t-i}}\left(e_{t-i}\right)\right|\underline{F}_3,$$

in order to give a full-scale overview, consider for instance the case $i = 2$, we then have

$$\left\{\prod_{j=0}^{2-1}\left|\Lambda_{\delta_{t-j}}\left(e_{t-j}\right)\right|\right\}\left|\eta_{\delta_{t-2}}\left(e_{t-2}\right)\right|\underline{F}_3 = \lambda_{1,\delta_{t-1}}\left(e_{t-2}\right)\left|\eta_{\delta_{t-2}}\left(e_{t-2}\right)\right|\left(\begin{array}{l}\lambda_{1,\delta_t}\left(e_{t-1}\right)\\ 1\\ \underline{O}_{(p\vee q-2)}\end{array}\right) + \left|\eta_{\delta_{t-2}}\left(e_{t-2}\right)\right|\left(\begin{array}{l}\lambda_{2,\delta_t}\left(e_{t-2}\right)\\ 0\\ 1\\ \underline{O}_{(p\vee q-3)}\end{array}\right),$$

and we get

$$E\left\{\left\{\prod_{j=0}^{2-1}\left|\Lambda_{\delta_{t-j}}\left(e_{t-j}\right)\right|\right\}\left|\eta_{\delta_{t-2}}\left(e_{t-2}\right)\right|\underline{F}_3\right\} = E\left\{\left\{\prod_{j=0}^{2-1}\left|\Lambda\left(\delta_{t-j}\right)\right|\right\}\left|\eta\left(\delta_{t-2}\right)\right|\underline{F}_3\right\} = \mathbb{I}'_{((p\vee q)d)}\mathbb{P}^2\left(|\Lambda|\right)\underline{\Pi}\left(\left|\eta\right|\underline{F}_3\right),$$

because $\eta_{\delta_{t-2}=k}\left(e_{t-2}\right)$ is a function of e_{t-3} and its past values for all $k \in \mathbb{S}$, where

$$\Lambda\left(\delta_t\right) := \left(\begin{array}{ccc}\lambda_1\left(\delta_t\right) & \dots & \lambda_{p\vee q}\left(\delta_t\right)\\ I_{(p\vee q-1)} & & \underline{O}_{(p\vee q-1)}\end{array}\right),$$

$\lambda_i\left(\delta_t = k\right) = E\left\{\lambda_{i,\delta_t=k}\left(e_t\right)\right\}, 1 \leq i \leq p \vee q$, $\eta\left(\delta_t = k\right) = E\left\{\eta_{s_t=k}\left(e_t\right)\right\}$ for all $k \in \mathbb{S}$ and $\left|\eta\right| := \left(\left|\eta\left(\delta_t = k\right)\right|, k \in \mathbb{S}\right)$. Thus, in general, we can be easily obtained

$$E\left\{\left\{\prod_{j=0}^{i-1}\left|\Lambda_{\delta_{t-j}}\left(e_{t-j}\right)\right|\right\}\left|\eta_{\delta_{t-i}}\left(e_{t-i}\right)\right|\underline{F}_3\right\} = E\left\{\left\{\prod_{j=0}^{i-1}\left|\Lambda\left(\delta_{t-j}\right)\right|\right\}\left|\eta\left(\delta_{t-i}\right)\right|\underline{F}_3\right\} = \mathbb{I}'_{((p\vee q)d)}\mathbb{P}^i\left(|\Lambda|\right)\underline{\Pi}\left(\left|\eta\right|\underline{F}_3\right),$$

the condition $\rho\left(\mathbb{P}\left(|\Lambda|\right)\right) < 1$ entails that $E\left\{\left|\underline{\sigma}_t\right|\right\}$ is finite. □

Remark 3.12. *In Theorem* 3.11 *we have got the* log −*moment conditions around a smaller size of sequence* $\left(\Lambda_{\delta_t}\left(e_t\right)\right)$ *of matrices (i.e.,* $q \vee p$ *size) but dependent, while Theorem* 3.10 *gives the* log −*moment conditions with* $\left(\Gamma_{\delta_t}\left(e_t\right)\right)$ *is a sequence of i.i.d. and the* $(2q + p)$ *size.*

Corollary 3.13. *Consider the* $MS - A\log GARCH_d(1,1)$ *model and under the conditions of Theorem* 3.10, *if*

$$\rho\left(\mathbb{P}\left(\underline{\varphi}^{(m)}\right)\right) < 1, \text{ where } \underline{\varphi}^{(m)} := \left(\varphi^{(m)}(k) = s\left|\alpha_1(k) + \gamma_1(k)\right|^m + (1-s)\left|\beta_1(k) + \gamma_1(k)\right|^m, k \in \mathbb{S}\right), \tag{3.4}$$

then $E\left\{\left|\log \sigma_t^2\right|^m\right\} < \infty$ *and* $E\left\{\left|\log \varepsilon_t^2\right|^m\right\} < \infty$. *In particular, noting here that when* $d = 2$, *with* $p_{11} = p_{22} = 1 - q = \frac{4}{5}$, *then the condition* (3.4) *is equivalent to the following two conditions*

$$\left\{\begin{array}{l}(2q-1)\varphi^{(m)}(1)\varphi^{(m)}(2) + (1-q)\left(\varphi^{(m)}(1) + \varphi^{(m)}(2)\right) < 1\\ (1-q)\left(\varphi^{(m)}(1) + \varphi^{(m)}(2)\right) \leq 2\end{array}\right..$$

For $MS - A\log ARCH_2(1)$ *model with* $\alpha_1(1) = 0$, $\alpha_1(2) = \beta_1(1) = a$, $\beta_1(2) = b$ *and* $e_t \sim \mathcal{N}(0,1)$, *the regions are shown in Fig* 2. *below*

