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Certain fractional inequalities via the Caputo Fabrizio operator

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Abstract. The Caputo Fabrizio fractional integral operator is one of the key concepts in fractional calculus. It is involved in many concrete and practical issues. In the present study, we have discussed some novel ideas to fractional Hermite-Hadamard inequalities within a Caputo Fabrizio fractional integral framework. The fractional integral under investigation is used to establish some new fractional Hermite-Hadamard inequalities. The findings of this study can be seen as a generalization and extension of numerous earlier inequalities via convex function. In addition, we demonstrate a few applications of our findings to special means of real numbers.

1. Introduction

Fractional calculus, which is interested with differential and integral operators of non integer orders, is nearly as old as classical calculus, which is concerned with integer orders. Because the classical calculus operators cannot model the entirety of real-world phenomena, scientists and authors investigated generalizations of these operators. As of present now, a lot of researchers are very interested in the theory of fractional calculus. Particularly for the fractional calculus, such as the definitions of Riemann-Liouville and Caputo, there is a wide range of studies and literature. The Riemann-Liouville derivative is a general concept that, according to some definitions, is the most uniform and natural. Due to the fact that the necessary initial conditions are themselves fractional, which is probably incorrect for physical situations, it has significant drawbacks when used to modeling physical problems. The Caputo derivative has the advantage of being acceptable for physical conditions because it requires only typical type [1] initial conditions. However, these are not the only ways to define and describe fractional calculus. Convex functions have a famous and scientific history, and for almost a century, they have been the focus of study. Due to the quick development of the theory and the wide-ranging applications of fractional calculus, inequalities with unique convex functions have been a significant research problem for many researchers. By utilizing convex functions, mathematicians have proposed numerous types of inequalities or equalities, including the Hermite-Hadamard type, the Ostrowski type, the Hermite-Hadamard-Mercer type, the Bullen type, the Opial type, and other types. Many researchers are interested in the Hermite-Hadamard inequality [2] out

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of all of these integral inequalities.

$$h\left(\frac{x_1+x_2}{2}\right) \leq \frac{1}{x_2-x_1}\int_{x_1}^{x_2} h(x)\,dx \leq \frac{h(x_1)+h(x_2)}{2}.$$

It has been the most well-known and helpful inequality in mathematical analysis since its discovery in 1883. Other authors have also worked on improving this condition for different classes of convex functions and mappings, as shown in these articles[3–14]. It is crucial to note that the idea of fractional calculus was first introduced in 1695 by Leibniz and L'Hôspital. To the science of fractional calculus and its numerous applications, other researchers, such as Riemann, Liouville, Grünwald, Letnikov, Erdéli, and Kober, have made significant contributions. Fractional calculus has attracted the interest of many physical and engineering professionals due to its behavior and ability to resolve numerous real-world problems. The importance of fractional operators in the development of fractional calculus is highly crucial. Numerous engineering and science fields, including as physics [15], epidemiology [16], medicine [17], nanotechnology [18], economics [19], bioengineering [20], and fluid mechanics[21], utilized fractional calculus. Several researches have shown that fractional operators can explain complex multiscale phenomena that are difficult to model using traditional mathematical calculus. It has been known in recent years that presenting well-known inequalities employing various novel ideas of fractional integral operators is extremely popular among mathematicians. For other fractional-order integral inequalities, see the articles discussed in [22–30] in this connection.

2. Preliminaries

We recall some known concepts related to our main results.

Definition 2.1. *See [31, 32]. Let I be a convex subset of a real vector space $\mathbb{R}$ and let $h : I \to \mathbb{R}$ be a function. Then, a function h is said to be convex, if*

$$h(\eta x_1 + (1-\eta)x_2) \leq \eta h(x_1) + (1-\eta)h(x_2)$$

holds for all $x_1, x_2 \in I$ and $\eta \in [0,1]$.

In [33, 34], Hudzik and Maligranda considered among other, class of functions which are s-convex in the second sense

Definition 2.2. *[33, 34]. A function $h : \mathbb{R}^+ \to \mathbb{R}$ where $\mathbb{R}^+ = [0,\infty)$ is said to be $s-$convex in the second sense if*

$$h(\eta x_1 + (1-\eta)x_2) \leq \eta^s h(x_1) + (1-\eta)^s h(x_2)$$

holds for all $x_1, x_2 \in [0,\infty)$, $\eta \in [0,1]$ and for some fixed $s \in (0,1]$ is denoted by k_s^2. It can be easily seen that for $s = 1$, s-convex function reduces to the ordinary convex function.

Hudzik and Maligranda also established a result that if $s \in (0,1)$, $h \in k_s^2$ implies $h([0,\infty)) \subseteq [0,\infty)$, proved that all functions from k_s^2, $s \in (0,1)$ are positive.

Example 2.3. *[33, 34]. Let $s \in (0,1)$ and $x_1, x_2, x_3 \in \mathbb{R}$. We define the function $h : [0,\infty) \to \mathbb{R}$ as*

$$h(\eta) = \begin{cases} x_1 & \eta = 0, \\ x_2\eta^s + x_3 & \eta > 0\,. \end{cases}$$

We can easily checked that

(1) *if $x_2 \geq 0$ and $0 \leq x_3 \leq x_1$, then $h \in k_s^2$,*
(2) *if $x_2 > 0$ and $x_3 < 0$, then $h \notin k_s^2$.*

Theorem 2.4. *[35]. Suppose that $h : [0,\infty) \to [0,\infty)$ is an $s-$convex function in the second kind, where $s \in (0,1)$, and let $x_1,x_2 \in [0,\infty)$, $x_1 < x_2$. If $h \in L[x_1,x_2]$, then the following inequalities hold:*

$$\begin{aligned} 2^{s-1}h\left(\frac{\varkappa_1 + \varkappa_2}{2}\right) &\leq \frac{B(\alpha)}{\alpha(x_2 - x_1)}\left[\left({}^{CF}_{x_1}I^{\alpha}\, h\right)(k) + \left({}^{CF}I^{\alpha}_{x_2}\, h\right)(k) - \frac{2(1-\alpha)}{\alpha(x_2 - x_1)}h(k)\right] \\ &\leq \frac{h(\varkappa_1) + h(\varkappa_2)}{s+1} \end{aligned}$$

Lemma 2.5. *[35]. Let I be a real interval such that $x_1, x_2 \in I^0$, the interior of I with $x_1 < x_2$. Let $h : I^0 \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^0, $x_1,x_2 \in I$with $x_1 < x_2$. If $h' \in L[x_1,x_2]$, then following equality holds:*

$$\frac{h(\varkappa_1) + h(\varkappa_2)}{2} - \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} h(u)\,du = \frac{x_2 - x_1}{2}\int_0^1 (1-2\eta)\,h'(\eta x_1 + (1-\eta)x_2)\,d\eta.$$

Lemma 2.6. *[35]. Let I be a real interval such that $x_1, x_2 \in I^0$, the interior of I with $x_1 < x_2$. Let $h : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^0, $x_1,x_2 \in I$with $x_1 < x_2$. If $h' \in L[x_1,x_2]$, and $0 \leq \alpha \leq 1$, then following equality holds:*

$$\begin{aligned} &\frac{x_2 - x_1}{2}\int_0^1 (1-2\eta)\,h'(\eta x_1 + (1-\eta)x_2)\,d\eta - \frac{2(1-\alpha)}{\alpha(x_2 - x_1)}h(k) \\ = &\frac{h(\varkappa_1) + h(\varkappa_2)}{2} - \frac{B(\alpha)}{\alpha(x_2 - x_1)}\left(\left({}^{CF}_{x_1}I^{\alpha}\, h\right)(k) + \left({}^{CF}I^{\alpha}_{x_2}\, h\right)(k)\right), \end{aligned}$$

where $k \in [\varkappa_1, \varkappa_2]$ and $B(\alpha) > 0$ is a normalization function.

