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Approximation properties and q-statistical convergence of Kantorovich variant of Stancu type Lupaş operators

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Abstract. We introduce Kantorovich variant of Stancu-Lupaş operators and study convergence and q-statistical convergence properties using Korovkin theorem. Rate of convergence is analyzed in terms of modulus of continuity, elements of Lipschitz class and Peetre's K-functional. Direct theorems are proved and Voronovskaja type theorem is established. Graphical analysis of convergence and error estimations are presented with the help of MATLAB.

1. Introduction

Weierstrass approximation theorem is the foundation of approximation theory, which states that any continuous function defined on a closed and bounded interval can be uniformly approximated by some polynomials. In order to provide a clear, concise and elegant proof of this pre-eminent theorem, S.N. Bernstein pioneered the renowned polynomials known as the Bernstein polynomials in 1912. Thenceforth, various studies have emerged to investigate the approximation properties in different settings and spaces. A bunch of operators have been constructed, e.g. Kantorovich [9], Mirakjan [13], Szász [24], Stancu [23] and many more. These operators offer the enhancement of approximating functions of various kinds and produce better and finer estimates. For example, an imperative discrete operator was suggested by renowned Romanian mathematician Alexandru Lupaş [12], which is for $u \geq 0,\ m \in \mathbb{N}$ defined by

$$L_m(h;u) = (1-a)^{mu} \sum_{j=0}^{\infty} \frac{(mu)_j}{j!} h\left(\frac{j}{m}\right) a^j, \quad u \geq 0,$$

where $(\beta)_j = \beta(\beta+1)(\beta+2)\ldots(\beta+j-1)$, $j \geq 1$ and $(\beta)_0 = 1$, for the function $h : [0,\infty) \to \mathbb{R}$.
By considering $a = \frac{1}{2}$, O. Agratini [1] obtained the following operators

$$\mathfrak{L}_m(h;u) = 2^{-mu} \sum_{j=0}^{\infty} \frac{(mu)_j}{2^j j!} h\left(\frac{j}{m}\right), \quad u \geq 0 \tag{1}$$

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and extended these operators to Kantotrovich type operators

$$\mathcal{K}_m(h;u) = \frac{m}{2^{mu}} \sum_{j=0}^{\infty} \frac{(mu)_j}{2^j j!} \int_{\frac{j}{m}}^{\frac{j+1}{m}} h\Big(\frac{j}{m}\Big), \ \ u \geq 0 \tag{2}$$

in order to approximate the Lebesgue integrable functions and studied their approximation properties. One can refer to ([10],[14],[16],[19],[20],[21],[22],[25]) for more results about Kantorovich type operators. Stancu type generalizations of various operators have been constructed and investigated, *e.g.*, in ([8],[6], [15]).

Khan and Khan [11] constructed the Stancu type generalization of Lupaş operators in (1) as follows:

$$\mathfrak{L}_m^{\gamma,\eta}(h;u) = \frac{1}{2^{mu}} \sum_{j=0}^{\infty} \frac{(mu)_j}{2^j j!} g\left(\frac{j+\gamma}{m+\eta}\right), \tag{3}$$

where $h:[0,\infty) \to \mathbb{R},\ 0 \leq \gamma \leq \eta$ and studied approximation properties for these operators in [11].

We introduce the following Kntorovich type generalization of Lupaş operators defined in (3) and proves some approximation results for them

$$\mathcal{K}_m^{*\gamma,\eta}(h;u) = \frac{(m+\eta)}{2^{mu}} \sum_{j=0}^{\infty} \frac{(mu)_j}{2^j j!} \int_{\frac{j+\gamma}{m+\eta}}^{\frac{j+\gamma+1}{m+\eta}} h(s)ds, \tag{4}$$

where

$$h:[0,\infty) \to \mathbb{R},\ 0 \leq \gamma \leq \eta.$$

Plugging in to $\gamma = \eta = 0$, the operators presented in (4) reduce to the Kantorovich-Lupaş operators given in (2). Clearly, our operators are more general.

2. Auxiliary results

We have the following lemma.

Lemma 2.1. *For $e_i(t) = t^i,\ (i = 0,1,2,3)$, the following estimates are obtained*

$$
\begin{aligned}
(i)\ \mathcal{K}_m^{*\gamma,\eta}(e_0;u) &= 1\\
(ii)\ \mathcal{K}_m^{*\gamma,\eta}(e_1;u) &= \frac{m}{m+\eta}u + \frac{2\gamma+1}{2(m+\eta)}\\
(iii)\ \mathcal{K}_m^{*\gamma,\eta}(e_2;u) &= \frac{m^2}{(m+\eta)^2}u^2 + \frac{m(2\gamma+3)}{(m+\eta)^2}u + \frac{(3\gamma^2+3\gamma+1)}{3(m+\eta)^2}\\
(iv)\ \mathcal{K}_m^{*\gamma,\eta}(e_3;u) &= \frac{m^3}{(m+\eta)^3}u^3 + \frac{3(2\gamma+5)m^2}{2(m+\eta)^3}u^2 + \frac{(6\gamma^2+18\gamma+17)m}{2(m+\eta)^3}u + \frac{(4\gamma^3+6\gamma^2+4\gamma+1)}{4(m+\eta)^3}
\end{aligned}
$$

Proof. With easy calculations, (i) is obtained.

(ii) We have

$$\begin{aligned}
\mathcal{K}_m^{*\gamma,\eta}(e_1;u) &= \frac{(m+\eta)}{2^{mu}}\sum_{j=0}^{\infty}\frac{(mu)_j}{2^j j!}\int_{\frac{(j+\gamma)}{(m+\eta)}}^{\frac{(j+\gamma+1)}{(m+\eta)}} s ds\\
&= \frac{1}{2^{mu+1}(m+\eta)}\sum_{j=0}^{\infty}\frac{(mu)_j}{2^j j!}[(j+\gamma+1)^2-(j+\gamma)^2]\\
&= \frac{1}{2^{mu+1}(m+\eta)}\sum_{j=0}^{\infty}\frac{(mu)_j}{2^j j!}j+\frac{2\gamma+1}{2^{mu}(m+\eta)}\sum_{j=0}^{\infty}\frac{(mu)_j}{2^j j!}\\
&= \frac{mu}{2^{mu+1}(m+\eta)}\sum_{j=1}^{\infty}\frac{(mu+1)_{j-1}}{2^{j-1}(j-1)!}+\frac{2\gamma+1}{2^{mu+1}(m+\eta)}\sum_{j=0}^{\infty}\frac{(mu)_j}{2^j j!}\\
&= \frac{m}{m+\eta}u+\frac{2\gamma+1}{2(m+\eta)}.
\end{aligned}$$

(iii)

$$\begin{aligned}
\mathcal{K}_m^{*\gamma,\eta}(e_2;u) &= \frac{(m+\eta)}{2^{mu}}\sum_{j=0}^{\infty}\frac{(mu)_j}{2^j j!}\int_{\frac{(j+\gamma)}{(m+\eta)}}^{\frac{(j+\gamma+1)}{(m+\eta)}} s^2 ds\\
&= \frac{1}{3.2^{mu}(m+\eta)^2}\sum_{j=0}^{\infty}\frac{(mu)_j}{2^j j!}[(j+\gamma+1)^3-(j+\gamma)^3]\\
&= \frac{1}{2^{mu}(m+\eta)^2}\sum_{j=0}^{\infty}\frac{(mu)_j}{2^j j!}j^2+\frac{2\gamma+1}{2^{mu}(m+\eta)^2}\sum_{j=0}^{\infty}\frac{(mu)_j}{2^j j!}j+\frac{3\gamma^2+3\gamma+1}{3.2^{mu}(m+\eta)^2}\sum_{j=0}^{\infty}\frac{(mu)_j}{2^j j!}\\
&= \frac{mu(mu+1)}{2^{mu+2}(m+\eta)^2}\sum_{j=2}^{\infty}\frac{(mu+2)_{j-2}}{2^{j-2}(j-2)!}+\frac{2mu(2\gamma+1)}{2^{mu+1}(m+\eta)^2}\sum_{j=1}^{\infty}\frac{(mu+1)_{j-1}}{2^{j-1}(j-1)!}\\
&+\frac{3\gamma^2+3\gamma+1}{3.2^{mu}(m+\eta)^2}\sum_{j=0}^{\infty}\frac{(mu)_j}{2^j j!}\\
&= \frac{m^2}{(m+\eta)^2}u^2+\frac{m(2\gamma+3)}{(m+\eta)^2}u+\frac{(3\gamma^2+3\gamma+1)}{3(m+\eta)^2}.
\end{aligned}$$