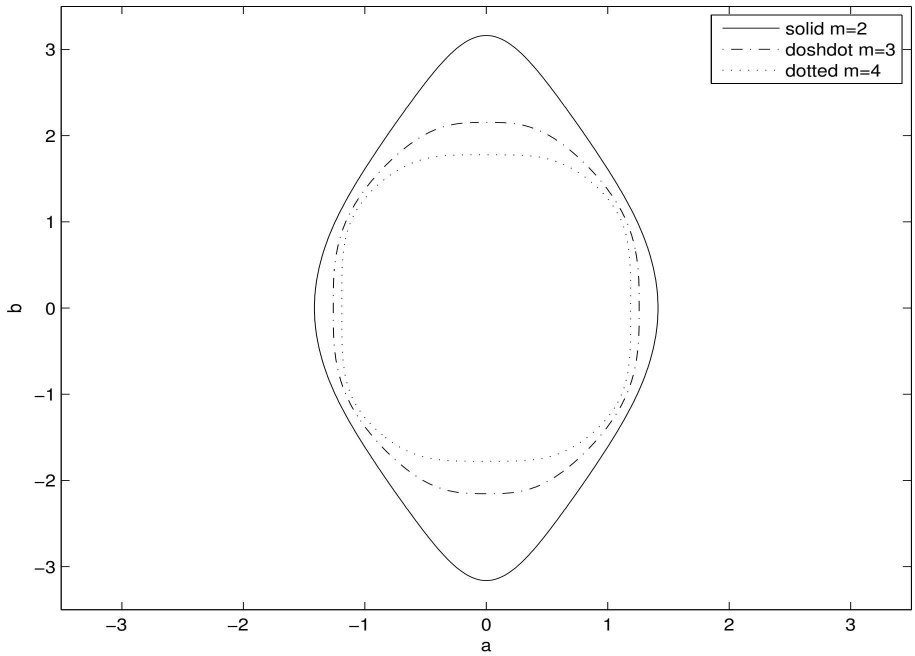

Fig 2. Plots of the boundary curves for $MS-A\log ARCH_2(1)$ *model.*

4. Estimation of the $MS-A\log GARCH$ model using QML method

The statistical inference for $MS-A\log GARCH$ model is rather difficult, and one of the key issues because that observations at any time are then dependent on the whole regime path, whereas the likelihood becomes quickly intractable as the total length of observations increases. But by transforming it into an infinite order $MS-A\log ARCH$ model, we get the possibility of writing a likelihood that can be handled directly. Some specific models were considered in the literature (see., for instance, Francq and Zakoïan [18], Xie [43], Ghezal et al. [5] – [7], [24], for further discussions). In this section, we consider the quasi-maximum likelihood estimator ($QMLE$) for estimating the parameters of $MS-A\log GARCH$ model gathered in vector $\underline{\theta}' := \left(\underline{\alpha}', \underline{\beta}', \underline{\gamma}', \underline{p}'\right) \in \Theta \subset \mathbb{R}^{d(d+2q+p)}$ where $\underline{\alpha}' := \left(\underline{\omega}', \underline{\alpha}_1', ..., \underline{\alpha}_q'\right)$, $\underline{\beta}' := \left(\underline{\beta}_1', ..., \underline{\beta}_q'\right)$, $\underline{\gamma}' := \left(\underline{\gamma}_1', ..., \underline{\gamma}_p'\right)$ and $\underline{p}' := \left(\underline{p}_1', ..., \underline{p}_d'\right)$ with $\underline{\omega}' := (\omega(1), ..., \omega(d))$, $\underline{\alpha}_i' := (\alpha_i(1), ..., \alpha_i(d))$, $\underline{\beta}_i' := (\beta_i(1), ..., \beta_i(d))$, $\underline{\gamma}_j' := \left(\gamma_j(1), ..., \gamma_j(d)\right)$ and $\underline{p}_l' := (p_{lk}, 1 \leq k \leq d, l \neq k)$ for all $1 \leq i \leq q$, $1 \leq j \leq p$ and $1 \leq l \leq d$. The true parameter value denoted by $\underline{\theta}_0 \in \Theta$ is unknown and should be estimated. For this purpose, let $\{\varepsilon_1, ..., \varepsilon_n\}$ be a realization from the unique, causal and strictly stationary solution of (2.1) and (2.3) and assume that the orders p, q and the number of regime d are assumed to be known and fixed and (e_t) is standard Gaussian with mean zero and variance one. A $QMLE$ of $\underline{\theta}$ is defined as any measurable solution $\widehat{\underline{\theta}}_n$ of

$$\widehat{\underline{\theta}}_n = \arg\max_{\underline{\theta}\in\Theta} L_n\left(\underline{\theta}\right), \tag{4.1}$$

where $L_n\left(\underline{\theta}\right)$ is the Gaussian likelihood function, given by summing, over all the possible paths of the Markov chain,

$$L_n\left(\underline{\theta}\right) = \sum_{s_1,...,s_n\in\mathbb{S}} \pi(s_1) \left\{\prod_{i=2}^{n} p_{s_{i-1},s_i}\right\} \left\{\prod_{l=1}^{n} g_{s_l}(\varepsilon_1, ..., \varepsilon_l)\right\}, \tag{4.2}$$

where

$$g_{s_l}(\varepsilon_1,...,\varepsilon_l)=\frac{1}{(2\pi)^{1/2}\,\sigma_{s_l}(\varepsilon_1,...,\varepsilon_{l-1})}\exp\left\{-\frac{\varepsilon_l^2}{2\sigma_{s_l}^2(\varepsilon_1,...,\varepsilon_{l-1})}\right\},$$

with the conditional log −variance process satisfy

$$\log\sigma_{s_l}^2(\varepsilon_1,...,\varepsilon_{l-1})=\widetilde{\omega}(s_l)+\sum_{j=1}^{l-1}\left(\widetilde{\alpha}_j(s_l)\,\mathbb{I}_{\{\varepsilon_{l-j}>0\}}+\widetilde{\beta}_j(s_l)\,\mathbb{I}_{\{\varepsilon_{l-j}<0\}}\right)\log\varepsilon_{l-j}^2,$$

with $\log\sigma_{s_1}^2=\widetilde{\omega}(s_1)$ if $l=1$, where $\widetilde{\omega}(s_l)=\left(1-\sum_{i=1}^{p}\gamma_i(s_l)\right)^{-1}\omega(s_l)$, $\widetilde{\alpha}_j(s_l)=\frac{d^j}{dz^j}\left(\frac{\sum_{i=1}^{q}\alpha_i(s_l)z^i}{1-\sum_{i=1}^{p}\gamma_i(s_l)z^i}\right)_{z=0}\mathbb{I}_{\{j>0\}}$ and $\widetilde{\beta}_j(s_l)=\frac{d^j}{dz^j}\left(\frac{\sum_{i=1}^{q}\beta_i(s_l)z^i}{1-\sum_{i=1}^{p}\gamma_i(s_l)z^i}\right)_{z=0}\mathbb{I}_{\{j>0\}}$. We can write the likelihood function (4.2) as a product of matrices, as follows

$$L_n\left(\underline{\theta}\right)=\underline{\mathbf{1}}'_{(d)}\left\{\prod_{l=2}^{n}\mathbb{P}_{\underline{\theta}}\left(g(\varepsilon_1,...,\varepsilon_l)\right)\right\}\underline{\Pi}(g(\varepsilon_1)),\tag{4.3}$$