Definition 2.7. *See [36]. Let $[x_1,x_2] \to \mathbb{R}$. Then, Riemann-Liouville fractional integrals $I^{\alpha}_{x_1^+}h(\eta)$ and $I^{\alpha}_{x_2^-}h(\eta)$ of order $\alpha > 0$ are defined by*

$$\begin{aligned} I^{\alpha}_{x_1^+}h(x) &= \frac{1}{\Gamma(\alpha)}\int_{x_1}^{x} (\eta - x)^{\alpha-1}\,h(\eta)\,d\eta,\ x > x_1, \\ I^{\alpha}_{x_2^-}h(x) &= \frac{1}{\Gamma(\alpha)}\int_{x}^{x_2} (x - \eta)^{\alpha-1}\,h(\eta)\,d\eta,\ x < x_2. \end{aligned}$$

Definition 2.8. *See [37–39]. Let $h \in H^1(x_1,x_2)$, $x_1 < x_2$, $\alpha \in [0,1]$, then the notion of left and right Caputo-Fabrizio fractional integrals are defined by*

$$\begin{aligned} \left({}^{CF}_{x_1}I^{\alpha}\, h\right)(x) &= \frac{1-\alpha}{B(\alpha)}h(x) + \frac{\alpha}{B(\alpha)}\int_{x_1}^{x} h(\eta)\,d\eta \\ \left({}^{CF}I^{\alpha}_{x_2}\, h\right)(x) &= \frac{1-\alpha}{B(\alpha)}h(x) + \frac{\alpha}{B(\alpha)}\int_{x}^{x_2} h(\eta)\,d\eta, \end{aligned}$$

where $B(\alpha) > 0$ is a normalization function that satisfies $B(0) = B(1) = 1$.

3. Caputo Fractional Integral type inequalities

This section explains how to use the Caputo-Fabrizio fractional integral operator to derive a new identity for differentiable convex functions. Then, taking this identity into consideration, numerous improvements are shown using some basic integral inequalities.

Lemma 3.1. *Let $h : I \to \mathbb{R}$ be three times differentiable function on I^0. If $h''' \in L[\varkappa_1,\varkappa_2]$, then we have the following equality for Caputo-Fabrizio fractional integral operator:*

$$\frac{(\varkappa_2 - \varkappa_1)^3}{96}\int_0^1 (1-\eta)^3\left[h'''\left(\frac{1-\eta}{2}\varkappa_1 + \frac{1+\eta}{2}\varkappa_2\right) - h'''\left(\frac{1+\eta}{2}\varkappa_1 + \frac{1-\eta}{2}\varkappa_2\right)\right]d\eta$$

$$
\begin{aligned}
&= \frac{B(\alpha)}{\alpha(\varkappa_2-\varkappa_1)}\left[\left\{\left({}^{CF}_{\varkappa_1}I^{\alpha}\,h\right)(k)+\left({}^{CF}I^{\alpha}_{\frac{\varkappa_1+\varkappa_2}{2}}\,h\right)(k)\right\}+\left\{\left({}^{CF}_{\frac{\varkappa_1+\varkappa_2}{2}}I^{\alpha}\,\,h\right)(k)+\left({}^{CF}I^{\alpha}_{\varkappa_2}\,h\right)(k)\right\}\right]\\
&\quad -h\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\frac{(\varkappa_2-\varkappa_1)^2}{24}h''\left(\frac{\varkappa_1+\varkappa_2}{2}\right),
\end{aligned} \tag{1}
$$

$k \in [\varkappa_1, \varkappa_2]$ and $B(\alpha) > 0$ is a normalization function.

Proof. Let

$$
\begin{aligned}
I &= \frac{(\varkappa_2-\varkappa_1)^3}{96}\int_0^1(1-\eta)^3\left[h'''\left(\frac{1-\eta}{2}\varkappa_1+\frac{1+\eta}{2}\varkappa_2\right)-h'''\left(\frac{1+\eta}{2}\varkappa_1+\frac{1-\eta}{2}\varkappa_2\right)\right]d\eta\\
&= \int_0^1(1-\eta)^3h'''\left(\frac{1-\eta}{2}\varkappa_1+\frac{1+\eta}{2}\varkappa_2\right)d\eta-\int_0^1(1-\eta)^3h'''\left(\frac{1+\eta}{2}\varkappa_1+\frac{1+\eta}{2}\varkappa_2\right)d\eta\\
&= I_1-I_2.
\end{aligned}
$$

By using integration by parts, we get

$$
\begin{aligned}
I_1 &= \int_0^1(1-\eta)^3h'''\left(\frac{1-\eta}{2}\varkappa_1+\frac{1+\eta}{2}\varkappa_2\right)d\eta\\
&= \left.\frac{2(1-\eta)^3h''\left(\frac{1-\eta}{2}\varkappa_1+\frac{1+\eta}{2}\varkappa_2\right)}{-\varkappa_1+x_2}\right|_0^1-\int_0^1\frac{2(1-\eta)^3h''\left(\frac{1-\eta}{2}\varkappa_1+\frac{1+\eta}{2}\varkappa_2\right)}{-\varkappa_{1+}x_2}d\eta\\
&= \frac{-2}{\varkappa_2-\varkappa_1}h''\left(\frac{\varkappa_1+\varkappa_2}{2}\right)+\frac{6}{\varkappa_2-\varkappa_1}\int_0^1(1-\eta)^2h''\left(\frac{1-\eta}{2}\varkappa_1+\frac{1+\eta}{2}\varkappa_2\right)d\eta\\
&= \frac{-2}{\varkappa_2-\varkappa_1}h''\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\frac{12}{(\varkappa_2-\varkappa_1)^2}h'\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\\
&\quad+\frac{24}{(\varkappa_2-\varkappa_1)^2}\int_0^1(1-\eta)^2h'\left(\frac{1-\eta}{2}\varkappa_1+\frac{1+\eta}{2}\varkappa_2\right)d\eta\\
&= \frac{-2}{\varkappa_2-\varkappa_1}h''\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\frac{12}{(\varkappa_2-\varkappa_1)^2}h'\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\frac{48}{(\varkappa_2-\varkappa_1)^3}h\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\\
&\quad+\frac{96}{(\varkappa_2-\varkappa_1)^4}\left(\int_{\frac{\varkappa_1+\varkappa_2}{2}}^k h(u)\,du+\int_k^{\varkappa_2}h(u)\,du\right).
\end{aligned} \tag{2}
$$