(iv)

$$\begin{aligned}
\mathcal{K}_m^{*\gamma,\eta}(e_3;u) &= \frac{(m+\eta)}{2^{mu}}\sum_{j=0}^{\infty}\frac{(mu)_j}{2^j j!}\int_{\frac{(j+\gamma)}{(m+\eta)}}^{\frac{(j+\gamma+1)}{(m+\eta)}} s^3 ds\\
&= \frac{1}{2^2.2^{mu}(m+\eta)^3}\sum_{j=0}^{\infty}\frac{(mu)_j}{2^j j!}[(j+\gamma+1)^4-(j+\gamma)^4]\\
&= \frac{4}{4.2^{mu}(m+\eta)^3}\sum_{j=0}^{\infty}\frac{(mu)_j}{2^j j!}j^3+\frac{6(2\gamma+1)}{4.2^{mu}(m+\eta)^3}\sum_{j=0}^{\infty}\frac{(mu)_j}{2^j j!}j^2\\
&+\frac{6(2\gamma^2+2\gamma+1)}{4.2^{mu}(m+\eta)^3}\sum_{j=0}^{\infty}\frac{(mu)_j}{2^j j!}j+\frac{4\gamma^3+6\gamma^2+4\gamma+1}{4.2^{mu}(m+\eta)^3}\sum_{j=0}^{\infty}\frac{(mu)_j}{2^j}\\
&= \frac{mu(mu+1)(mu+2)}{2^{mu+3}(m+\eta)^3}\sum_{j=3}^{\infty}\frac{(mu+3)_{j-3}}{2^{j-3}(j-3)!}+\frac{3(2\gamma+3)mu(mu+1)}{2^{mu+3}(m+\eta)^3}\sum_{j=2}^{\infty}\frac{(mu+2)_{j-2}}{2^{j-2}(j-2)!}
\end{aligned}$$

$$+\frac{mu(3\gamma^2+6\gamma+4)}{2^{mu}(m+\eta)^3}\sum_{j=1}^{\infty}\frac{(mu+1)_{j-1}}{2^{j-1}(j-1)!}+\frac{4\gamma^3+6\gamma^2+4\gamma+1}{2^{mu+2}(m+\eta)^3}\sum_{j=0}^{\infty}\frac{(mu)_j}{2^j j!}$$
$$=\frac{m^3}{(m+\eta)^3}u^3+\frac{3(2\gamma+5)m^2}{2(m+\eta)^3}u^2+\frac{(6\gamma^2+18\gamma+17)m}{2(m+\eta)^3}u+\frac{(4\gamma^3+6\gamma^2+4\gamma+1)}{4(m+\eta)^3}.$$

This proves the lemma completely. □

We compute moments in the following.

Lemma 2.2. *The first, second and the third moments for the operators* (4) *are obtained as*

$$(i)\ \mathcal{K}_m^{*\gamma,\eta}((t-u);u)=\left(\frac{m}{m+\eta}-1\right)u+\frac{2\gamma+1}{2(m+\eta)}$$
$$(ii)\ \mathcal{K}_m^{*\gamma,\eta}((t-u)^2;u)=\left(\frac{m}{(m+\eta)}-1\right)^2u^2+\left(\frac{2m-(2\gamma+1)(\eta+1)}{(m+\eta)^2}\right)u+\frac{(3\gamma^2+3\gamma+1)}{3(m+\eta+1)^2}$$
$$(iii)\ \mathcal{K}_m^{*\gamma,\eta}((t-u)^3;u)=\left(\frac{m}{(m+\eta)}-1\right)^3u^3+\frac{3(\eta+1)(-4m+(\eta+1)(2\gamma\eta+1))}{(m+\eta)^3}u^2$$
$$+\frac{(4\gamma+5-2(\eta+1)(3\gamma^2+3\gamma+1))}{2(m+\eta)^3}u+\frac{(4\gamma^3+6\gamma^2+4\gamma+1)}{4(m+\eta)^3}$$

Proof. (i) follows by the linearity of the operators $\mathcal{K}_m^{*\gamma,\eta}(.;u)$.

(ii) Making use of Lemma 2.1 and linearity, we have

$$\mathcal{K}_m^{*\gamma,\eta}((t-u)^2;u)=\mathcal{K}_m^{*\gamma,\eta}(t^2;u)-2u\mathcal{K}_m^{*\gamma,\eta}(t;u)+u^2\mathcal{K}_m^{*\gamma,\eta}(1;u)$$
$$=\frac{m^2}{(m+\eta)^2}u^2+\frac{m(2\gamma+3)}{(m+\eta)^2}u+\frac{(3\gamma^2+3\gamma+1)}{3(m+\eta)^2}$$
$$-2u\left(\frac{m}{m+\eta}u+\frac{2\gamma+1}{2(m+\eta)}\right)+u^2$$
$$=\left(\frac{m^2}{(m+\eta)^2}-\frac{2m}{m+\eta}+1\right)u^2+\left(\frac{m(2\gamma+3)}{(m+\eta)^2}-\frac{2\gamma+1}{(m+\eta)}\right)u+\frac{(3\gamma^2+3\gamma+1)}{3(m+\eta)^2}$$
$$=\left(\frac{m}{(m+\eta)}-1\right)^2u^2+\left(\frac{2m-(2\gamma+1)(\eta+1)}{(m+\eta)^2}\right)u+\frac{(3\gamma^2+3\gamma+1)}{3(m+\eta)^2}.$$

(iii)

$$\mathcal{K}_m^{*\gamma,\eta}((t-u)^3;u)=\mathcal{K}_m^{*\gamma,\eta}(t^3;u)-3u\mathcal{K}_m^{*\gamma,\eta}(t^2;u)+3u^2\mathcal{K}_m^{*\gamma,\eta}(t;u)-u^3L_m^{*\gamma,\eta}(1;u)$$
$$=\left(\frac{m^3}{(m+\eta)^3}-1\right)u^3+\frac{3m^2(2\gamma+5)}{2(m+\eta)^3}u^2+\frac{(6\gamma^2+18\gamma+17)m}{2(m+\eta)^3}u$$
$$+\frac{(4\gamma^3+6\gamma^2+4\gamma+1)}{4(m+\eta)^3}-\frac{3m^2}{(m+\eta)^2}u^3-\frac{3m(2\gamma+3)}{(m+\eta)^2}u^2$$
$$-\frac{3\gamma^2+3\gamma+1}{(m+\eta)^2}u+\frac{3m}{m+\eta}u^3+\frac{3(2\gamma+1)}{2(m+\eta)}$$
$$=\left(\frac{m}{(m+\eta)}-1\right)^3u^3+\left[\frac{3m^2(2\gamma+5)}{2(m+\eta)^3}-\frac{3m(2\gamma+3)}{(m+\eta)^2}+\frac{3m(2\gamma+1)}{(m+\eta)}\right]u^2$$
$$+\left[\frac{m(6\gamma^2+18\gamma+17)m}{2(m+\eta)}-\frac{3\gamma^2+3\gamma+1}{(m+\eta)^2}\right]u+\frac{(4\gamma^3+6\gamma^2+4\gamma+1)}{4(m+\eta)^3}$$
$$=\left(\frac{m}{(m+\eta)}-1\right)^3u^3+\frac{3(\eta+1)(-4m+(\eta+1)(2\gamma\eta+1))}{(m+\eta)^3}u^2$$

$$+\frac{(4\gamma+5-2(\eta+1)(3\gamma^2+3\gamma+1))}{2(m+\eta)^3}u+\frac{(4\gamma^3+6\gamma^2+4\gamma+1)}{4(m+\eta)^3},$$

which proves the Lemma 2.2. □

Lemma 2.3. *For any $h \in [0,\infty)$, we have*

$$|\mathcal{K}_m^{*\gamma,\eta}(h;u)| \le \|h\|.$$

Proof. From the definition of $\mathcal{K}_m^{*\gamma,\eta}$ given in (4), we have

$$\begin{aligned}
\|\mathcal{K}_m^{*\gamma,\eta}(h;u)| &\le \frac{(m+\eta)}{2^{mu}}\sum_{j=0}^{\infty}\frac{(mu)_j}{2^j j!}\int_{\frac{j+\gamma}{m+\eta}}^{\frac{j+\gamma+1}{m+\eta}}|h(s)|ds\\
&\le \|h\|\frac{(m+\eta)}{2^{mu}}\sum_{j=0}^{\infty}\frac{(mu)_j}{2^j j!}\int_{\frac{j+\gamma}{m+\eta}}^{\frac{j+\gamma+1}{m+\eta}}1ds\\
&\le \|h\|\sum_{k=0}^{\infty}\frac{(mu)_j}{2^{mu+j}j!}=\|h\|.
\end{aligned}$$