where $\underline{\mathbf{1}}_{(d)}$ denotes the vector of order d whose entries are all ones. Now, we define here $g_{s_t}(\varepsilon_t|\varepsilon_{t-1|\infty})$ (resp. $g_{s_t}(\varepsilon_t|\varepsilon_{t-1|1})$) as the density of ε_t given the all previous observations until infinite past (resp. previous observations until ε_1) and let $h_{\underline{\theta}}(\varepsilon_t|\varepsilon_{t-1|\infty})$ (resp. $h_{\underline{\theta}}(\varepsilon_t|\varepsilon_{t-1|1})$) be the corresponding logarithms. For this purpose, we define the conditional likelihood function based on all observations from infinite past noted $\widetilde{L}_n\left(\underline{\theta}\right)$ defined as similar to equation (4.2) except replacing the density $g_{s_l}(\varepsilon_1,...,\varepsilon_l)$ by $g_{s_t}(\varepsilon_t|\varepsilon_{t-1|\infty})$, we have

$$\widetilde{L}_n\left(\underline{\theta}\right)=\underline{\mathbf{1}}'_{(d)}\left\{\prod_{l=2}^{n}\mathbb{P}_{\underline{\theta}}\left(g(\varepsilon_l|\varepsilon_{l-1|\infty})\right)\right\}\underline{\Pi}(g(\varepsilon_1|\varepsilon_{0|\infty})),\tag{4.4}$$

where $\mathbb{P}_{\underline{\theta}}(g(\varepsilon_l|\varepsilon_{l-1|\infty}))$ and $\underline{\Pi}(g(\varepsilon_1|\varepsilon_{0|\infty}))$ replace $g_{s_l}(\varepsilon_1,...,\varepsilon_l)$ by $g_{s_l}(\varepsilon_l|\varepsilon_{l-1|\infty})$, $s_l=1,..,d$ and $l=1,..,n$ in matrix $\mathbb{P}_{\underline{\theta}}(g(\varepsilon_1,...,\varepsilon_l))$ and vector $\underline{\Pi}(g(\varepsilon_1))$.

Remark 4.1. *It should be noted that asymptotically, the stationary distribution $\pi(k)$, $k\in\mathbb{S}$ will not affect the estimation (see., Leroux* [34] *for more details).*

Remark 4.2. *Francq et al.* [19] *proved the inference for stationary standard asymmetric* $\log GARCH_1(p,q)$ *models (i.e., $d=1$) based likelihood. So, it would be interesting to study the strong consistency of the QMLE for the $MS-A\log GARCH_d(p,q)$ model while some parameters are locally outside the stationarity domain, thus we can generalize those results.*

Remark 4.3. *Numerous authors have pointed out the choice of the initial values is unimportant for the asymptotic behavior of the QMLE. However, it may be significant from a practical point of view.*

The next results in this paper establish the strong consistency of $\widehat{\underline{\theta}}_n$.

4.1. Strong consistency of QMLE

The following assumptions will be used to establish the strong consistency of the *QMLE* estimator.

A1. $\underline{\theta}_0\in\Theta$ and Θ is a compact.

A2. $\gamma\left(\Gamma^0\right)<0$ for all $\underline{\theta}\in\Theta$ where Γ^0 denotes the sequence $(\Gamma_{\delta_t}(e_t),t\in\mathbb{Z})$ when the parameters $\underline{\theta}$ are replaced by their true values $\underline{\theta}_0$.

A3. For any $\underline{\theta}, \underline{\theta}^* \in \Theta$, if almost surely $h_{\underline{\theta}}\left(\varepsilon_t | \varepsilon_{t-1|\infty}\right) = h_{\underline{\theta}^*}\left(\varepsilon_t | \varepsilon_{t-1|\infty}\right)$ then $\underline{\theta} = \underline{\theta}^*$.

A4. The observed process satisfies $E\left\{\left|\log \varepsilon_0^2\right|\right\} < \infty$.

A5. The support of e_0 contains at least two positive values and two negative values and $E\left\{\left|\log e_0^2\right|^{\nu}\right\} < \infty$ for some $\nu > 0$.

Assumption **A1**. the compactness of Θ is assumed in order that several results from the real analysis may be used. As seen in Theorem 3.1 Assumption **A2.,** ensure that the process $(\varepsilon_t, t \in \mathbb{Z})$ admits a strictly stationary and ergodic solution. Assumption **A3.,** is made to guarantee the identifiability of parameter $\underline{\theta}$. Assumption **A4.** can be replaced by the sufficient conditions given in Theorem 3.8. Assumption **A5.** eliminates a mass at zero for the innovation, and, for identifiability reasons, imposes non-degeneracy of the positive and negative parts of e_0.
Now, we will appear the consistency of *QMLE* for the $MS - \log GARCH$ model (Theorem 4.7). Our method has availed from the work of Ghezal [24], which depends on Lemmas 4.4 – 4.6, below. First, the following Lemma 4.4 presents that the uniform asymptotic forgetting of initial values

Lemma 4.4. *Under the assumptions **A1-A5**, almost surely, uniformly with respect to* $\underline{\theta} \in \Theta$

$$\lim_{n\longrightarrow\infty} \frac{1}{n} \log L_n\left(\underline{\theta}\right) = \lim_{n\longrightarrow\infty} \frac{1}{n} \log \widetilde{L}_n\left(\underline{\theta}\right) = E_{\underline{\theta}_0}\left\{h_{\underline{\theta}}\left(\varepsilon_t | \varepsilon_{t-1|\infty}\right)\right\}.$$

Proof. Note that $\log \widetilde{L}_n\left(\underline{\theta}\right) = \sum_{t=1}^{n} h_{\underline{\theta}}\left(\varepsilon_t | \varepsilon_{t-1|\infty}\right)$ and $\log L_n\left(\underline{\theta}\right) = \sum_{t=1}^{n} h_{\underline{\theta}}\left(\varepsilon_t | \varepsilon_{t-1|1}\right)$, so,

$$\frac{1}{n}\sum_{t=1}^{n} h_{\underline{\theta}}\left(\varepsilon_t | \varepsilon_{t-1|1}\right) = \frac{1}{n}\sum_{t=1}^{n} h_{\underline{\theta}}\left(\varepsilon_t | \varepsilon_{t-1|\infty}\right) + \frac{1}{n}\sum_{t=1}^{n}\left(h_{\underline{\theta}}\left(\varepsilon_t | \varepsilon_{t-1|1}\right) - h_{\underline{\theta}}\left(\varepsilon_t | \varepsilon_{t-1|\infty}\right)\right).$$