Multiplying both sides of the equality (2) by $\frac{(\varkappa_2-\varkappa_1)^3}{96}$ and adding $\frac{2(1-\alpha)}{(\varkappa_2-\varkappa_1)B(\alpha)}h(k)$, we have

$$
\begin{aligned}
&\frac{(\varkappa_2-\varkappa_1)^3}{96}\int_0^1(1-\eta)^3h'''\left(\frac{1-\eta}{2}\varkappa_1+\frac{1+\eta}{2}\varkappa_2\right)d\eta+\frac{2(1-\alpha)}{B(\alpha)}h(k)\\
&= \frac{(\varkappa_2-\varkappa_1)^3}{96}\left[\frac{-2}{\varkappa_2-\varkappa_1}h''\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\frac{12}{(\varkappa_2-\varkappa_1)^2}h'\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\right.\\
&\quad\left.-\frac{48}{(\varkappa_2-\varkappa_1)^3}h\left(\frac{\varkappa_1+\varkappa_2}{2}\right)+\frac{96}{(\varkappa_2-\varkappa_1)^4}\left(\int_{\frac{\varkappa_1+\varkappa_2}{2}}^k h(u)\,du+\int_k^{\varkappa_2}h(u)\,du\right)\right]+\frac{2(1-\alpha)}{B(\alpha)}h(k)\\
&= -\frac{(\varkappa_2-\varkappa_1)^2}{48}h''\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\frac{(\varkappa_2-\varkappa_1)}{8}h'\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\frac{1}{2}h\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\\
&\quad+\left(\int_{\frac{\varkappa_1+\varkappa_2}{2}}^k h(u)\,du+\frac{(1-\alpha)}{B(\alpha)}h(k)\right)+\left(\int_k^{\varkappa_2}h(u)\,du+\frac{(1-\alpha)}{B(\alpha)}h(k)\right)\\
&= -\frac{(\varkappa_2-\varkappa_1)^2}{48}h''\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\frac{(\varkappa_2-\varkappa_1)}{8}h'\left(\frac{\varkappa_1+\varkappa_2}{2}\right)
\end{aligned}
$$

$$-\frac{1}{2}h\left(\frac{\varkappa_1 + \varkappa_2}{2}\right) + \frac{B(\alpha)}{\alpha(\varkappa_2-\varkappa_1)}\left(\left({}^{CF}_{\frac{x_1+x_2}{2}}I^{\alpha}\ h\right)(k) + \left({}^{CF}I^{\alpha}_{x_2}\ h\right)(k)\right). \tag{3}$$

Similarly, we obtain

$$\begin{aligned} I_2 &= \int_0^1 (1-\eta)^3 h'''\left(\frac{1+\eta}{2}\varkappa_1 + \frac{1-\eta}{2}\varkappa_2\right) d\eta \\ &= \frac{2}{\varkappa_2-\varkappa_1}h''\left(\frac{\varkappa_1+\varkappa_2}{2}\right) - \frac{12}{(\varkappa_2-\varkappa_1)^2}h'\left(\frac{\varkappa_1+\varkappa_2}{2}\right) + \frac{48}{(\varkappa_2-\varkappa_1)^3}h\left(\frac{\varkappa_1+\varkappa_2}{2}\right) \\ &\quad -\frac{96}{(\varkappa_2-\varkappa_1)^4}\left(\int_{x_1}^{k} h(u)\,du + \int_{k}^{\frac{x_1+x_2}{2}} h(u)\,du\right). \end{aligned} \tag{4}$$

Multiplying both sides of the equality (4) by $\frac{(\varkappa_2-\varkappa_1)^3}{96}$ and adding $\frac{2(1-\alpha)}{(\varkappa_2-\varkappa_1)B(\alpha)}h(k)$, we have

$$\begin{aligned} &\frac{(\varkappa_2-\varkappa_1)^3}{96}\int_0^1 (1-\eta)^3 h'''\left(\frac{1+\eta}{2}\varkappa_1 + \frac{1-\eta}{2}\varkappa_2\right) d\eta + \frac{2(1-\alpha)}{(\varkappa_2-\varkappa_1)B(\alpha)}h(k) \\ =\ &\frac{(\varkappa_2-\varkappa_1)^2}{48}h''\left(\frac{\varkappa_1+\varkappa_2}{2}\right) - \frac{(\varkappa_2-\varkappa_1)}{8}h'\left(\frac{\varkappa_1+\varkappa_2}{2}\right) + \frac{1}{2}h\left(\frac{\varkappa_1+\varkappa_2}{2}\right) \\ &-\frac{B(\alpha)}{\alpha(\varkappa_2-\varkappa_1)}\left(\left({}^{CF}_{x_1}I^{\alpha}\ h\right)(k) + \left({}^{CF}I^{\alpha}_{\frac{x_1+x_2}{2}}\ h\right)(k)\right). \end{aligned} \tag{5}$$

From equalities (3) and (5), we obtain the equality (1).
This completes the proof . □

Theorem 3.2. *Let $h: I \to \mathbb{R}$ be three times differentiable function on I^0. If $h''' \in L[\varkappa_1, \varkappa_2]$ and $|h'''|$ is $s-$convex function, we have the following inequality for Caputo-Fabrizio fractional integral operator:*

$$\begin{aligned} &\left|\frac{B(\alpha)}{\alpha(\varkappa_2-\varkappa_1)}\left[\left\{\left({}^{CF}_{x_1}I^{\alpha}\ h\right)(k) + \left({}^{CF}I^{\alpha}_{\frac{x_1+x_2}{2}}\ h\right)(k)\right\} + \left\{\left({}^{CF}_{\frac{x_1+x_2}{2}}I^{\alpha}\ h\right)(k) + \left({}^{CF}I^{\alpha}_{x_2}\ h\right)(k)\right\}\right]\right. \\ &\left.-h\left(\frac{\varkappa_1+\varkappa_2}{2}\right) - \frac{(\varkappa_2-\varkappa_1)^2}{24}h''\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\right| \\ \leq\ &\frac{(\varkappa_2-\varkappa_1)^3}{3\times 2^{s+5}}\left[\frac{3.2^{s+5}-6s^2-42s-84}{(s+1)(s+2)(s+3)(s+4)}\right]\left(|h'''(\varkappa_1)| + |h'''(\varkappa_2)|\right), \end{aligned}$$

where $k \in [\varkappa_1, \varkappa_2]$ and $B(\alpha) > 0$ is a normalization function.

Proof. Using the Lemma 3 and fact that $|h'''|$ is $s-$convex function, we have

$$\begin{aligned} &\left|\frac{B(\alpha)}{\alpha(\varkappa_2-\varkappa_1)}\left[\left\{\left({}^{CF}_{x_1}I^{\alpha}\ h\right)(k) + \left({}^{CF}I^{\alpha}_{\frac{x_1+x_2}{2}}\ h\right)(k)\right\} + \left\{\left({}^{CF}_{\frac{x_1+x_2}{2}}I^{\alpha}\ h\right)(k) + \left({}^{CF}I^{\alpha}_{x_2}\ h\right)(k)\right\}\right]\right. \\ &\left.-h\left(\frac{\varkappa_1+\varkappa_2}{2}\right) - \frac{(\varkappa_2-\varkappa_1)^2}{24}h''\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\right| \\ \leq\ &\frac{(\varkappa_2-\varkappa_1)^3}{96}\int_0^1 (1-\eta)^3\left(\left|h'''\left(\frac{1-\eta}{2}\varkappa_1 + \frac{1+\eta}{2}\varkappa_2\right) - h'''\left(\frac{1+\eta}{2}\varkappa_1 + \frac{1-\eta}{2}\varkappa_2\right)\right|\right) d\eta \\ \leq\ &\frac{(\varkappa_2-\varkappa_1)^3}{96}\int_0^1 (1-\eta)^3\left(\left|h'''\left(\frac{1-\eta}{2}\varkappa_1 + \frac{1+\eta}{2}\varkappa_2\right)\right|\right) d\eta \\ &+\frac{(\varkappa_2-\varkappa_1)^3}{96}\int_0^1 (1-\eta)^3\left(\left|h'''\left(\frac{1+\eta}{2}\varkappa_1 + \frac{1-\eta}{2}\varkappa_2\right)\right|\right) d\eta \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(\varkappa_2-\varkappa_1)^3}{96}\int_0^1 (1-\eta)^3\left(\left(\frac{1-\eta}{2}\right)^s |h'''(\varkappa_1)| + \left(\frac{1+\eta}{2}\right)^s |h'''(\varkappa_2)|\right)d\eta \\
&\quad + \frac{(\varkappa_2-\varkappa_1)^3}{96}\int_0^1 (1-\eta)^3\left(\left(\frac{1+\eta}{2}\right)^s |h'''(\varkappa_1)| + \left(\frac{1-\eta}{2}\right)^s |h'''(\varkappa_2)|\right)d\eta \\
&= \frac{(\varkappa_2-\varkappa_1)^3}{96\times 2^s}\left(\int_0^1 (1-\eta)^{s+3}\,d\eta\,|h'''(\varkappa_1)| + \int_0^1 (1-\eta)^3(1+\eta)^s\,d\eta\,|h'''(\varkappa_2)|\right) \\
&\quad + \frac{(\varkappa_2-\varkappa_1)^3}{96\times 2^s}\left(\int_0^1 (1-\eta)^3(1+\eta)^s\,d\eta\,|h'''(\varkappa_1)| + \int_0^1 (1-\eta)^{s+3}\,d\eta\,|h'''(\varkappa_2)|\right) \\
&= \frac{(\varkappa_2-\varkappa_1)^3}{3\times 2^{s+5}}\left[\frac{3.2^{s+5}-6s^2-42s-84}{(s+1)(s+2)(s+3)(s+4)}\right](|h'''(\varkappa_1)|+|h'''(\varkappa_2)|).
\end{aligned}
$$