□

3. Convergence in weighted space

This section is devoted to investigate convergence of our operators $\mathcal{K}_m^{*\gamma,\eta}(.;u)$ in weighted space of functions. Let $\mathcal{B}_\rho[0,\infty)=\{h \mid h:[0,\infty)\to\mathbb{R}\}$ such that $|h(u)|\le\mathcal{M}_h\rho(u)$, where $\mathcal{M}_h$ is a constant associated with the function h and $\rho(u)=1+u^2$ is a weight function. Define

$$\|h\|_\rho=\sup_{u\in[0,\infty)}\frac{|h(u)|}{\rho(u)}.$$

Also, let $C_\rho[0,\infty)=\{h\in\mathcal{B}_\rho[0,\infty):\text{h is continuous on }[0,\infty)\}$, and
$C_\rho^0[0,\infty)=\left\{h\in C_\rho[0,\infty):\lim_{u\to\infty}\frac{|h(u)|}{\rho(u)}\text{ is finite}\right\}$.
We have

Lemma 3.1. *Let $\mathcal{K}_m^{*\gamma,\eta}(.;u)$ be operators defined by (4). Then for the weight function $\rho(u)=1+u^2$, we obtain*

$$\|\mathcal{K}_m^{*\gamma,\eta}(\rho;u)\|_\rho\le K,$$

where K is a positive constant greater than 1.

Proof. Using linearity and Lemma 2.1, we obtain

$$\begin{aligned}
\mathcal{K}_m^{*\gamma,\eta}(\rho;u) &= \mathcal{K}_m^{*\gamma,\eta}(1+u^2;u)\\
&= \mathcal{K}_m^{*\gamma,\eta}(1;u)+\mathcal{K}_m^{*\gamma,\eta}(u^2;u)\\
&= 1+\frac{m^2}{(m+\eta)^2}u^2+\frac{m(2\gamma+3)}{(m+\eta)^2}u+\frac{(3\gamma^2+3\gamma+1)}{3(m+\eta)^2}.
\end{aligned}$$

Then, $\|\mathcal{K}_m^{*\gamma,\eta}(\rho;u)\|_\rho$

$$\begin{aligned} &= \sup_{u\geq 0}\left\{\frac{1}{1+u^2}+\frac{m^2}{(m+\eta)^2}\frac{u^2}{1+u^2}+\frac{m(2\gamma+3)}{(m+\eta)^2}\frac{u}{1+u^2}+\frac{(3\gamma^2+3\gamma+1)}{3(m+\eta)^2}\frac{1}{1+u^2}\right\} \\ &< 1+\frac{m^2}{(m+\eta)^2}+\frac{m(2\gamma+3)}{(m+\eta)^2}+\frac{(3\gamma^2+3\gamma+1)}{3(m+\eta)^2}. \end{aligned}$$

Noting that $\lim_{m\to\infty}\frac{m^2}{(m+\eta)^2}=1$, $\lim_{m\to\infty}\frac{m(2\gamma+3)}{(m+\eta)^2}=0=\lim_{m\to\infty}\frac{(3\gamma^2+3\gamma+1)}{3(m+\eta)^2}$, we find that there exists a constant $K>1$ such that

$$\|\mathcal{K}_m^{*\gamma,\eta}(\rho;u)\|_\rho\leq K,$$

which proves the lemma. □

The following theorem is proved.

Theorem 3.2. *Let $\mathcal{K}_m^{*\gamma,\eta}(.;u)$ be the operators defined by* (2.1) *and $\rho(u)=1+u^2$ be the weight function. Then for each $g\in C_\rho^0[0,\infty)$, we obtain*

$$\lim_{m\to\infty}\|\mathcal{K}_m^{*\gamma,\eta}(g;u)-g(u)\|_\rho=0.$$

Proof. It is sufficient to show that

$$\lim_{m\to\infty}\|\mathcal{K}_m^{*\gamma,\eta}(t^i;u)-u^i\|_\rho=0,\ \text{for } i=0,1,2.$$

From Lemma 2.1 (i), we have $\mathcal{K}_m^{*\gamma,\eta}(1;u)=1$, so

$$\|\mathcal{K}_m^{*\gamma,\eta}(1;u)-1\|_\rho=0.$$

By Lemma 2.1 (ii), we get $\|\mathcal{K}_m^{*\gamma,\eta}(e_1;u)-e_1\|_\rho$

$$\begin{aligned} &= \sup_{u\geq 0}\left|\left(\frac{m}{m+\eta}-1\right)\frac{u}{1+u^2}+\frac{2\gamma+1}{2(m+\eta)}\frac{u}{1+u^2}\right| \\ &\leq \left(\frac{m}{m+\eta}-1\right)+\frac{2\gamma+1}{2(m+\eta)}. \end{aligned}$$

Hence, we obtain

$$\lim_{m\to\infty}\|\mathcal{K}_m^{*\gamma,\eta}(e_1;u)-e_1\|_\rho=0.$$

In view of Lemma 2.1 (iii), one infers that
$\|\mathcal{K}_m^{*\gamma,\eta}(e_2;u)-e_2\|_\rho$

$$\begin{aligned} &= \sup_{u\geq 0}\left|\left(\frac{m^2}{(m+\eta)^2}-1\right)\frac{u^2}{1+u^2}+\frac{m(2\gamma+3)}{(m+\eta)^2}\frac{u}{1+u^2}+\frac{(3\gamma^2+3\gamma+1)}{3(m+\eta)^2}\frac{1}{1+u^2}\right| \\ &\leq \left(\frac{m^2}{(m+\eta)^2}-1\right)+\frac{m(2\gamma+3)}{(m+\eta)^2}+\frac{(3\gamma^2+3\gamma+1)}{3(m+\eta)^2}\to 0\ \textit{as}\ m\to\infty, \end{aligned}$$

and this leads to

$$\lim_{m\to\infty}\|\mathcal{K}_m^{*\gamma,\eta}(e_2;u)-e_2\|_\rho=0,$$

and this completes the proof in view of [5]. □

4. Main Results

In this section, we compute the rate of convergence of the operators $\mathcal{K}_m^{*\gamma,\eta}(,;u)$ in (4)to the function h in terms of modulus of continuity, elements of Lipschitz class and Peetre's K-functional.
Let $\delta > 0$, $C_B^2[0,\infty) = \{f \in C_B[0,\infty);\ f',\ f'' \in C_B[0,\infty)\}$.
We compute the approximate order of operator $\mathcal{K}_m^{*\gamma,\eta}(.;u)$ in terms of the elements of the usual Lipschitz class. Let $h \in C[0,\infty)$ and $0 < \sigma \leq 1$. We recall that f belongs to $Lip_{\mathcal{M}}(\sigma)$, when

$$|h(u) - h(v)| \leq \mathcal{M}|u - v|^{\sigma};\quad for\ all\ u, v \in [0,\infty). \tag{5}$$

We prove

Theorem 4.1. *For all $h \in Lip_{\mathcal{M}}(\sigma)$, we have*

$$||\mathcal{K}_m^{*\gamma,\eta}(h;u) - h(u)||_{C[0,\infty)} \leq \mathcal{M}\delta_m^{\sigma}$$

where

$$\delta_m = \left(\left(\frac{m}{(m+\eta)} - 1\right)^2 + \frac{2m - (2\gamma+1)(\eta+1)}{(m+\eta)^2} + \frac{3\gamma^2 + 3\gamma + 1}{3(m+\eta)^2}\right)^{\frac{1}{2}}$$

and $\mathcal{M}$ is a positive constant.