Now for any $l \geq 0$, define the process $H_t(m) = \sup_{l \geq m}\left|h_{\underline{\theta}}\left(\varepsilon_t | \varepsilon_{t-1|t-l}\right) - h_{\underline{\theta}}\left(\varepsilon_t | \varepsilon_{t-1|\infty}\right)\right|$, then for each fixed m, the process $(H_t(m), t \in \mathbb{Z})$ is stationary, ergodic and $E_{\underline{\theta}_0}\{H_t(m)\} < +\infty$. We have

$$\begin{aligned}
&\limsup_{n\longrightarrow\infty}\left|\frac{1}{n}\sum_{t=1}^{n}\left(h_{\underline{\theta}}\left(\varepsilon_t | \varepsilon_{t-1|1}\right) - h_{\underline{\theta}}\left(\varepsilon_t | \varepsilon_{t-1|\infty}\right)\right)\right| \\
&\leq \limsup_{n\longrightarrow\infty}\frac{1}{n}\sum_{t=1}^{n}\left|h_{\underline{\theta}}\left(\varepsilon_t | \varepsilon_{t-1|1}\right) - h_{\underline{\theta}}\left(\varepsilon_t | \varepsilon_{t-1|\infty}\right)\right| \\
&\leq \limsup_{n\longrightarrow\infty}\frac{1}{n}\sum_{t=m+1}^{n} H_t(m) = E_{\underline{\theta}_0}\{H_0(m)\}.
\end{aligned}$$

Since, $\lim_{m\longrightarrow\infty} E_{\underline{\theta}_0}\{H_0(m)\} = 0$, then the result of the first assertion follows. □

We will next compare the likelihood $L_n\left(\underline{\theta}\right)$ (resp. $\widetilde{L}_n\left(\underline{\theta}\right)$) with the one evaluated at the true parameter $\underline{\theta}_0$. Write $I_n\left(\underline{\theta}\right) = \frac{1}{n}\log\left(\frac{L_n\left(\underline{\theta}\right)}{L_n\left(\underline{\theta}_0\right)}\right)$, and the following lemma follows from Lemma 4.4, Jensen's inequality and identifiability assumption.

Lemma 4.5. *Under the assumptions **A1-A5**,* $\lim_{n\to\infty}\frac{1}{n}\log\left(\frac{\widetilde{L}_n\left(\underline{\theta}\right)}{\widetilde{L}_n\left(\underline{\theta}_0\right)}\right) = \lim_{n\longrightarrow\infty} I_n\left(\underline{\theta}\right) \leq 0$ *with equality iff* $\underline{\theta} = \underline{\theta}_0$ *for all* $\underline{\theta} \in \Theta$.

Proof. Under conditions **A1-A5** almost surely $I_n\left(\underline{\theta}\right)$ is well defined. Moreover, by Lemma 4.4 and Jensen's inequality, we have

$$\lim_{n\longrightarrow\infty} I_n\left(\underline{\theta}\right) = E_{\underline{\theta}_0}\left\{\log\frac{h_{\underline{\theta}}\left(\varepsilon_t|\,\varepsilon_{t-1|\infty}\right)}{h_{\underline{\theta}_0}\left(\varepsilon_t|\,\varepsilon_{t-1|\infty}\right)}\right\} \leq \log E_{\underline{\theta}_0}\left\{\frac{h_{\underline{\theta}}\left(\varepsilon_t|\,\varepsilon_{t-1|\infty}\right)}{h_{\underline{\theta}_0}\left(\varepsilon_t|\,\varepsilon_{t-1|\infty}\right)}\right\} \leq \log 1 = 0.$$

By the condition **A3,** $I_n\left(\underline{\theta}\right)$ converge to Kullback-Leibler information which equals to zero iff $\underline{\theta} = \underline{\theta}^0$. □

Theorem 4.7 below follows easily from the following lemma together with the identifiability assumption.

Lemma 4.6. *Under the assumptions **A1-A5**. For all $\underline{\theta}^* \neq \underline{\theta}_0$, there exists a neighborhood $\mathcal{V}\left(\underline{\theta}^*\right)$ of $\underline{\theta}^*$such that, almost surely*

$$\lim\sup_{n\to+\infty}\sup_{\underline{\theta}\in\mathcal{V}(\underline{\theta}^*)} I_n\left(\underline{\theta}\right) < 0.$$

Proof. In view of equation (4.4), we have

$$\min_k \pi(k) g_k\left(\varepsilon_1|\,\varepsilon_{0|\infty}\right)\left\|\left\{\prod_{t=2}^{n}\mathbb{P}_{\underline{\theta}}\left(g\left(\varepsilon_t|\,\varepsilon_{t-1|\infty}\right)\right)\right\}\right\| \leq \widetilde{L}_n\left(\underline{\theta}\right) \leq \max_k \pi(k) g_k\left(\varepsilon_1|\,\varepsilon_{0|\infty}\right)\left\|\left\{\prod_{t=2}^{n}\mathbb{P}_{\underline{\theta}}\left(g\left(\varepsilon_t|\,\varepsilon_{t-1|\infty}\right)\right)\right\}\right\|.$$

So, we obtain $\lim\limits_{n\to\infty}\frac{1}{n}\log\widetilde{L}_n\left(\underline{\theta}\right) = \lim\limits_{n\to\infty}\frac{1}{n}\log\left\|\left\{\prod\limits_{t=2}^{n}\mathbb{P}_{\underline{\theta}}\left(g\left(\varepsilon_t|\,\varepsilon_{t-1|\infty}\right)\right)\right\}\right\| = E_{\underline{\theta}^0}\left\{h_{\underline{\theta}}\left(\varepsilon_t|\,\varepsilon_{t-1|\infty}\right)\right\}$.

Let $\mathcal{V}_m\left(\widetilde{\underline{\theta}}\right) = \left\{\underline{\theta} : \left\|\underline{\theta}-\widetilde{\underline{\theta}}\right\| \leq \frac{1}{m}\right\}$ and set $\Sigma_{2:n}^m = \sup\limits_{\underline{\theta}\in\mathcal{V}_m\left(\widetilde{\underline{\theta}}\right)}\left\|\prod\limits_{t=2}^{n}\mathbb{P}_{\underline{\theta}}\left(g\left(\varepsilon_t|\,\varepsilon_{t-1|\infty}\right)\right)\right\|$. Because the matrix norm is multiplicative, we obtain on $\mathcal{V}_m\left(\widetilde{\underline{\theta}}\right)$

$$\sup_{\underline{\theta}}\left\|\prod_{t=2}^{n+k}\mathbb{P}_{\underline{\theta}}\left(g\left(\varepsilon_t|\,\varepsilon_{t-1|\infty}\right)\right)\right\| \leq \sup_{\underline{\theta}}\left\|\prod_{t=2}^{n}\mathbb{P}_{\underline{\theta}}\left(g\left(\varepsilon_t|\,\varepsilon_{t-1|\infty}\right)\right)\right\|\sup_{\underline{\theta}}\left\|\prod_{t=n+1}^{n+k}\mathbb{P}_{\underline{\theta}}\left(g\left(\varepsilon_t|\,\varepsilon_{t-1|\infty}\right)\right)\right\|,$$