Note that

$$
\begin{aligned}
\int_0^1 (1-\eta)^{s+3}\,d\eta &= \frac{1}{s+4}, \\
\int_0^1 (1-\eta)^3(1+\eta)^s\,d\eta &= \frac{3.2^{s+5}-s^3-12s^2-53s-90}{(s+1)(s+2)(s+3)(s+4)}.
\end{aligned}
$$

This completes the proof. □

Corollary 3.3. *If we choose $s=1$ in Theorem 3, then we have*

$$
\begin{aligned}
&\left|\frac{B(\alpha)}{\alpha(\varkappa_2-\varkappa_1)}\left[\left\{\left({}^{CF}_{x_1}I^{\alpha}\,h\right)(k) + \left({}^{CF}I^{\alpha}_{\frac{x_1+x_2}{2}}\,h\right)(k)\right\} + \left\{\left({}^{CF}_{\frac{x_1+x_2}{2}}I^{\alpha}\,h\right)(k) + \left({}^{CF}I^{\alpha}_{x_2}\,h\right)(k)\right\}\right]\right. \\
&\left. -h\left(\frac{\varkappa_1+\varkappa_2}{2}\right) - \frac{(\varkappa_2-\varkappa_1)^2}{24}h''\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\right| \\
&\leq \frac{(\varkappa_2-\varkappa_1)^3}{384}\left[|h'''(\varkappa_1)| + |h'''(\varkappa_2)|\right].
\end{aligned}
$$

Example 3.4. *Clarification related to the following expression occurs in Theorem 2*

$$
P := \frac{(\varkappa_2-\varkappa_1)^2}{24}h''\left(\frac{\varkappa_1+\varkappa_2}{2}\right), \tag{6}
$$

we consider the function $h(x)=\frac{x}{x^2+2}$ on the interval $[x_1,x_2]=[0,1]$. Then, we have

$$
\begin{aligned}
y(x) \quad : &= h''(x) = \frac{2x\left(4x^2-3\right)}{\left(x^2+2\right)^2}; \\
P &= \frac{1}{24}h''\left(\frac{1}{2}\right) = -\frac{46}{2187}.
\end{aligned}
$$

The Figure 1 represents the relationship between the functions $h(x)$ and $y(x)$.

Theorem 3.5. *Let $h: I \to \mathbb{R}$ be three times differentiable function on I^0. If $h''' \in L[\varkappa_1,\varkappa_2]$ and $|h'''|^q$ is s-convex function where $\frac{1}{p}+\frac{1}{q}=1$, $p,q\geq 1$, then we have the following inequality for Caputo-Fabrizio fractional integral operator:*

$$
\begin{aligned}
&\left|\frac{B(\alpha)}{\alpha(\varkappa_2-\varkappa_1)}\left[\left\{\left({}^{CF}_{x_1}I^{\alpha}\,h\right)(k) + \left({}^{CF}I^{\alpha}_{\frac{x_1+x_2}{2}}\,h\right)(k)\right\} + \left\{\left({}^{CF}_{\frac{x_1+x_2}{2}}I^{\alpha}\,h\right)(k) + \left({}^{CF}I^{\alpha}_{x_2}\,h\right)(k)\right\}\right]\right. \\
&\left. -h\left(\frac{\varkappa_1+\varkappa_2}{2}\right) - \frac{(\varkappa_2-\varkappa_1)^2}{24}h''\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\right|
\end{aligned}
$$

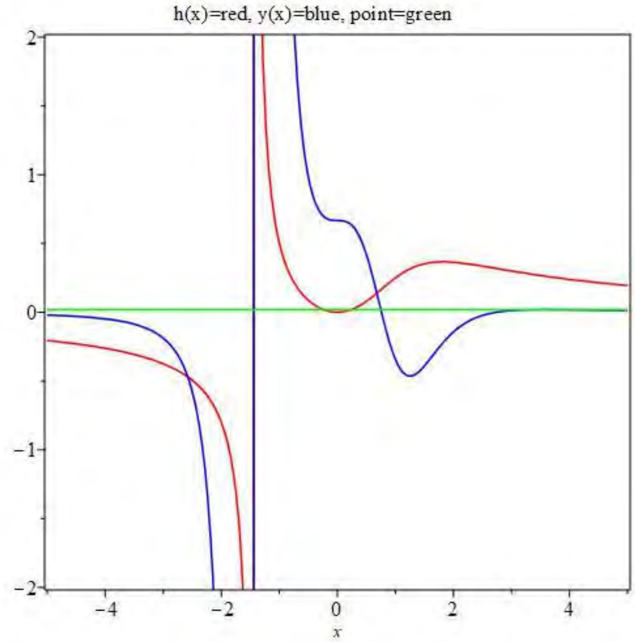

Figure 1: Graphical description of (6).

$$\leq \frac{(\varkappa_2-\varkappa_1)^3}{96\times 2^{\frac{s}{q}}}\left(\frac{1}{3p+1}\right)^{\frac{1}{p}}\left(\frac{2^{s+1}}{s+1}\right)^{\frac{1}{q}}\left(|h'''(\varkappa_1)|^q+|h'''(\varkappa_2)|^q\right)^{\frac{1}{q}},$$

where $k\in[\varkappa_1,\varkappa_2]$ *and* $B(\alpha)>0$ *is a normalization function.*