Proof. Let $h \in Lip_{\mathcal{M}}(\sigma)$ and $0 < \sigma \leq 1$. Using the properties of linearity, monotonicity of $\mathcal{K}_m^{*\gamma,\eta}(.;u)$ and (5), we get

$$\begin{aligned} |\mathcal{K}_m^{*\gamma,\eta}(h;u) - h(u)| &\leq \mathcal{K}_m^{*\gamma,\eta}(|h(t) - h(u)|;u) \\ &\leq \mathcal{M}\mathcal{K}_m^{*\gamma,\eta}(|t-u|^{\sigma};u) \end{aligned}$$

Applying the Hölder's inequality with $n = \frac{2}{\sigma}$ and $m = \frac{2}{2-\sigma}$, we obtain

$$|\mathcal{K}_m^{*\gamma,\eta}(h;u) - h(u)| \leq \mathcal{M}(\mathcal{K}_m^{*\gamma,\eta}(|t-u|^2;u))^{\frac{\sigma}{2}}$$

For $u \in [0,\infty)$, if we take the maximum of both sides of above equation, then we have

$$||\mathcal{K}_m^{*\gamma,\eta}(h;u) - h(u)|| \leq \mathcal{M}(\max_u \mathcal{K}_m^{*\gamma,\eta}(|t-u|^2;u))^{\frac{\sigma}{2}}$$

if

$$\delta_m = \left(\left(\frac{m}{(m+\eta)} - 1\right)^2 + \frac{2m - (2\gamma+1)(\eta+1)}{(m+\eta)^2} + \frac{3\gamma^2 + 3\gamma + 1}{3(m+\eta)^2}\right)^{\frac{1}{2}},$$

Taking into consideration Lemma 2.2 (ii), the proof is completed. □

Next, we will study the rate of convergence of the positive linear operators $\mathcal{K}_m^{*\gamma,\eta}(.;u)$ by means of the following Peetre's K-functional for $h \in C_B[0,\infty]$ defined as

$$K_2(h,\delta) = inf\{||h - f|| + \delta||f''|| : f \in C_B^2[0,\infty)\}. \tag{6}$$

Then there exists a constant $\mathcal{D} > 0$ such that

$$K_2(h,\delta) \leq \mathcal{D}\omega_2(h, \sqrt{\delta}), \tag{7}$$

and the modulus of continuity of second order is defined as

$$\omega_2(h;\sqrt{\delta}) = \sup_{0\leq k\leq\sqrt{\delta}}\ \sup_{u\in[0,\infty)} \{|h(u+2k) - 2h(u+k) + h(u)|\}. \tag{8}$$

For $h \in C_B^2[0,\infty)$, the usual modulus of continuity is defined as

$$\omega(h;\delta) = \sup_{0\le k\le\sqrt{\delta}} \sup_{u\in[0,\infty)} \{|h(u+k)-h(u)|\}.$$

The following properties are also met by $\omega(h;\delta)$
(1) $\lim_{\delta\to 0}\omega(h;\delta) = 0,$
(2) $|h(u)-h(v)| \le \omega(h;\delta)\Big(1+\frac{(u-v)^2}{\delta^2}\Big)$
We have

Theorem 4.2. *Let $h \in C[0,\infty)$ and $v \in [0,\infty)$, then the following holds*

$$|\mathcal{K}_m^{*\gamma,\eta}(h;u) - h(v)| \le 2\omega(h;\sqrt{\delta_m^{*\gamma,\eta}(v)}).$$

Proof. Using the linearity and positivity of the operators as well as applying the property (2) of $\omega(h;\delta)$, we have

$$\begin{aligned}|\mathcal{K}_m^{*\gamma,\eta}(h;u) - h(v)| &\le \mathcal{K}_m^{*\gamma,\eta}(|h(u)-h(v)|;v)\\ &\le \omega(h;\delta)\Big(1+\frac{1}{\delta^2}\mathcal{K}_m^{*\gamma,\eta}((u-v)^2;v)\Big).\end{aligned}$$

The intended result is easily followed by selecting $\delta^2 = \delta_m^{*\gamma,\eta} = \mathcal{K}_m^{*\gamma,\eta}((u-v)^2;v)$. □

Theorem 4.3. *Let $h \in C[0,\infty)$. Then, there exists a constant $\mathcal{D} > 0$ such that*

$$|\mathcal{K}_m^{*\gamma,\eta}(h;u) - h(u)| \le \mathcal{D}\omega_2(h,\sqrt{\xi_m^{*(\gamma,\eta)}(u)}) + \omega\Big(h;\frac{\gamma-\eta u}{m+\eta}+\frac{1}{2(m+\eta)}\Big)$$

where

$$\xi_m^{*(\gamma,\eta)}(u) = \mathcal{K}_m^{*\gamma,\eta}((t-u)^2;u) + \Big(\frac{mu}{m+\eta}+\frac{2\gamma+1}{2(m+\eta)}\Big)^2$$

Proof. For $u \in [0,\infty)$, we consider the auxiliary operators $\overline{\mathcal{K}}_m^{*\gamma,\eta}$ defined by

$$\overline{\mathcal{K}}_m^{*\gamma,\eta}(h;u) = \mathcal{K}_m^{*\gamma,\eta}(h;u) + h(u) - h(\mathcal{K}_m^{*\gamma,\eta}(t;u)) \tag{9}$$

By the linearity and construction of $\overline{\mathcal{K}}_m^{*\gamma,\eta}$, we have $\overline{\mathcal{K}}_m^{*\gamma,\eta}(1;u) = 1$ and $\overline{\mathcal{K}}_m^{*\gamma,\eta}(t;u) = u$.
Let $f \in C_B^2[0,\infty)$. Using Taylor's expansion of f at $s = u$, we have

$$f(s) = f(u) + (s-u)f''(u) + \int_u^s \frac{(s-u)^2}{2} f''(u)du$$

On applying $\overline{\mathcal{K}}_m^{*\gamma,\eta}(,;u)$ on both the sides, immediately follows

$$\begin{aligned}|\overline{\mathcal{K}}_m^{*\gamma,\eta}(f;u) - f(u)| &\le \mathcal{K}_m^{*\gamma,\eta}\Big(\Big|\int_u^s (s-v)f''(v)dv\Big|;u\Big)\\ &\quad + \Big|\int_u^{\frac{mu}{m+\eta}+\frac{2\gamma+1}{2(m+\eta)}} \Big|\Big(\frac{mu}{m+\eta}+\frac{2\gamma+1}{2(m+\eta)} - v\Big)\Big||f''(v)|dv\Big|\\ &\le \mathcal{K}_m^{*\gamma,\eta}((s-u)^2;u)\|f''\| + \Big(\frac{mu}{m+\eta}+\frac{2\gamma+1}{2(m+\eta)}\Big)^2\|f''\|\\ &= \xi_m^{*(\gamma,\eta)}(u)\|f''\|\end{aligned} \tag{10}$$

From (6), we have

$$|\overline{\mathcal{K}}_m^{*\gamma,\eta}(h;u)| \leq 3\|h\| \tag{11}$$

For $g \in C_B^2[0,\infty)$ and using (8) and (7) in (6) we obtain

$$\begin{aligned}|\mathcal{K}_m^{*\gamma,\eta}(h;u)-h(u)| &\leq |\overline{\mathcal{K}}_m^{*\gamma,\eta}(h-f;u)|+|\overline{\mathcal{K}}_m^{*\gamma,\eta}(f;u)-f(u)| \\ &\quad +|f(u)-h(u)|+\left|h\Big(\frac{mu}{m+\eta}+\frac{2\gamma+1}{2(m+\eta)}\Big)-h(u)\right| \\ &\leq 4\|h-f\|+|\overline{\mathcal{K}}_m^{*\gamma,\eta}(f;u)-f(u)|+\left|h\Big(\frac{mu}{m+\eta}+\frac{2\gamma+1}{2(m+\eta)}\Big)-h(u)\right| \\ &\leq 4\|h-f\|+\xi_m^{*(\gamma,\eta)}\|f''\|+\omega\Big(h;\frac{\gamma-\eta u}{m+\eta}+\frac{1}{2(m+\eta)}\Big)\end{aligned}$$