that implies $\log\Sigma_{2:n+k}^m \leq \log\Sigma_{2:n}^m + \log\Sigma_{n+1:n+k}^m$ for any positive integers n and k. Hence $\left(\log\Sigma_{2:n}^m\right)$ is subadditive, stationary, ergodic process and $E_{\underline{\theta}_0}\left\{\log\Sigma_{2:n}^m\right\}$ is finite. From the subadditive ergodic theorem we can get $\phi_m\left(\underline{\theta}^*\right) = \lim\limits_{n\longrightarrow\infty}\frac{1}{n}\log\Sigma_{2:n}^m = \inf\limits_{n>1}\frac{1}{n}E_{\underline{\theta}_0}\left\{\log\Sigma_{2:n}^m\right\}$ *a.s.* Note that if $\phi\left(\underline{\theta}\right)$ denotes the Lyapunov exponent associated with the sequence of random matrices $\left(\mathbb{P}_{\underline{\theta}}\left(g\left(\varepsilon_t|\,\varepsilon_{t-1|\infty}\right)\right), t\in\mathbb{Z}\right)$ i,e.,

$$\phi\left(\underline{\theta}\right) = \inf_{n>1}\frac{1}{n}E_{\underline{\theta}^0}\left\{\log\left\|\prod_{t=2}^{n}\mathbb{P}_{\underline{\theta}}\left(g\left(\varepsilon_t|\,\varepsilon_{t-1|\infty}\right)\right)\right\|\right\} \overset{a.s.}{=} \lim_{n\longrightarrow\infty}\frac{1}{n}\log\left\|\prod_{t=2}^{n}\mathbb{P}_{\underline{\theta}}\left(g\left(\varepsilon_t|\,\varepsilon_{t-1|\infty}\right)\right)\right\|,$$

then, from Lemma 4.5, there exist $\epsilon > 0$ and $n_\epsilon \in \mathbb{N}$ such that $\dfrac{1}{n_\epsilon}E_{\underline{\theta}_0}\left\{\log\left\|\prod\limits_{t=2}^{n_\epsilon}\mathbb{P}_{\underline{\theta}_0}\left(g\left(\varepsilon_t|\,\varepsilon_{t-1|\infty}\right)\right)\right\|\right\} < \phi\left(\underline{\theta}_0\right)-\varepsilon$.
By dominated convergence theorem, it follows that for m large enough, we obtain

$$\phi_m\left(\underline{\theta}^*\right) \leq \frac{1}{n_\epsilon}E_{\underline{\theta}_0}\left\{\log\left\|\prod_{t=1}^{n_\epsilon}\mathbb{P}_{\underline{\theta}^*}\left(g\left(\varepsilon_t|\,\varepsilon_{t-1|\infty}\right)\right)\right\|\right\} < \phi\left(\underline{\theta}_0\right) - \frac{\varepsilon}{2},$$

and the rest follows by Lemma 4.4. □

Second, we are now in a position to state the following consistency theorem

Theorem 4.7. [*Strong consistency*] *Let* $\left(\widehat{\underline{\theta}}_n\right)_n$ *be a sequence of QML estimators satisfying* (4.1). *Then under assumptions* ***A1-A5***, $\widehat{\underline{\theta}}_n$ *is strongly consistent in the sense that* $\widehat{\underline{\theta}}_n \to \underline{\theta}_0$ *a.s. when* $n \to +\infty$.

Proof. The proof of Theorem 4.7 is similar in Bibi and Ghezal [5] for the $MS-BL$ model. For this purpose, suppose $(\widehat{\underline{\theta}}_n)$ does not converge to $\underline{\theta}_0$ a.s. So, $\exists \epsilon > 0$ such that for each integer n_0 there is an integer $n = n(n_0) \geq n_0$ with $\left\|\hat{\underline{\theta}}_n - \underline{\theta}_0\right\| \geq \epsilon$. Using Lemma 4.6, it follows that $L_n\left(\hat{\underline{\theta}}_n\right) < L_n\left(\underline{\theta}_0\right)$. Moreover, we use the definition of $QMLE$ given by (4.1), we have $L_n\left(\hat{\underline{\theta}}_n\right) = \sup_{\underline{\theta}\in\Theta^*} L_n\left(\underline{\theta}\right) \geq L_n\left(\underline{\theta}_0\right)$ for any compact subset Θ^* of Θ containing $\underline{\theta}_0$ and this contradiction completes the proof. □

Remark 4.8. *Chan* [12] *and Jensen and Rahbek* [31] *established statistical inference of QMLE for nonstationary standard GARCH models i.e., when* $\gamma\left(\Gamma^0\right) \geq 0$. *Consequently, it would be profitable for* $MS-A\log GARCH$ *to generalize the strong consistency of QMLE in which all regimes are explosive.*

Remark 4.9. *Francq and Zakoïan* [17] *established asymptotic distribution of the quasi-maximum likelihood QMLE when some coefficients are equal to zero, thus, it would be beneficial to generalize the strong consistency of QMLE for* $MS-A\log GARCH$ *when* $\underline{\theta}_0$ *is on the boundary.*

Remark 4.10. *It is worth noting that the* $MS-EGARCH$ *model, i.e.,*

$$\begin{cases} \varepsilon_t = \sigma_t e_t \\ \log \sigma_t^2 = \omega\left(\delta_t\right) + \sum_{i=1}^{q}\left(\alpha_i\left(\delta_t\right)e_{t-i} + \beta_i\left(\delta_t\right)\left|e_{t-i}\right|\right) + \sum_{j=1}^{p}\gamma_j\left(\delta_t\right)\log\sigma_{t-j}^2, \end{cases} , t \in \mathbb{Z}, \tag{4.5}$$

has similarities with the $MS-A\log GARCH$, *but no similar results, proving the asymptotic properties (see., Wintenberger* [41] *for further details).*

5. Simulation studies

To evaluate the performance of the *QML* method for parameters estimation, we carried out a simulation study based on two stationary $MS-A\log GARCH_d(1,1)$ models ($d = 1$ and $d = 2$), for innovation errors, we use Gaussian $\mathcal{N}(0,1)$ and Student's t_5 innovations. We simulated 500 data samples with different lengths. The sample sizes to be examined in this simulation study are $n = 1000$, $n = 3000$ and $n = 10000$. The corresponding parameter values are chosen to be satisfied the stationary condition (3.2) . For each trajectory the vector $\underline{\theta}$ of parameters of interest has been estimated with *QMLE* noted as $\widehat{\underline{\theta}}$. The *QMLE* algorithm has been executed for these series under *MATLAB*8 using "*fminsearch.m*" as a minimizer function. In the tables below, the roots mean square errors (*RMSE*) of $\widehat{\theta}(i)$, $i = 1, .., d(d+3)$, are displayed in parenthesis in each table, the true values (TV) of the parameters of each of the considered data-generating process are reported.