Proof. Using the Lemma 3, Hölder inequality and fact that $|h'''|^q$ is $s-$convex function, we have

$$\Bigg|\frac{B(\alpha)}{\alpha(\varkappa_2-\varkappa_1)}\left[\left\{\left({}^{CF}_{\varkappa_1}I^{\alpha}h\right)(k)+\left({}^{CF}I^{\alpha}_{\frac{\varkappa_1+\varkappa_2}{2}}h\right)(k)\right\}+\left\{\left({}^{CF}_{\frac{\varkappa_1+\varkappa_2}{2}}I^{\alpha}h\right)(k)+\left({}^{CF}I^{\alpha}_{\varkappa_2}h\right)(k)\right\}\right]$$
$$-h\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\frac{(\varkappa_2-\varkappa_1)^2}{24}h''\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\Bigg|$$
$$\leq \frac{(\varkappa_2-\varkappa_1)^3}{96}\int_0^1(1-\eta)^3\left[\left|h'''\left(\frac{1-\eta}{2}\varkappa_1+\frac{1+\eta}{2}\varkappa_2\right)+h'''\left(\frac{1+\eta}{2}\varkappa_1+\frac{1-\eta}{2}\varkappa_2\right)\right|\right]d\eta$$
$$=\frac{(\varkappa_2-\varkappa_1)^3}{96}\int_0^1(1-\eta)^3\left|h'''\left(\frac{1-\eta}{2}\varkappa_1+\frac{1+\eta}{2}\varkappa_2\right)\right|d\eta$$
$$+\frac{(\varkappa_2-\varkappa_1)^3}{96}\int_0^1(1-\eta)^3\left|h'''\left(\frac{1+\eta}{2}\varkappa_1+\frac{1-\eta}{2}\varkappa_2\right)\right|d\eta$$
$$\leq \frac{(\varkappa_2-\varkappa_1)^3}{96}\left[\left(\int_0^1(1-\eta)^{3p}\right)^{\frac{1}{p}}\left(\int_0^1\left|h'''\left(\frac{1-\eta}{2}\varkappa_1+\frac{1+\eta}{2}\varkappa_2\right)^q\right|d\eta\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1(1-\eta)^{3p}\right)^{\frac{1}{p}}\left(\int_0^1\left|h'''\left(\frac{1+\eta}{2}\varkappa_1+\frac{1-\eta}{2}\varkappa_2\right)^q\right|d\eta\right)^{\frac{1}{q}}\right]$$
$$=\frac{(\varkappa_2-\varkappa_1)^3}{96}\left(\frac{1}{3p+1}\right)^{\frac{1}{p}}\left[\left(\int_0^1\left|h'''\left(\frac{1-\eta}{2}\varkappa_1+\frac{1+\eta}{2}\varkappa_2\right)^q\right|d\eta\right)^{\frac{1}{q}}\right.$$

$$
\begin{aligned}
&+\left(\int_0^1\left|h'''\left(\frac{1\;+\;\eta}{2}\varkappa_1+\frac{1\;-\;\eta}{2}\varkappa_2\right)^q\right|d\eta\right)^{\frac{1}{q}}\Bigg]\\
\leq\;& \frac{(\varkappa_2-\varkappa_1)^3}{96}\left(\frac{1}{3p+1}\right)^{\frac{1}{p}}\left[\int_0^1\left(\frac{1\;-\;\eta}{2}\right)^s|h'''(\varkappa_1)|^q+\int_0^1\left(\frac{1\;+\;\eta}{2}\right)^s|h'''(\varkappa_2)|^q\right.\\
&\left.+\int_0^1\left(\frac{1\;+\;\eta}{2}\right)^s|h'''(\varkappa_1)|^q+\int_0^1\left(\frac{1\;-\;\eta}{2}\right)^s|h'''(\varkappa_2)|^q\right]^{\frac{1}{q}}\\
=\;& \frac{(\varkappa_2\;-\;\varkappa_1)^3}{96\times 2^{\frac{s}{q}}}\left(\frac{1}{3p\;+\;1}\right)^{\frac{1}{p}}\left[\left(\frac{1}{s\;+\;1}\right)|h'''(\varkappa_1)|^q+\left(\frac{2^{s+1}-1}{s+1}\right)|h'''(\varkappa_2)|^q\right.\\
&\left.+\left(\frac{2^{s+1}-1}{s+1}\right)|h'''(\varkappa_1)|^q+\left(\frac{1}{s+1}\right)|h'''(\varkappa_2)|^q\right]^{\frac{1}{q}}\\
=\;& \frac{(\varkappa_2-\varkappa_1)^3}{96\times 2^{\frac{s}{q}}}\left(\frac{1}{3p+1}\right)^{\frac{1}{p}}\left(\frac{2^{s+1}}{s+1}\right)^{\frac{1}{q}}\left[|h'''(\varkappa_1)|^q\;\;+\;\;|h'''(\varkappa_2)|^q\right]^{\frac{1}{q}}.
\end{aligned}
$$

This completes the proof. □

Theorem 3.6. *Let $h: I\to\mathbb{R}$ be three times differentiable function on I^0. If $h'''\in L[\varkappa_1,\varkappa_2]$ and $|h'''|^q$ is $s-$convex function where $q>1$, then we have the following inequality for Caputo-Fabrizio fractional integral operator:*

$$
\begin{aligned}
&\left|\frac{B(\alpha)}{\alpha(\varkappa_2-\varkappa_1)}\left[\left\{\left({}^{CF}_{x_1}I^\alpha h\right)(k)+\left({}^{CF}I^\alpha_{\frac{x_1+x_2}{2}}h\right)(k)\right\}+\left\{\left({}^{CF}_{\frac{\varkappa_1+\varkappa_2}{2}}I^{\alpha}h\right)(k)+\left({}^{CF}I^\alpha_{x_2}h\right)(k)\right\}\right]\right.\\
&\left.-h\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\frac{(\varkappa_2-\varkappa_1)^2}{24}h''\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\frac{4(1-\alpha)}{\alpha(\varkappa_2-\varkappa_1)}h(k)\right|\\
\leq\;& \frac{(\varkappa_2-\varkappa_1)^3}{96}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left[\left(\left(\frac{3.2^{s+5}-s^3-12s^2-53s-90}{2^s(s+1)(s+2)(s+3)(s+4)}\right)|h'''(\varkappa_2)|^q+\frac{1}{2^s(s+4)}|h'''(\varkappa_1)|^q\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{1}{2^s(s+4)}|h'''(\varkappa_2)|^q+\left(\frac{3.2^{s+5}-s^3-12s^2-53s-90}{2^s(s+1)(s+2)(s+3)(s+4)}\right)|h'''(\varkappa_1)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

where $k\in[\varkappa_1,\varkappa_2]$ and $B(\alpha)>0$ is a normalization function.

Proof. Using the Lemma 3, power -mean inequality and fact that $|h'''|^q$ is s-convex function, we have