On taking the infimum of the right side over all $f \in C_B^2[0,\infty)$, we get

$$|\mathcal{K}_m^{*\gamma,\eta}(h;u)-h(u)| \leq 4K_2(h;\xi_m^{*(\gamma,\eta)}(u))+\omega\Big(h;\frac{\gamma-\eta u}{m+\eta}+\frac{1}{2(m+\eta)}\Big)$$

Thus by using (7), we get

$$|\mathcal{K}_m^{*\gamma,\eta}(h;u)-h(u)| \leq \mathcal{D}(h;\sqrt{\xi_m^{*(\gamma,\eta)}(u)})+\omega\Big(h;\frac{\gamma-\eta u}{m+\eta}+\frac{1}{2(m+\eta)}\Big)$$

which completes the proof. □

Lemma 4.4. *For the operators in (4), we further have*

(a) $\lim_{m\to\infty} m(\mathcal{K}_m^{*\gamma,\eta}((t-u);u)) = \gamma-\eta-\frac{1}{2}$

(b) $\lim_{m\to\infty} m(\mathcal{K}_m^{*\gamma,\eta}((t-u)^2;u)) = 1.$

Theorem 4.5. *Let $\mathcal{K}_m^{*\gamma,\eta}(.;u)$ be operators in (4) and if h'' exists at a point $u \in [0,\infty)$, for $h \in C[0,\infty)$. Then the following holds*

$$\lim_{m\to\infty} m(\mathcal{K}_m^{*\gamma,\eta}(h;u)-h(u)) = (\gamma-\eta+\frac{3}{2})h'(u)+h''(u).$$

Proof. We write Taylor's expansion

$$h(t) = h(u)+(t-u)h'(u)+\frac{1}{2}(t-u)^2\,h''(u)+\varphi(t,u)(t-u)^2, \tag{12}$$

where $\varphi(t,u)$ is the remainder and $\varphi(t,u)\to 0$ as $t\to u$. Applying the operators $\mathcal{K}_m^{*\gamma,\eta}(.;u)$ on both the sides of (12), we obtain

$$\begin{aligned}\mathcal{K}_m^{*\gamma,\eta}(h;u)-h(u) &= \mathcal{K}_m^{*\gamma,\eta}((t-u);u)h'(u) \\ &+ \frac{1}{2}\,\mathcal{K}_m^{*\gamma,\eta}((t-u)^2;u)h''(u)+\mathcal{K}_m^{*\gamma,\eta}(\varphi(t,u)(t-u)^2;u).\end{aligned}$$

By Cauchy-Schwarz inequality and Lemma 2.2 (ii), we get

$$\mathcal{K}_m^{*\gamma,\eta}\Big(\varphi(t,u)(t-u)^2;u\Big) \leq \Big([\mathcal{K}_m^{*\gamma,\eta}(\varphi^2(t,u);u)]\Big)^{\frac{1}{2}}\Big(\mathcal{K}_m^{*\gamma,\eta}((t-u)^4;u)\Big)^{\frac{1}{2}}.$$

Since $\mathcal{K}_m^{*\gamma,\eta}(h;u) \to h(u)$, we have

$$\begin{aligned}\lim_{m\to\infty} \mathcal{K}_m^{*\gamma,\eta}(\varphi^2(t,u);u) &= \varphi^2(u,u) = 0,\\ \lim_{m\to\infty} m\mathcal{K}_m^{*\gamma,\eta}(\varphi^2(t,u);u) &= 0.\end{aligned}$$

Combining the above equations using the Lemma 4.4, it is easily seen that

$$\begin{aligned}\lim_{m\to\infty} m(\mathcal{K}_m^{*\gamma,\eta}(h;u) - h(u)) &= \lim_{m\to\infty} m \lim_{m\to\infty} m(\mathcal{K}_m^{*\gamma,\eta}((t-u);u))\\ &+ \lim_{m\to\infty} m(\mathcal{K}_m^{*\gamma,\eta}((t-u)^2;u)) + \lim_{m\to\infty} m(\mathcal{K}_m^{*\gamma,\eta}(\varphi(t,u)(t-u)^2))\\ &= (\gamma - \eta + \frac{3}{2})h'(u) + h''(u)\end{aligned}$$

which proves the theorem. □

5. q-statistical convergence

In this section, we prove q-statistical convergence of operators $\mathcal{K}_m^{*\gamma,\eta}(.;u)$. Recently, q-statistical convergence was introduced in [4] and used in [3], [7], [6], [17] and [18]. Let us first review some definitions from q-calculus. For any non-negative integer s, the q-integer of the number s is defined as

$$[s]_q = \begin{cases} \frac{1-q^s}{1-q} & \text{if } q \neq 1\\ s & \text{if } q = 1,\end{cases}$$

where q is a positive real number.
The q-factorial is defined as:

$$[s]_q! = \begin{cases} [1]_q[2]_q\cdots[s]_q & \text{if } s = 1,2,\ldots\\ 1 & \text{if } s = 0.\end{cases}$$

and

$$(1+u)_q^s := \begin{cases} \prod_{k=0}^{s-1}\left(1+q^k u\right) & s = 1,2,\ldots\\ 1 & s = 0.\end{cases}$$

For integers $0 \leq l \leq s$, the q-binomial coefficient is defined by

$$\begin{bmatrix} s\\ l\end{bmatrix}_q = \frac{[s]!}{[l]![s-l]!}.$$

The q-analogue of integration, introduced by Jackson [7] in the interval $[0,a]$, is defined by

$$\int_0^a f(x)d_qx := a(1-q)\sum_{n=0}^{\infty} f(aq^n)\,q^n, 0 < q < 1 \text{ and } a > 0.$$

The q-analog of *Cesàro* matrix $\mathfrak{C}_1$ is defined as

$$\mathfrak{C}_1(q) = (c_{nk}^1(q^k))_{n,k=0}^{\infty},$$

where

$$c_{nk}^1(q^k) = \left\{\begin{array}{cc} \frac{q^k}{[n+1]_q}, & \text{if } k \leq n\\ 0, & \text{otherwise}\end{array}\right\}$$

which is regular for $q \geq 1$.
Let $\mathfrak{F}$ be a subset of $\mathbb{N}$. Then, the asymptotic density of $\mathfrak{F}$ is defined as

$$\eta(\mathfrak{F}) = \lim_{n} \frac{1}{n} \#\{k \leq n : k \in \mathfrak{F}\},$$

where # represents the cardinality of the set.
The q-density is defined as

$$\begin{aligned} \eta_q(\mathfrak{F}) = \eta_{\mathfrak{C}_1^q}(\mathfrak{F}) &= \lim_{n\to\infty} \inf (\mathfrak{C}^q_{1}\chi_{\mathfrak{F}})_n \\ &= \lim_{n\to\infty} \inf \sum_{j\in\mathfrak{F}} \frac{q^{j-1}}{[n]}\ , q \geq 1 \end{aligned}$$

A sequence $\xi = (\xi_k)$ is q-statistically convergent to $\mathfrak{r}$, written as $st_q - \lim \xi_k = \mathfrak{r}$, if for every $\epsilon > 0$, $\eta_q(\mathfrak{F}_\epsilon) = 0$, where $\mathfrak{F}_\epsilon = \{k \leq n : |\xi_k - \mathfrak{r}| > \epsilon\}$. That is, for each $\epsilon > 0$,

$$\lim_{n} \frac{1}{[n]} \#\left\{k \leq n : q^{k-1}|\xi_k - \mathfrak{r}| \geq \epsilon\right\} = 0.$$

If $\eta(\mathfrak{F}) = 0$ for an infinite set $\mathfrak{F}$, then $\eta_q(\mathfrak{F}) = 0$. Therefore, statistical convergent sequence is always q-statistical convergence but not conversely (see [2], Example 15). We have