5.1. *Standard asymmetric* log *GARCH model*

First, we present an example illustrating our theoretical analysis, which is the standard $A\log GARCH_1(1,1)$ model, the vector of parameter $\underline{\theta}' = (\omega, \alpha_1, \beta_1, \gamma_1)$ is chosen to subject the following condition

$$\left|\alpha_1(1) + \gamma_1(1)\right|^s \left|\beta_1(1) + \gamma_1(1)\right|^{1-s} < 1.$$

The results of the simulation are shown in Table 2,

	Tv	$\mathcal{N}(0,1)$	t_5
	$n = 1000$		
$\widehat{\omega}$	0.40	0.4084 (0.0269)	0.4035 (0.0403)
$\widehat{\alpha}_1$	0.05	0.0528 (0.0173)	0.0522 (0.0297)
$\widehat{\beta}_1$	0.35	0.3535 (0.0176)	0.3526 (0.0291)
$\widehat{\gamma}_1$	0.70	0.6928 (0.0195)	0.6939 (0.0329)
	$n = 3000$		
$\widehat{\omega}$	0.40	0.4036 (0.0245)	0.4033 (0.0275)
$\widehat{\alpha}_1$	0.05	0.0510 (0.0113)	0.0521 (0.0163)
$\widehat{\beta}_1$	0.35	0.3517 (0.0130)	0.3519 (0.0183)
$\widehat{\gamma}_1$	0.70	0.6973 (0.0167)	0.6972 (0.0214)
	$n = 10000$		
$\widehat{\omega}$	0.40	0.4019 (0.0103)	0.4007 (0.0156)
$\widehat{\alpha}_1$	0.05	0.0506 (0.0064)	0.0506 (0.0104)
$\widehat{\beta}_1$	0.35	0.3508 (0.0063)	0.3507 (0.0103)
$\widehat{\gamma}_1$	0.70	0.6987 (0.0072)	0.6989 (0.0119)

Table 2. Average and *RMSE* of *QMLE* for standard $A\log GARCH_1(1,1)$ model.

5.2. $MS - A\log GARCH$ model

Second, we will present an example illustrating our theoretical analysis to estimate $MS{-}A\log GARCH_d(1,1)$. Therefore, we consider the most widely used two-regime switching model (i.e., $d = 2$). The vector of interest parameters to be estimated is $\underline{\theta}' = \left(\underline{\omega}', \underline{\alpha}_1', \underline{\beta}_1', \underline{\gamma}_1', \underline{p}'\right)$ where $\underline{\omega}' = (\omega(1), \omega(2))$, $\underline{\alpha}_1' = (\alpha_1(1), \alpha_1(2))$, $\underline{\beta}_1' = (\beta_1(1), \beta_1(2))$, $\underline{\gamma}_1' = (\gamma_1(1), \gamma_1(2))$ and $\underline{p}' = (p_{12}, p_{21})$ is chosen to ensure the stationary condition, i.e.,

$$\prod_{k=1}^{2} \left|\alpha_1(k) + \gamma_1(k)\right|^{\pi(k)} \left|\beta_1(k) + \gamma_1(k)\right|^{-(1-s)\pi(k)} < 1.$$

The results of the simulation are gathered in Table 3,

n		1000		3000		10000	
	Tv	$\mathcal{N}(0,1)$	t_5	$\mathcal{N}(0,1)$	t_5	$\mathcal{N}(0,1)$	t_5
$\underline{\omega}$	0.35	0.3478 (0.1248)	0.3428 (0.2345)	0.3515 (0.0628)	0.3578 (0.1299)	0.3513 (0.0322)	0.3516 (0.0680)
	0.10	0.0961 (0.1286)	0.1072 (0.2577)	0.0982 (0.0623)	0.0985 (0.1426)	0.0992 (0.0333)	0.0978 (0.0719)
$\underline{\alpha}_1$	0.05	0.0499 (0.0272)	0.0536 (0.0400)	0.0498 (0.0146)	0.0521 (0.0219)	0.0503 (0.0079)	0.0504 (0.0123)
	0.60	0.6007 (0.0234)	0.6024 (0.0398)	0.6001 (0.0180)	0.6014 (0.0259)	0.6003 (0.0090)	0.6007 (0.0154)
$\underline{\beta}_1$	0.30	0.2995 (0.0237)	0.3012 (0.0349)	0.3009 (0.0148)	0.3022 (0.0227)	0.3008 (0.0078)	0.3012 (0.0116)
	0.40	0.4024 (0.0227)	0.4040 (0.0364)	0.4005 (0.0163)	0.4023 (0.0210)	0.4002 (0.0081)	0.4004 (0.0140)
$\underline{\gamma}_1$	0.50	0.4982 (0.0318)	0.4952 (0.0485)	0.4993 (0.0223)	0.4980 (0.0327)	0.4994 (0.0100)	0.4994 (0.0150)
	0.75	0.7497 (0.0418)	0.7520 (0.0692)	0.7498 (0.0279)	0.7493 (0.0429)	0.7494 (0.0124)	0.7491 (0.0208)
$\underline{p}$	0.80	0.7987 (0.0356)	0.7973 (0.0449)	0.7991 (0.0192)	0.7985 (0.0295)	0.7993 (0.0120)	0.7990 (0.0176)
	0.25	0.2529 (0.0387)	0.2538 (0.0598)	0.2511 (0.0268)	0.2526 (0.0364)	0.2503 (0.0116)	0.2508 (0.0205)

Table 3. Average and *RMSE* of *QMLE* for $MS - A\log GARCH_2(1,1)$ model.

Now let us devote a few comments in order, Table 2 shows that the strong consistency of *QMLE* of standard models is fairly satisfying and the associated *RMSE* decreases closely as the sample size increases. Regarding outcomes associated with *MS*−Models reported in Table 3, it is obvious that the strong consistency is fully approved.

6. Empirical application

In this section, we apply our model for modeling the foreign exchange rate series $(\varepsilon_t)_{t\geq 1}$ of the Algerian Dinar against the U.S. Dollar (USD/DZD) from January 3, 2000, to September 22, 2011 (3050 observations) and $e_t \sim \mathcal{N}(0,1)$. We then removed all dates when the market was closed (i.e., holidays and weekends). The graphics of prices, the daily returns series of prices $(r_t = 100\log(\varepsilon_t/\varepsilon_{t-1}))_{t\geq 1}$, squared, absolute and log −absolute returns are plotted in Fig3.