$$
\begin{aligned}
&\left|\frac{B(\alpha)}{\alpha(\varkappa_2-\varkappa_1)}\left[\left\{\left({}^{CF}_{x_1}I^\alpha h\right)(k)+\left({}^{CF}I^\alpha_{\frac{x_1+x_2}{2}}h\right)(k)\right\}+\left\{\left({}^{CF}_{\frac{\varkappa_1+\varkappa_2}{2}}I^{\alpha}h\right)(k)+\left({}^{CF}I^\alpha_{x_2}h\right)(k)\right\}\right]\right.\\
&\left.-h\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\frac{(\varkappa_2-\varkappa_1)^2}{24}h''\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\right|\\
\leq\;& \frac{(\varkappa_2-\varkappa_1)^3}{96}\int_0^1(1\;-\;\eta)^3\left[\left|h'''\left(\frac{1\;-\;\eta}{2}\varkappa_1+\frac{1\;+\;\eta}{2}\varkappa_2\right)+h'''\left(\frac{1\;+\;\eta}{2}\varkappa_1+\frac{1\;-\;\eta}{2}\varkappa_2\right)\right|\right]d\eta\\
=\;& \frac{(\varkappa_2-\varkappa_1)^3}{96}\int_0^1(1-\eta)^3\left|h'''\left(\frac{1-\eta}{2}\varkappa_1+\frac{1+\eta}{2}\varkappa_2\right)\right|d\eta\\
&+\frac{(\varkappa_2-\varkappa_1)^3}{96}\int_0^1(1-\eta)^3h'''\left(\frac{1+\eta}{2}\varkappa_1+\frac{1-\eta}{2}\varkappa_2\right)d\eta\\
\leq\;& \frac{(\varkappa_2-\varkappa_1)^3}{96}\left[\left(\int_0^1(1-\eta)^3\right)^{1-\frac{1}{q}}\left(\int_0^1(1\;-\;\eta)^3\left|h'''\left(\frac{1\;-\;\eta}{2}\varkappa_1+\frac{1\;+\;\eta}{2}\varkappa_2\right)\right|^q d\eta\right)^{\frac{1}{q}}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\int_0^1 (1-\eta)^3\right)^{1-\frac{1}{q}} \left(\int_0^1 (1-\eta)^3 \left|h'''\left(\frac{1+\eta}{2}\varkappa_1+\frac{1-\eta}{2}\varkappa_2\right)\right|^q d\eta\right)^{\frac{1}{q}} \Bigg]\\
\leq\ & \frac{(\varkappa_2-\varkappa_1)^3}{96}\left(\frac{1}{4}\right)^{1-\frac{1}{q}} \int_0^1 (1-\eta)^3 \left[\left(\frac{1-\eta}{2}\right)^s |h'''(\varkappa_1)|^q + \left(\frac{1+\eta}{2}\right)^s |h'''(\varkappa_2)|^q\right.\\
&\left.+\left(\frac{1+\eta}{2}\right)^s |h'''(\varkappa_1)|^q + \left(\frac{1-\eta}{2}\right)^s |h'''(\varkappa_2)|^q\right]^{\frac{1}{q}} d\eta\\
=\ & \frac{(\varkappa_2-\varkappa_1)^3}{96}\left(\frac{1}{4}\right)^{1-\frac{1}{q}} \left[\left(\left(\frac{3.2^{s+5}-s^3-12s^2-53s-90}{2^s(s+1)(s+2)(s+3)(s+4)}\right)|h'''(\varkappa_2)|^q + \frac{1}{2^s(s+4)}|h'''(\varkappa_1)|^q\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{1}{2^s(s+4)}|h'''(\varkappa_2)|^q + \left(\frac{3.2^{s+5}-s^3-12s^2-53s-90}{2^s(s+1)(s+2)(s+3)(s+4)}\right)|h'''(\varkappa_1)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

This completes the proof. □

Theorem 3.7. *Let $h : [\varkappa_1, \varkappa_2] \to \mathbb{R}$ be three times differentiable function on $(\varkappa_1, \varkappa_2)$ with $\varkappa_1 < \varkappa_2$. If $|h'''|$ is concave on $[\varkappa_1, \varkappa_2]$, then we have the following inequality for Caputo-Fabrizio fractional integral operator:*

$$
\begin{aligned}
&\left|\frac{B(\alpha)}{\alpha(\varkappa_2-\varkappa_1)}\left[\left\{\left({}^{CF}_{x_1}I^\alpha h\right)(k)+\left({}^{CF}I^\alpha_{\frac{x_1+x_2}{2}} h\right)(k)\right\}+\left\{\left({}^{CF}_{\frac{x_1+x_2}{2}}I^\alpha h\right)(k)+\left({}^{CF}I^\alpha_{x_2} h\right)(k)\right\}\right]\right.\\
&\left.-h\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\frac{(\varkappa_2-\varkappa_1)^2}{24}h''\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\right|\\
\leq\ & \frac{(\varkappa_2-\varkappa_1)^3}{384}\left[\left|h'''\left(\frac{2\varkappa_1+3\varkappa_2}{5}\right)\right|+\left|h'''\left(\frac{3\varkappa_1+2\varkappa_2}{5}\right)\right|\right],
\end{aligned}
$$

where $k \in [\varkappa_1, \varkappa_2]$ and $B(\alpha) > 0$ is a normalization function.

Proof. From Lemma 3, we have

$$
\begin{aligned}
&\left|\frac{B(\alpha)}{\alpha(\varkappa_2-\varkappa_1)}\left[\left\{\left({}^{CF}_{x_1}I^\alpha h\right)(k)+\left({}^{CF}I^\alpha_{\frac{x_1+x_2}{2}} h\right)(k)\right\}+\left\{\left({}^{CF}_{\frac{x_1+x_2}{2}}I^\alpha h\right)(k)+\left({}^{CF}I^\alpha_{x_2} h\right)(k)\right\}\right]\right.\\
&\left.-h\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\frac{(\varkappa_2-\varkappa_1)^2}{24}h''\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\right|\\
\leq\ & \frac{(\varkappa_2-\varkappa_1)^3}{96}\int_0^1 \left|(1-\eta)^3\right|\left|h'''\left(\frac{1-\eta}{2}\varkappa_1+\frac{1+\eta}{2}\varkappa_2\right)\right|d\eta\\
&+\frac{(\varkappa_2-\varkappa_1)^3}{96}\int_0^1 \left|(1-\eta)^3\right|\left|h'''\left(\frac{1+\eta}{2}\varkappa_1+\frac{1-\eta}{2}\varkappa_2\right)\right|d\eta.
\end{aligned}
$$

By the Jensen integral inequality, we have

$$
\begin{aligned}
&\int_0^1 \left|(1-\eta)^3\right| h'''\left(\frac{1-\eta}{2}\varkappa_1+\frac{1+\eta}{2}\varkappa_2\right)d\eta\\
\leq\ & \left(\int_0^1 \left|(1-\eta)^3\right|d\eta\right)\left[\left|h'''\left(\frac{\int_0^1 \left|(1-\eta)^3\right|\left(\frac{1-\eta}{2}\varkappa_1+\frac{1+\eta}{2}\varkappa_2\right)d\eta}{\int_0^1 \left|(1-\eta)^3\right|d\eta}\right)\right|\right]\\
=\ & \frac{1}{4}\left|h'''\left(\frac{2\varkappa_1+3\varkappa_2}{5}\right)\right|, \qquad (7)
\end{aligned}
$$

and analogously

$$\begin{aligned}
&\int_0^1 \left|(1-\eta)^3\right| h'''\left(\frac{1+\eta}{2}\varkappa_1+\frac{1-\eta}{2}\varkappa_2\right)d\eta\\
\leq\ &\left(\int_0^1 \left|(1-\eta)^3\right| d\eta\right)\left[\left|h'''\left(\frac{\int_0^1 \left|(1-\eta)^3\right|\left(\frac{1+\eta}{2}\varkappa_1+\frac{1-\eta}{2}\varkappa_2\right)d\eta}{\int_0^1 \left|(1-\eta)^3\right| d\eta}\right)\right|\right]\\
=\ &\frac{1}{4}\left|h'''\left(\frac{3\varkappa_1+2\varkappa_2}{5}\right)\right|. \qquad (8)
\end{aligned}$$

Combination of the above inequalities (7)and (8) give the result. That is

$$\begin{aligned}
&\left|\frac{B(\alpha)}{\alpha(\varkappa_2-\varkappa_1)}\left[\left\{\left({}^{CF}_{x_1}I^{\alpha}\,h\right)(k)+\left({}^{CF}I^{\alpha}_{\frac{x_1+x_2}{2}}\,h\right)(k)\right\}+\left\{\left({}^{CF}_{\frac{x_1+x_2}{2}}I^{\alpha}\,h\right)(k)+\left({}^{CF}I^{\alpha}_{x_2}\,h\right)(k)\right\}\right]\right.\\
&\left.-h\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\frac{(\varkappa_2-\varkappa_1)^2}{24}h''\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\right|\\
\leq\ &\frac{(\varkappa_2-\varkappa_1)^3}{384}\left[\left|h'''\left(\frac{2\varkappa_1+3\varkappa_2}{5}\right)\right|+\left|h'''\left(\frac{3\varkappa_1+2\varkappa_2}{5}\right)\right|\right]
\end{aligned}$$