Theorem 5.1. *For all $h \in C^0_\rho[0,\infty]$, we have*

$$st_q - \lim_{m} \|\mathcal{K}_m^{*\gamma,\eta}(h;u) - h(u)\|_\rho = 0,\ u \in [0,\infty).$$

Proof. In order to prove the theorem, it is suffices to show that

$$st_q - \lim_{m} \|\mathcal{K}_m^{*\gamma,\eta}(e_i;u) - e_i\|_\rho = 0, \quad for\ i = 0,1,2$$

Since $\mathcal{K}_m^{*\gamma,\eta}(e_0;u) = 1$ then $st_q - \lim_m \|\mathcal{K}_m^{*\gamma,\eta}(e_0;u) - e_0\|_\rho = 0$.
By Lemma 2.1, we have

$$\begin{aligned} \|\mathcal{K}_m^{*\gamma,\eta}(e_1;u) - u\|_\rho &= \sup_{u\in[0,\infty)} \frac{1}{1+u^2}\left|\left(\frac{m}{m+\eta} - 1\right)u + \frac{2\gamma+1}{2(m+\eta)u}\right| \\ &\leq \left|\frac{m}{m+\eta} - 1\right| \sup_{u\in[0,\infty)} \frac{u}{1+u^2} + \frac{2\gamma+1}{2(m+\eta)} \sup_{u\in[0,\infty)} \frac{1}{1+u^2} \\ &\leq \frac{1}{2}\left|\frac{m}{m+\eta} - 1\right| + \left|\frac{2\gamma+1}{2(m+\eta)}\right|. \end{aligned}$$

Now, for a given $\epsilon > 0$, the following sets are considered:

$$\begin{aligned} \mathcal{H}_1 : &= \left\{m : \|\mathcal{K}_m^{*\gamma,\eta}(e_1;u) - u\| \geq \epsilon\right\}, \\ \mathcal{H}_2 : &= \left\{m : \frac{1}{2}\left|\frac{m}{m+\eta} - 1\right| \geq \frac{\epsilon}{2}\right\}, \\ \mathcal{H}_3 : &= \left\{m : \left|\frac{2\gamma+1}{2(m+\eta)}\right| \geq \frac{\epsilon}{2}\right\}. \end{aligned}$$

This implies that $\mathcal{H}_1 \subseteq \mathcal{H}_2 \cup \mathcal{H}_3$, which shows that $\delta_q(\mathcal{H}_1) \leq \delta_q(\mathcal{H}_2) + \delta_q(\mathcal{H}_3)$.
Hence by taking the limit $m \to \infty$, we will obtain

$$st_q - \lim_{m} \|\mathcal{K}_m^{*\gamma,\eta}(e_1;u) - u\|_\rho = 0.$$

By Lemma 2.1, we have

$$\begin{aligned} \|\mathcal{K}_m^{*\gamma,\eta}(e_2;u)-u^2\|_\rho &= \sup_{u\in[0,\infty)}\frac{1}{1+u^2}\left|\left(\frac{m^2}{(m+\eta)^2}-1\right)u^2+\frac{m(2\gamma+3)}{(m+\eta)^2u}+\frac{3\gamma^2+3\gamma+1}{3(m+\eta)^2}\right| \\ &\leq \left|\frac{m^2}{(m+\eta)^2}-1\right|\sup_{u\in[0,\infty)}\frac{u^2}{1+u^2} \\ &+ \frac{m(2\gamma+3)}{(m+\eta)^2}\sup_{u\in[0,\infty)}\frac{u}{1+u^2}+\frac{3\gamma^2+3\gamma+1}{3(m+\eta)^2}\sup_{u\in[0,\infty)}\frac{1}{1+u^2}. \end{aligned}$$

For a given $\epsilon > 0$, the following sets are considered.

$$\begin{aligned} \mathcal{T}_1: &= \left\{m : \|\mathcal{K}_m^{*\gamma,\eta}(e_2;u)-u^2\| \geq \epsilon\right\}, \\ \mathcal{T}_2: &= \left\{m : \left|\frac{m^2}{(m+\eta)^2}-1\right| \geq \frac{\epsilon}{3}\right\}, \\ \mathcal{T}_3: &= \left\{m : \frac{m(2\gamma+3)}{(m+\eta)^2} \geq \frac{\epsilon}{3}\right\}, \\ \mathcal{T}_4: &= \left\{m : \frac{3\gamma^2+3\gamma+1}{3(m+\eta)^2} \geq \frac{\epsilon}{3}\right\}. \end{aligned}$$

This implies that $\mathcal{T}_1 \subseteq \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{T}_4$, which shows that $\delta_q(\mathcal{T}_1) \leq \delta_q(\mathcal{T}_2) + \delta_q(\mathcal{T}_3) + \delta_q(\mathcal{T}_4)$.
Letting $m \to \infty$, we have

$$st_q - \lim_m \|\mathcal{K}_m^{*\gamma,\eta}(e_2;u)-u^2\|_\rho = 0.$$

This completes the proof of the theorem. □

6. Graphical analysis and error estimation

In this section we demonstrate approximation by our operators in (4) through a number of numerical examples with graphical visuals and compare error estimates with the help of MATLAB to substantiate theoretical results using different values of parameters. These operators have the benefit of allowing approximation process in spaces of integrable functions on $[0,\infty)$. Additionally it offers approximating operators greater flexibility.

Example 6.1. *Let* $h(u) = u^4 - 9u^3 + 23u^2 - 15u$, $\gamma = 2$, $\eta = 3$ *and* $m \in \{15, 30, 50, 80\}$. *We demonstrate absolute error,* $E_m^{*\gamma,\eta}(h;u) = |\mathcal{K}_m^{*\gamma,\eta}(h;u) - h(u)|$ *of the operators* $\mathcal{K}_m^{*\gamma,\eta}(h;u)$ *with the function* $h(u)$. *Figures 8.1 and 8.2 depict the operators* $\mathcal{K}_m^{*\gamma,\eta}(h;u)$ *approaching towards the function* $h(u)$ *and the errors respectively. Absolute error at various points is given in Table 1.*

	u									
m	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
15	0.5447	0.0451	1.2183	1.5425	0.3614	2.2043	5.2575	7.1243	5.3543	3.2795
30	0.3348	0.0133	0.7099	0.8893	0.1615	0.3882	3.1988	4.2342	2.9823	2.5439
50	0.2200	0.0051	0.4561	0.5684	0.0898	0.9205	2.0912	2.7391	1.8692	1.8253
80	0.1451	0.0021	0.2968	0.3688	0.0530	0.6100	1.3744	1.7891	1.1972	1.2630

Table 1 Absolute Error of approximation by $\mathcal{K}_m^{*\gamma,\eta}(h;u)$ for $h(u) = u^4 - 9u^3 + 23u^2 - 15u$ for $\gamma = 2$, $\eta = 3$.