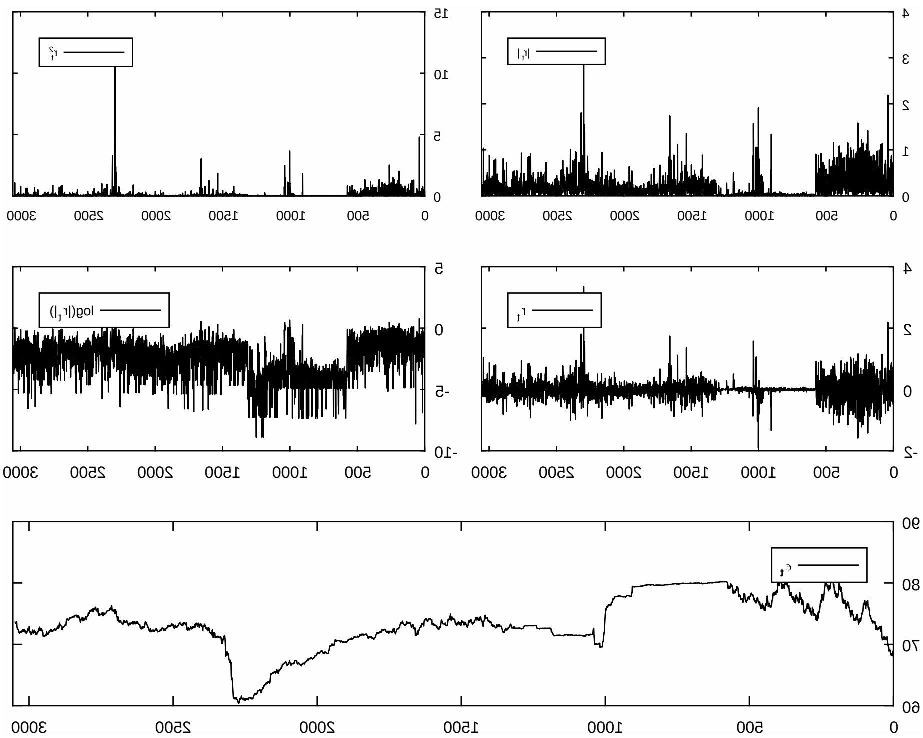

Fig3. The plots of the price series (ε_t), (r_t), $(|r_t|)$, $\left(r_t^2\right)$ and $\log(|r_t|)$.

In Table 4 below, we present some elementary descriptive statistics of the series $(\varepsilon_t)_{t\geq 1}$, $(r_t)_{t\geq 1}$, $(|r_t|)_{t\geq 1}$ and $\left(r_t^2\right)_{t\geq 1}$,

Series	mean	Std. Dev	Median	Skewness	Kurtosis	Min	Max	Arch(300)	J. Bera	LBtest
(ε_t)	73.451	4.2424	73.126	−0.6005	3.7642	60.345	81.281	100%	2.58×10^2	100%
(r_t)	0.0000	3.0000	0.0000	1.0000	13.000	−19.000	33.000	100%	1.3×10^4	100%
$(\|r_t\|)$	0.0000	2.0000	1.0000	3.0000	21.000	0.0000	32.000	90, 3%	4.49×10^4	100%
$\left(r_t^2\right)$	0.0000	0.0000	0.0000	0.0000	1000.0	0.0000	1000.0	00%	3.57×10^7	00%

Table 4: Summary statistics for daily crude oil prices series $(\varepsilon_t)_{t\geq 1}$, their returns, $(r_t)_{t\geq 1}$, $(|r_t|)_{t\geq 1}$ and $\left(r_t^2\right)_{t\geq 1}$.

The result shown in Table 4 for the kurtosis of the log −return series is 13, which indicates that models based on the Gaussian assumption may not well describe the data. Thus could not reject the null hypothesis $H0$: "The residuals of $(r_t)_{t\geq 1}$ are not correlated" contrary to the series $\left(r_t^2\right)_{t\geq 1}$ which presents a significant *ARCH* effect in its residuals because there is not enough evidence, while by Arch(300) column, reported in Table 4 for testing $R0$: "No residuals heteroscedasticity of $(r_t)_{t\geq 1}$" shows that through the first three hundred lags, $R0$ should be rejected. Furthermore, by looking over at the graphics shown in Fig4 of the sample autocorrelations functions of the series $(|r_t|)_{t\geq 1}$, $\left(r_t^2\right)_{t\geq 1}$ and $(r_t)_{t\geq 1}$

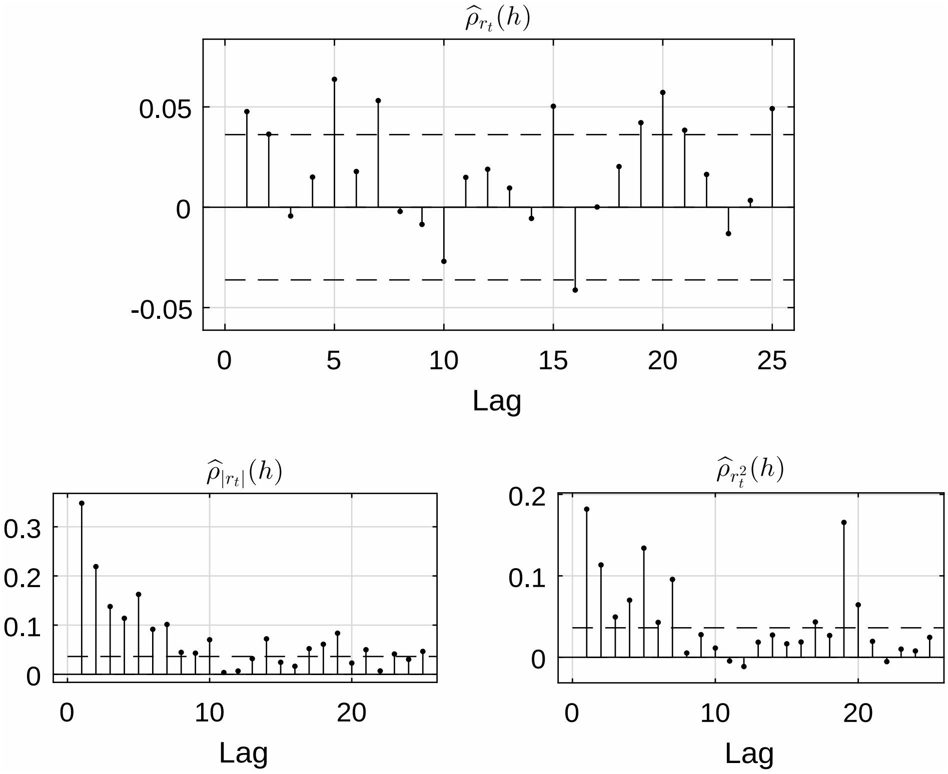

Fig4. The sample autocorrelations functions of the series $(r_t)_t$, $(|r_t|)_t$ and $\left(r_t^2\right)_t$.

one can observe that $(r_t)_{t\geq 1}$ presents a Taylor-effect (characterized by $\widehat{\rho}_{r_t^2}(k) < \widehat{\rho}_{|r_t|}(k)$ for some $k \geq 1$), and hence you must reject the modeling of the series $(r_t)_{t\geq 1}$ by standard *GARCH* models in favor of certain asymmetric models, such as $r_t = h_t e_t$ where h_t is the corresponding asymmetric volatility process which is a measurable function of $\{r_{m-1}, m \leq t\}$ and the innovation term (e_t) is subject to some theoretical distribution. Now, we assume that the volatility associated with $(r_t)_{t\geq 1}$ satisfies $MS-A\log GARCH(1,1)$ with two regimes, i.e., $\delta_t = 1$ corresponds to Monday with probability $\pi(1)$ and $\delta_t = 2$ for the other days with equiprobabilities $\pi(2)$. The estimated parameters of the 2−regimes $MS-A\log GARCH(1,1)$ model and their *RMSE* are reported in Table 5.