This completes the proof. □

Theorem 3.8. *Let $h : [\varkappa_1, \varkappa_2] \to \mathbb{R}$ be three times differentiable function on $(\varkappa_1, \varkappa_2)$ with $\varkappa_1 < \varkappa_2$. If $h''' \in L[\varkappa_1, \varkappa_2]$ and $|h'''|^q$ is $s-$convex on $[\varkappa_1, \varkappa_2]$, for some fixed $s \in (0, 1]$ and $q > 1$, then we have the following inequality for Caputo-Fabrizio fractional integral operator:*

$$\begin{aligned}
&\left|\frac{B(\alpha)}{\alpha(\varkappa_2-\varkappa_1)}\left[\left\{\left({}^{CF}_{x_1}I^{\alpha}\,h\right)(k)+\left({}^{CF}I^{\alpha}_{\frac{x_1+x_2}{2}}\,h\right)(k)\right\}+\left\{\left({}^{CF}_{\frac{x_1+x_2}{2}}I^{\alpha}\,h\right)(k)+\left({}^{CF}I^{\alpha}_{x_2}\,h\right)(k)\right\}\right]\right.\\
&\left.-h\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\frac{(\varkappa_2-\varkappa_1)^2}{24}h''\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\right|\\
\leq\ &\frac{(\varkappa_2-\varkappa_1)^3}{96}\left[\frac{1}{p(3p+1)}+\frac{q^{-1}}{2^s}\left(\frac{1}{s+1}+\frac{2^{s+1}-1}{s+1}\right)\right]\left[|h'''(\varkappa_1)|^q+|h'''(\varkappa_2)|^q\right],
\end{aligned}$$

where $k \in [\varkappa_1, \varkappa_2]$ and $B(\alpha) > 0$ is a normalization function $p^{-1} = 1 - q^{-1}$.

Proof. *Using Lemma* 3, *we have*

$$\begin{aligned}
&\left|\frac{B(\alpha)}{\alpha(\varkappa_2-\varkappa_1)}\left[\left\{\left({}^{CF}_{x_1}I^{\alpha}\,h\right)(k)+\left({}^{CF}I^{\alpha}_{\frac{x_1+x_2}{2}}\,h\right)(k)\right\}+\left\{\left({}^{CF}_{\frac{x_1+x_2}{2}}I^{\alpha}\,h\right)(k)+\left({}^{CF}I^{\alpha}_{x_2}\,h\right)(k)\right\}\right]\right.\\
&\left.-h\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\frac{(\varkappa_2-\varkappa_1)^2}{24}h''\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\right|\\
\leq\ &\frac{(\varkappa_2-\varkappa_1)^3}{96}\int_0^1 (1-\eta)^3\left[\left|h'''\left(\frac{1-\eta}{2}\varkappa_1+\frac{1+\eta}{2}\varkappa_2\right)+h'''\left(\frac{1+\eta}{2}\varkappa_1+\frac{1-\eta}{2}\varkappa_2\right)\right|\right]d\eta\\
=\ &\frac{(\varkappa_2-\varkappa_1)^3}{96}\int_0^1 (1-\eta)^3\left|h'''\left(\frac{1-\eta}{2}\varkappa_1+\frac{1+\eta}{2}\varkappa_2\right)\right|d\eta\\
&+\frac{(\varkappa_2-\varkappa_1)^3}{96}\int_0^1 (1-\eta)^3\left|h'''\left(\frac{1+\eta}{2}\varkappa_1+\frac{1-\eta}{2}\varkappa_2\right)\right|d\eta.
\end{aligned}$$

By using the Young′ s inequality stated as

$$\varkappa_1\varkappa_2 \leq \frac{1}{p}\varkappa_1^p + \frac{1}{q}\varkappa_2^q,$$

we obtain

$$
\begin{aligned}
& \left| \frac{B(\alpha)}{\alpha(\varkappa_2-\varkappa_1)} \left[\left\{ \left({}^{CF}_{x_1} I^{\alpha} h \right)(k) + \left({}^{CF} I^{\alpha}_{\frac{x_1+x_2}{2}} h \right)(k) \right\} + \left\{ \left({}^{CF}_{\frac{x_1+x_2}{2}} I^{\alpha} \ h \right)(k) + \left({}^{CF} I^{\alpha}_{x_2} h \right)(k) \right\} \right] \right. \\
& \left. -h\left(\frac{\varkappa_1+\varkappa_2}{2}\right) - \frac{(\varkappa_2-\varkappa_1)^2}{24} h''\left(\frac{\varkappa_1+\varkappa_2}{2}\right) \right| \\
\leq\ & \frac{(\varkappa_2-\varkappa_1)^3}{96} \left[\frac{1}{p}\int_0^1 (1-\eta)^{3p} d\eta + \frac{1}{q}\int_0^1 \left| h'''\left(\frac{1-\eta}{2}\varkappa_1 + \frac{1+\eta}{2}\varkappa_2\right)\right|^q d\eta \right. \\
& \left. + \frac{1}{p}\int_0^1 (1-\eta)^{3p} d\eta + \frac{1}{q}\int_0^1 \left| h'''\left(\frac{1+\eta}{2}\varkappa_1 + \frac{1-\eta}{2}\varkappa_2\right)\right|^q d\eta \right] \\
\leq\ & \frac{(\varkappa_2-\varkappa_1)^3}{96} \left[\frac{1}{p}\int_0^1 (1-\eta)^{3p} d\eta + \frac{1}{q}\int_0^1 \left\{ \left(\frac{1-\eta}{2}\right)^s |h'''(\varkappa_1)|^q + \left(\frac{1+\eta}{2}\right)^s |h'''(\varkappa_2)|^q \right\} d\eta \right. \\
& \left. + \frac{1}{p}\int_0^1 (1-\eta)^{3p} d\eta + \frac{1}{q}\int_0^1 \left\{ \left(\frac{1+\eta}{2}\right)^s |h'''(\varkappa_1)|^q + \left(\frac{1-\eta}{2}\right)^s |h'''(\varkappa_2)|^q \right\} d\eta \right] \\
\leq\ & \frac{(\varkappa_2-\varkappa_1)^3}{96} \left[\frac{1}{p(3p+1)} + \frac{q^{-1}}{2^s}\left(\frac{1}{s+1} + \frac{2^{s+1}-1}{s+1} \right) \right] \left[|h'''(\varkappa_1)|^q + |h'''(\varkappa_2)|^q \right].
\end{aligned}
$$

This completes the proof. □

4. Application to Special Means

We shall consider the following special means:
(*a*) Arithmetic Mean:

$$A = A(\varkappa_1, \varkappa_2) : = \frac{\varkappa_1 + \varkappa_2}{2}, \ \varkappa_1, \varkappa_2 \geq 0;$$

(*b*) Geometric Mean:

$$G = G(\varkappa_1, \varkappa_2) : = \sqrt{\varkappa_1 \varkappa_2}\, \varkappa_1, \varkappa_2 \geq 0;$$

(*c*) Harmonic Mean:

$$H = H(\varkappa_1, \varkappa_2) := \frac{2\varkappa_1\varkappa_2}{\varkappa_1 + \varkappa_2}, \ \varkappa_1, \varkappa_2 \geq 0;$$