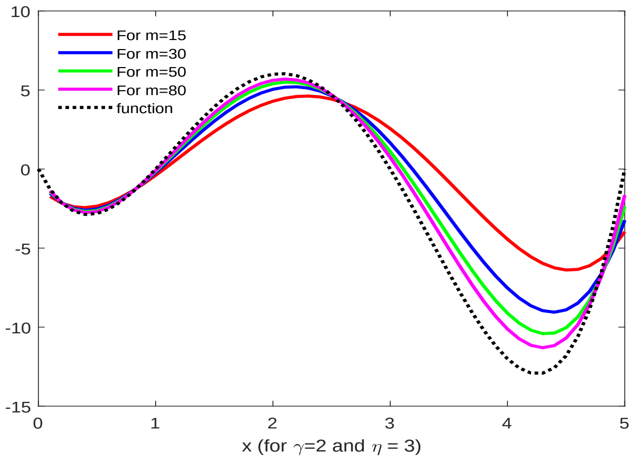

Fig 8.1 Convergence of $\mathcal{K}_m^{*\gamma,\eta}(h;u)$ to $h(u) = u^4 - 9u^3 + 23u^2 - 15u$

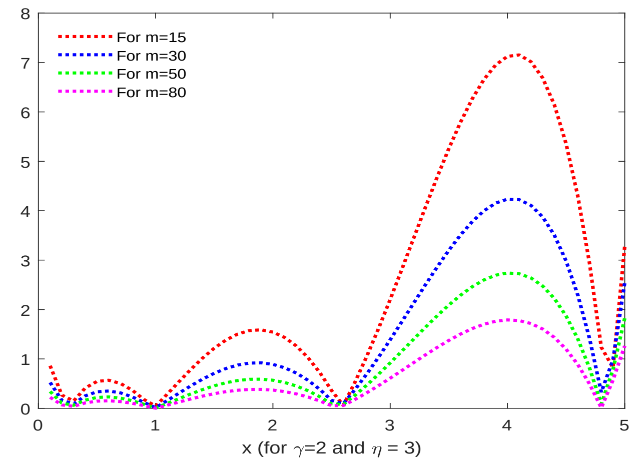

Fig 8.2 Absolute Error of approximation of $\mathcal{K}_m^{*\gamma,\eta}(h;u)$ with $h(u) = u^4 - 9u^3 + 23u^2 - 15u$

Example 6.2. *In this example, we consider the function* $h(u) = u^3 - 6u^2 + 8u$ *and take the values of parameters as* $\gamma = 1,\ \eta = 2$ *and keep the values of* m *as in previous example. Then the approximation and absolute error are graphed in Figures 8.3 and 8.2 respectively. Table 2 embodies absolute errors at various points for this function.*

	u									
m	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
15	0.0550	0.2454	0.1093	0.6931	1.1902	1.2844	0.6599	0.9993	4.0092	8.6857
30	0.0401	0.1486	0.0596	0.3980	0.6800	0.7191	0.3289	0.6772	2.4858	5.2832
50	0.0279	0.0967	0.0373	0.2538	0.4325	0.4531	0.1954	0.4609	1.6362	3.4507
80	0.0190	0.0634	0.0239	0.1644	0.2797	0.2914	0.1211	0.3096	1.0792	2.2662

Table 2 Absolute Error of approximation of $\mathcal{K}_m^{*\gamma,\eta}(h;u)$ with $h(u) = u^3 - 6u^2 + 8u$ for $\gamma = 1,\ \eta = 2$.

Example 6.3. *Figure 8.5 displays the approximation behaviour of operators (4) for* $h(u) = 2u^5 - 19u^4 + 54u^3 - 45u^2,\ \gamma = 1,\ \eta = 3$ *for the same values of* m *as considered in earlier examples. Error graphics is shown in Figure 8.6.*

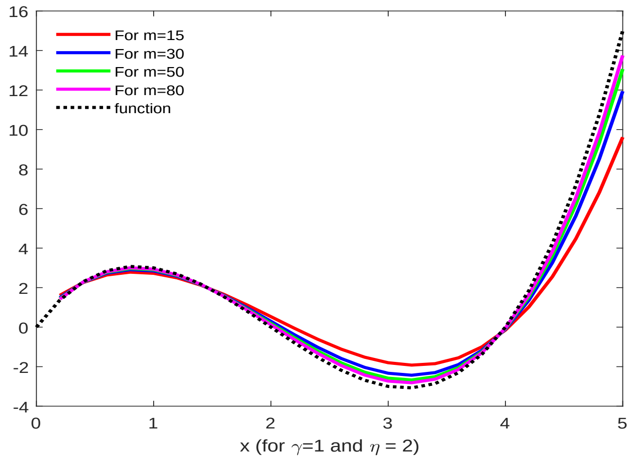

Fig 8.3 Convergence of $\mathcal{K}_m^{*\gamma,\eta}(h;u)$ to $h(u) = u^3 - 6u^2 + 8u$

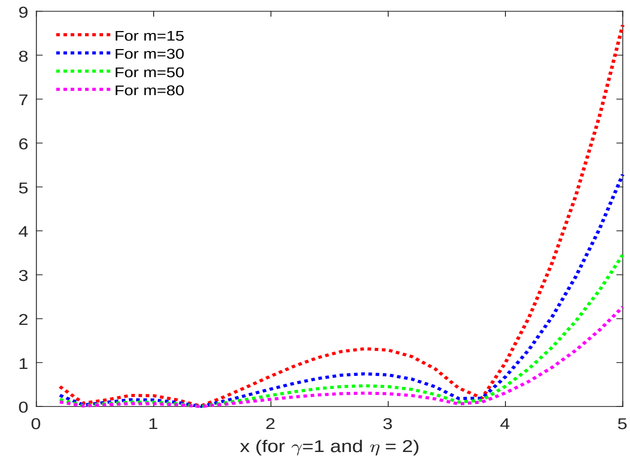

Fig 8.4 Absolute Error of approximation of $\mathcal{K}_m^{*\gamma,\eta}(h;u)$ with $h(u) = u^3 - 6u^2 + 8u$

The absolute error at various points is given in Table 3.

	u									
m	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
15	0.5189	1.5643	1.9965	7.0113	6.8195	4.2619	26.4522	50.0216	50.8049	14.2844
30	0.2937	0.9643	1.1905	4.2266	3.9877	3.0520	16.9444	31.4244	31.0672	11.5553
50	0.1853	0.6353	0.7725	2.7569	2.5655	2.1304	11.3398	20.8567	20.3646	8.4593
80	0.1192	0.4197	0.5057	1.8105	1.6711	1.4530	7.5580	13.8347	13.4079	5.9247

Table 3 Absolute Error of approximation of $\mathcal{K}_m^{*\gamma,\eta}(h;u)$ with $h(u) = 2u^5 - 19u^4 + 54u^3 - 45u^2$ for $\gamma = 1,\ \eta = 2$.

From Examples, 6.1, 6.2 and 6.3, it is easily observed that the approximation by $\mathcal{K}_m^{*\gamma,\eta}(h;u)$ betters on realizing bigger values of m.
In the ensuing examples, we compare approximation by our operators (4) with the approximation by Kantorovich-Lupaş operators defined in (2) and that of the Stancu type Lupaş operators given in (3).

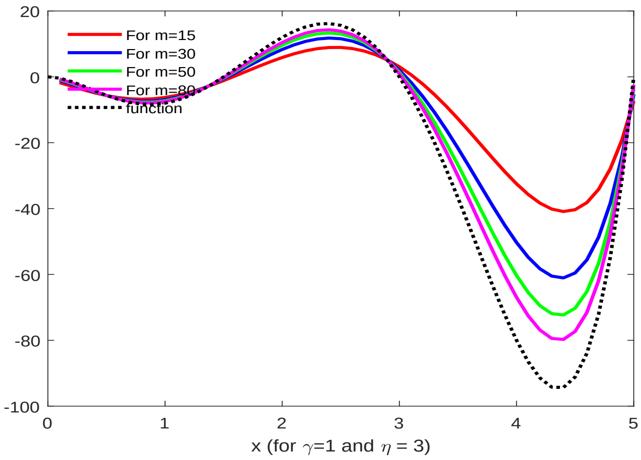

Fig 8.5 Convergence of $\mathcal{K}_m^{*\gamma,\eta}(h;u)$ to $h(u) = 2u^5 - 19u^4 + 54u^3 - 45u^2$

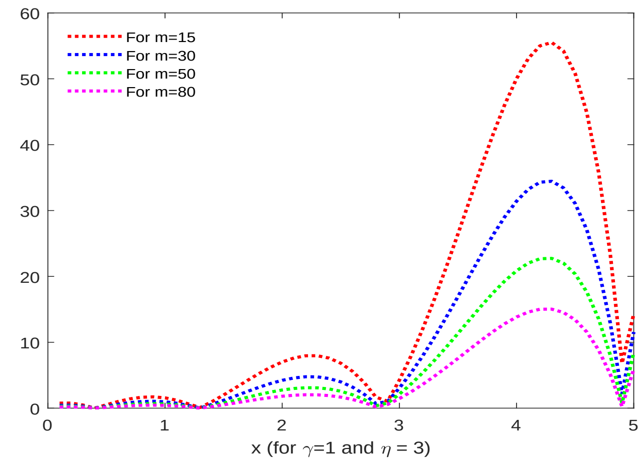

Fig 8.6 Absolute Error of approximation of $\mathcal{K}_m^{*\gamma,\eta}(h;u)$ with $h(u) = 2u^5 - 19u^4 + 54u^3 - 45u^2$