Days	$\underline{\widehat{\omega}}$	$\underline{\widehat{\alpha}}_1$	$\underline{\beta}_1$	$\underline{\gamma}_1$	$\mathbb{P}'$	
Monday	0.3392	0.0581	0.3242	0.7606	0.7439	0.2561
	(0.0081)	(0.0041)	(0.0015)	(0.0009)	(0.017)	(0.023)
Other days	0.0643	0.5808	0.7693	0.8870	0.2851	0.7149
	(0.0128)	(0.0018)	(0.0009)	(0.0008)	(0.044)	(0.039)

Table 5. *QMLE* estimate and their *RMSE*.

The plot of the corresponding volatility is shown in Fig. 5.

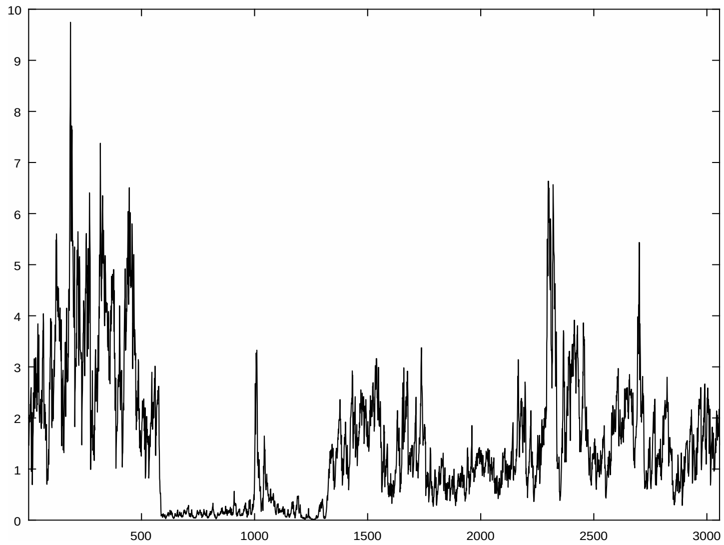

Fig5. The estimated volatility from $MS-A\log GARCH(1,1)$ model.

The diagnostic of residual associated is shown in Fig. 6.

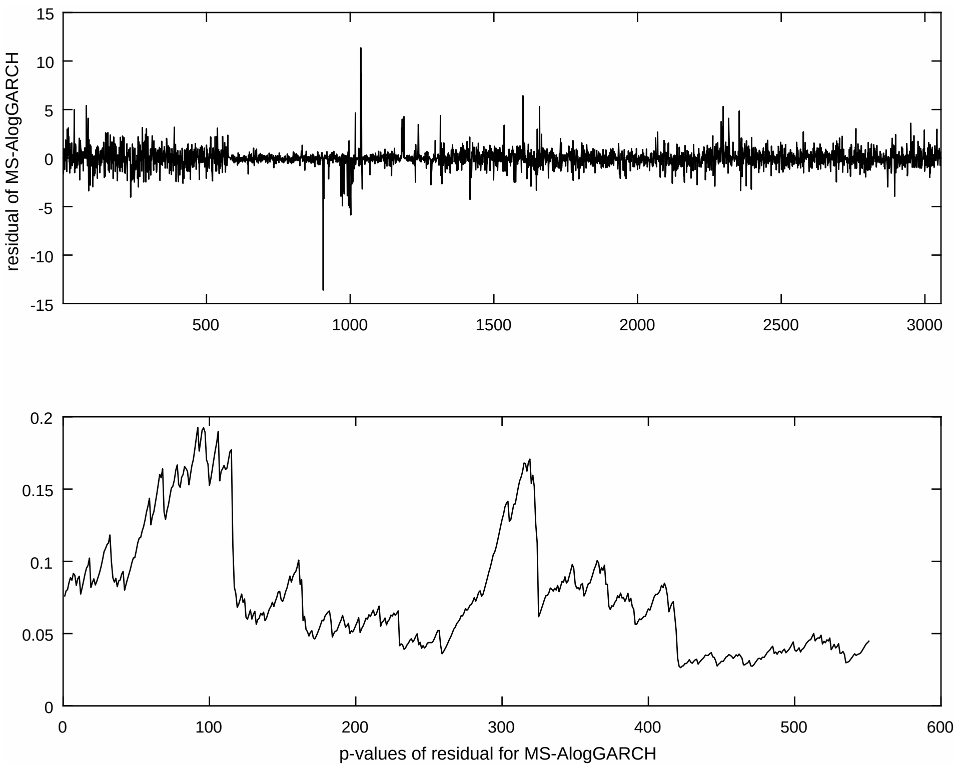

Fig6. Top panel: residual associated with $MS-A\log GARCH(1,1)$. Bottom panel: $p-$values of residual associated with $MS-A\log GARCH(1,1)$.

Table 5 displays the estimated $MS-A\log GARCH(1,1)$ models. This table shows that the $MS-A\log GARCH$ (1,1) has been accurately estimated. Note that the parameters mentioned in this table satisfy the strictly stationary condition and the estimated models also satisfy the assumptions **A1-A5** used to show the consistency.

7. Conclusion

This paper proposes a new Markov-switching asymmetric $\log GARCH$ model with constant transition probabilities by integrating the standard $A\log GARCH$ model with a hidden Markov chain, within each regime, there is a different $A\log GARCH-$type model. Therefore, this new model is an extension of the standard model with constant coefficients, where the positive coefficients are dropped and the volatility is not bounded below which needs an additional $\log -$moment assumption, compared to the $MS-GARCH$ model. Furthermore, this model can be used to capture three important dynamic characteristics of time series, regimes, asymmetric, and conditional heteroskedasticity. So, we provide some explicit results for the structural and asymptotic properties of the $MS-A\log GARCH$ process. First, we found sufficient conditions for the existence of moments and $\log -$moments of the strictly stationary solutions. Second, we showed the strong consistency of the $QMLE$ under mild assumptions. Lastly, the proposed methodology is illustrated through a simulation study helps to clarify the consistency of the estimators (for both Gaussian and Student's innovations) and an empirical application to the exchange rate of the Algerian Dinar against the U.S. Dollar.

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