(*d*) Logarithmic Mean:

$$L(\varkappa_1, \varkappa_2) := \frac{\varkappa_2 - \varkappa_1}{\ln \varkappa_2 - \ln \varkappa_1} \ \ \varkappa_1, \varkappa_2 > 0, \ \varkappa_1 \neq \varkappa_2;$$

(*e*) Generalized logarithmic Mean:

$$L_r^r = L_r^r(\varkappa_1, \varkappa_2) := \left[\frac{\varkappa_2^{r+1} - \varkappa_1^{r+1}}{(r+1)(\varkappa_2 - \varkappa_1)} \right]^{\frac{1}{r}} r \in R - \{-1, 0\}, \ \varkappa_1, \varkappa_2 \in \mathbb{R}, \ \varkappa_1 \neq \varkappa_2.$$

It is well know that L_r^r is monotonic nondecreasing over $r \in R$ with $L_{-1} = L$. In particular we have the following inequalities

$$H \leq G \leq L \leq A.$$

Proposition 4.1. *For some $n \in z\{-1,0\}$, $0 \le \varkappa_1 < \varkappa_2$, then we get*

$$\left|L(\varkappa_1,\varkappa_2) - A^n(\varkappa_1,\varkappa_2) - \frac{n(n-1)(\varkappa_2-\varkappa_1)^3}{24}A^{n-2}(\varkappa_1,\varkappa_2)\right| \le \frac{n(n-1)(n-2)(\varkappa_2-\varkappa_1)^3}{384}\left[|\varkappa_1|^{n-3}+|\varkappa_2|^{n-3}\right].$$

Proof. The assertion directly follows from Theorem 2 applying for $h(x) = x^n$ and $\alpha = s = 1$, & $B(0) = B(1) = 1$. □

Proposition 4.2. *For some $n \in z\{-1,0\}$, $0 \le \varkappa_1 < \varkappa_2$, and $\frac{1}{p}+\frac{1}{q} = 1$, $1 < q < \infty$, then*

$$\left|L(\varkappa_1,\varkappa_2) - A^n(\varkappa_1,\varkappa_2) - \frac{n(n-1)(\varkappa_2-\varkappa_1)^3}{24}A^{n-2}(\varkappa_1,\varkappa_2)\right| \le \frac{n(n-1)(n-2)(\varkappa_2-\varkappa_1)^3}{96}\left(\frac{1}{3p+1}\right)^{\frac{1}{p}}\left\{\left(\frac{1}{4}|\varkappa_1|^{q(n-3)} + \frac{3}{4}|\varkappa_2|^{q(n-3)}\right)^{\frac{1}{q}} + \left(\frac{3}{4}|\varkappa_1|^{q(n-3)} + \frac{1}{4}|\varkappa_2|^{q(n-3)}\right)^{\frac{1}{q}}\right\}.$$

Proof. The assertion directly follows from Theorem 3 applying for $h(x) = x^n$ and $\alpha = s = 1$, & $B(0) = B(1) = 1$. □

Proposition 4.3. *For some $n \in z\{-1,0\}$, $0 \le \varkappa_1 < \varkappa_2$, and $q > 1$, then*

$$\left|L(\varkappa_1,\varkappa_2) - A^n(\varkappa_1,\varkappa_2) - \frac{n(n-1)(\varkappa_2-\varkappa_1)^3}{24}A^{n-2}(\varkappa_1,\varkappa_2)\right| \le \frac{n(n-1)(n-2)(\varkappa_2-\varkappa_1)^3}{384}\left(\frac{2}{5}\right)^{\frac{1}{q}}\left\{\left(|\varkappa_1|^{q(n-3)} + \frac{3}{2}|\varkappa_2|^{q(n-3)}\right)^{\frac{1}{q}} + \left(\frac{3}{2}|\varkappa_1|^{q(n-3)} + |\varkappa_2|^{q(n-3)}\right)^{\frac{1}{q}}\right\}.$$

Proof. The assertion directly follows from Theorem 4 applying for $h(x) = x^n$ and $\alpha = s = 1$, & $B(0) = B(1) = 1$. □

Proposition 4.4. *For some $0 \le \varkappa_1 < \varkappa_2$, then*

$$\left|L^{-1}(\varkappa_1,\varkappa_2) - A^{-1}(\varkappa_1,\varkappa_2) - \frac{(\varkappa_2-\varkappa_1)^3}{12}A^{-3}(\varkappa_1,\varkappa_2)\right| \le \frac{(\varkappa_2-\varkappa_1)^3}{64}\left[|\varkappa_1|^{-4}+|\varkappa_2|^{-4}\right].$$

Proof. The assertion directly follows from Theorem 2 applying for $h(x) = x^{-1}$ and $\alpha = s = 1$, & $B(0) = B(1) = 1$. □

Proposition 4.5. *For some $0 \le \varkappa_1 < \varkappa_2$, and $\frac{1}{p}+\frac{1}{q} = 1$, $1 < q < \infty$, then*

$$\left|L^{-1}(\varkappa_1,\varkappa_2) - A^{-1}(\varkappa_1,\varkappa_2) - \frac{(\varkappa_2-\varkappa_1)^3}{12}A^{-3}(\varkappa_1,\varkappa_2)\right|$$

$$\leq \frac{(\varkappa_2-\varkappa_1)^3}{64}\left(\frac{1}{3p+1}\right)^{\frac{1}{p}}\left\{\left(\frac{1}{4}|\varkappa_1|^{-4q}+\frac{3}{4}|\varkappa_2|^{-4q}\right)^{\frac{1}{q}}+\left(\frac{3}{4}|\varkappa_1|^{-4q}+\frac{1}{4}|\varkappa_2|^{-4q}\right)^{\frac{1}{q}}\right\}.$$

Proof. The assertion directly follows from Theorem 3 applying for $h(x)=x^{-1}$ and $\alpha=s=1$, & $B(0)=B(1)=1$. □

Proposition 4.6. *For some $0\leq\varkappa_1<\varkappa_2$, and $q>1$, then*

$$\left|L^{-1}(\varkappa_1,\varkappa_2)-A^{-1}(\varkappa_1,\varkappa_2)-\frac{(\varkappa_2-\varkappa_1)^3}{12}A^{-3}(\varkappa_1,\varkappa_2)\right|$$
$$\leq \frac{(\varkappa_2-\varkappa_1)^3}{64}\left(\frac{2}{5}\right)^{\frac{1}{q}}\left\{\left(|\varkappa_1|^{-4q}+\frac{3}{2}|\varkappa_2|^{-4q}\right)^{\frac{1}{q}}+\left(\frac{3}{2}|\varkappa_1|^{-4q}+|\varkappa_2|^{-4q}\right)^{\frac{1}{q}}\right\}.$$

Proof. The assertion directly follows from Theorem 4 applying for $h(x)=x^{-1}$ and $\alpha=s=1$, & $B(0)=B(1)=1$. □

5. Conclusions

Fractional calculus is an intriguing subject with many applications in the modelling of natural events. We currently need to strengthen and improve our ability to generalize several recent results directly related to the subject of fractional calculus. Using fractional calculus tools and operators, many authors have generalized a variety of alternative fractional operators. Caputo-Fabrizio fractional integral is one of these operators. With regard to the Caputo Fabrizio fractional integral, Some new fractional integral Hermite Hadamard type inequalities for three times differentiable mapping have been established using the current fractional integral. Furthermore, we have obtained newly established inequalities using several special means. It is an intriguing and novel problem from which future scientists can obtain comparative inequalities for Atangana fractional.

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