Example 6.4. *We consider* $h(u) = u^4 - 9u^3 + 23u^2 - 15u$, *and* $m = 30$ *with* $\gamma = 1, \eta = 2$. *Figure 8.7 demonstrates a comparison of convergence of our operators* $\mathcal{K}_m^{*\gamma,\eta}(h;u)$*(magenta) (4) with the operators* $\mathcal{L}_m(h;u)$ *in (1) and operators* $\mathcal{K}_m(h;u)$ *in (2) to the function* $h(u)$. *Absolute errors are computed in Table 4 and depicted graphically in Figure 8.8. From these figures, it is clear that operators in (2) give better approximation than operators* $\mathcal{L}_m(,;u)$ *and* $\mathcal{K}_m(,;u)$.

	u									
m=30	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
$\mathcal{L}_m$	0.3471	0.1143	0.3986	0.8914	1.0643	0.6171	0.7500	3.3372	7.4443	13.3715
$\mathcal{K}_m$	0.3690	0.2323	0.3017	0.8828	1.1612	0.7868	0.5904	3.3204	7.7531	14.2386
$\mathcal{K}_m^{*\gamma,\eta}$	0.3275	0.0100	0.6404	0.8811	0.3908	0.8305	2.4415	3.7596	3.7610	1.0805

Table 4 Comparison of Absolute Error of approximation by operators $\mathcal{K}_m^{*\gamma,\eta}(h;u)$ with $\mathcal{L}_m(,;u)$ and $\mathcal{K}_m(,;u)$ for $h(u) = u^4 - 9u^3 + 23u^2 - 15u$

Also, Table 5 presents error of convergence for our operators for the function of Example 6.4 for various pairs of parameters, for equally spaced points between 0.5 – 5.0. Converging behaviour and absolute errors are displayed graphically in Figures 8.9 and 8.10 respectively.

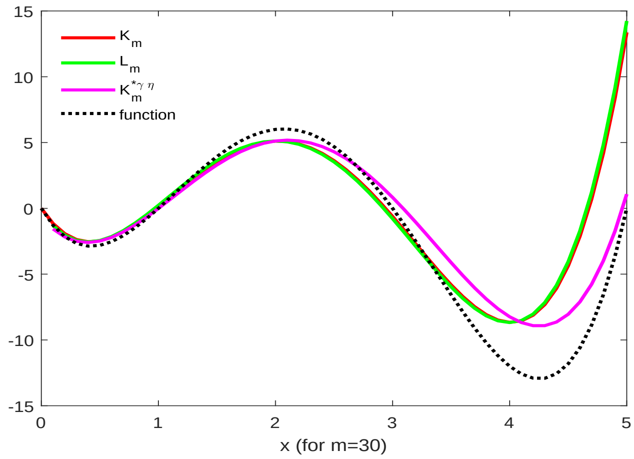

Fig 8.7 Comparison of convergence of approximation by operators $\mathcal{K}_m^{*\gamma,\eta}(h;u)$ with $\mathcal{L}_m(,;u)$ and $\mathcal{K}_m(,;u)$

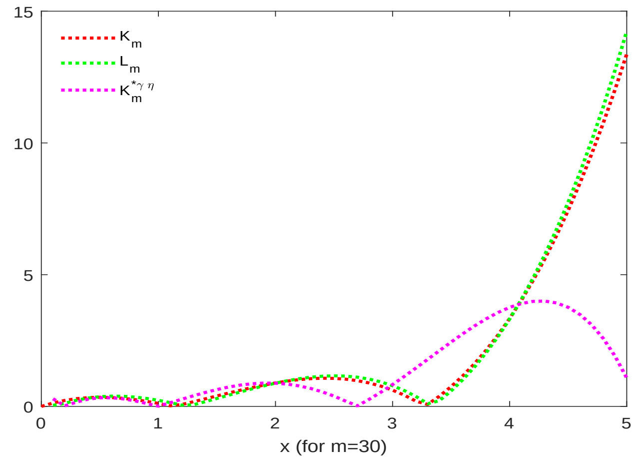

Fig 8.8 Comparison of Absolute Error of approximation by operators $\mathcal{K}_m^{*\gamma,\eta}(h;u)$ with $\mathcal{L}_m(,;u)$ and $\mathcal{K}_m(,;u)$

u	$\gamma = 0, \eta = 0$	$\gamma = 1, \eta = 1$	$\gamma = 1, \eta = 2$	$\gamma = 2, \eta = 2$
0.5	0.3690	0.3736	0.3275	0.3348
1.0	0.2323	0.2223	0.0100	0.0133
1.5	0.3017	0.3745	0.6404	0.7099
2.0	0.8828	0.8507	0.8811	0.8893
2.5	1.1612	0.8246	0.3908	0.1615
3.0	0.7868	0.0988	0.8305	1.3882
3.5	0.5904	1.3396	2.4415	3.1988
4.0	3.3204	3.3192	3.7596	4.2342
4.5	7.7531	5.4842	3.7610	2.9823
5.0	14.2386	7.2944	1.0805	2.5439

Table 5 Absolute Error of approximation of $\mathcal{K}_m^{*\gamma,\eta}(h;u)$ with $h(u) = u^4 - 9u^3 + 23u^2 - 15u$ for different values of γ ana η.

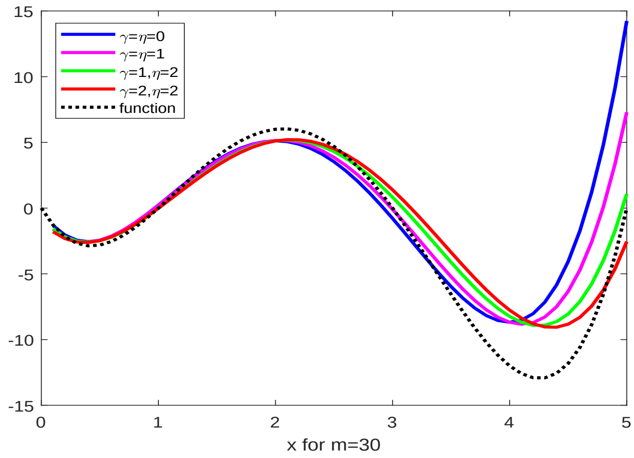

Fig 8.9 Convergence of $\mathcal{K}_m^{*\gamma,\eta}(h;u)$ to $h(u) = u^4 - 9u^3 + 23u^2 - 15u$

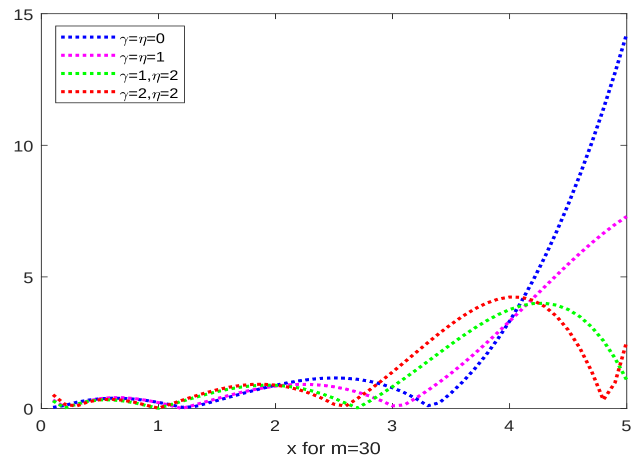

Fig 8.10 Absolute Error of approximation of $\mathcal{K}_m^{*\gamma,\eta}(h;u)$ with $h(u) = u^4 - 9u^3 + 23u^2 - 15u$

7. Conclusions

We constructed a generalization of Kantorovich variant of Stancu-Lupaş operators and investigated approximating properties. Basic estimates and convergence theorems have been established using Korovkin theorem. Approximation results are obtained in weighted space. It is shown that error estimation results of the generalized operators are significantly sharper.

8. Declarations

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Authors contributions

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