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Superconvergence of Hermite rule for third order hypersingular integrals on interval

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Abstract. The boundary element method has been widely applied to a lot of practical problems,such as fluid mechanics and fracture mechanics. As one of the important topics in boundary element method, the numerical calculation of hypersingular integrals is of great importance. This article deals with the composite Hermite rule of the third order hypersingular integrals. Based on the error expansion, the superconvergence result of the composite Hermite formula is obtained. We show that the convergence rate is $O(h^3)$ when the local coordinate of the singular point is $\tau = 0$, which is one order higher than the global convergence. The accuracy of the result is verified by several numerical examples.

1. Introduction

A large number of practical problems in modern science and technology engineering, such as fracture mechanics, elasticity mechanics, mathematical physics, all involve the calculation of hypersingular integrals (also known as Hadamard Finite-Part Integral). Therefore, the calculation of supersingular integrals is of important research significance.

Hypersingular integrals are different from Riemann integrals and Lebesgue integrals. We take third order hypersingular integrals as an example, the following hypersingular integrals are considered

$$\mathcal{I}(f,s) = \fint_a^b \frac{f(x)}{(x-s)^3}dx, \tag{1}$$

there are various ways to define the equation (1), mathematically, these definitions have been proven to be equivalent.

In this article, we use the following definition

$$\fint_a^b \frac{f(x)}{(x-s)^3}dx = \lim_{\varepsilon\to 0}\left\{\int_a^{s-\varepsilon} \frac{f(x)}{(x-s)^3} + \int_{s-\varepsilon}^b \frac{f(x)}{(x-s)^3} - \frac{2f'(s)}{\varepsilon}\right\}, \quad s \in (a,b), \tag{2}$$

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where $f''(x)$ is Hölder continuous on interval $[a,b]$, $⨍_a^b$ denotes a hypersingular integral, s is the singular point, $f(x)$ is the density function,$1/(x-s)^3$ is the singular kernel.

The numerical calculation of hypersingular integrals is an important research contents of the boundary element method [9]. Approximation of weakly and strongly singular integrals [19, 20] were presented which was based on the spline quasi-interpolation quadrature rules. As the research moves along, a number of methods for calculating singular integrals have been proposed [8] such as the Gaussian method [4–6], the Newton-Cotes method [2, 13, 17], Extrapolation method [11] and some other methods [12, 14]. In reference [7], it is the first time to calculate the singular point as an independent variable, and the hypersingular integral is calculated when the density function is approximated by the Lagrange interpolation functions. The modified Newton-Cotes formula is studied when the singular point s coincides with interpolation point [10]. As for reference [15], it mainly studies the solution of third order hypersingular integrals by Newton-Cotes formula, and focuses on the superconvergence phenomenon based on the error expansion. The superconvergence phenomenon of second order Newton-Cotes formula for solving hypersingular integrals was first studied in [18]. Subsequently, the superconvergence phenomenons of the Newton-Cotes formula for Hadamard Finite-Part integrals were studied in [1].

In this paper, we mainly study the numerical solution of composite Hermite interpolation function for the third order hypersingular integral. Let

$$\phi(\tau) = (\tau-1)^2(\tau+1)^2, \quad \tau \in (-1,1), \tag{3}$$

and we define

$$\psi(t) = \begin{cases} -\dfrac{1}{2}⨍_{-1}^{1} \dfrac{\phi(\tau)}{\tau - t}\,\mathrm{d}\tau, & |t| < 1, \\ -\dfrac{1}{2}\displaystyle\int_{-1}^{1} \dfrac{\phi(\tau)}{\tau - t}\,\mathrm{d}\tau, & |t| > 1. \end{cases} \tag{4}$$

Based on the error analysis of the error functional, we study the superconvergence phenomenon with the convergence rate reach to $O(h^3)$ from $O(h^2)$, when the special function $\mathcal{S}'(\tau) = 0$ defined by

$$\mathcal{S}'(\tau) := \psi''(\tau) + \sum_{i=1}^{\infty} \left[\psi''(2i+\tau) + \psi''(-2i+\tau)\right], \quad \tau \in (-1,1), \tag{5}$$

where ψ is a function of second kind associated with apolynomial of equally distributed zeros[3], when the local coordinate of the singular point s is the zero of $\mathcal{S}'(\tau)$, the convergence rate is one higher than that of the global convergence order.

The content of this article is organized as following. In the second section, the general composite Hermite rule of hypersingular integral is proposed. In the third section, main conclusions of superconvergence are given and the proof of the main conclusion is obtained. In the fourth section, several numerical examples are given to verify the theoretical analysis.

2. Composite Hermite rule

Let $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be a uniform partition of the interval $[a,b]$ with mesh size $h = (b-a)/n$, we define a linear transformation

$$x = \hat{x}_i(\tau) := (\tau+1)\frac{x_{i+1}-x_i}{2} + x_i, \quad i = 0,1,\cdots,n-1, \quad \tau \in (-1,1), \tag{6}$$

from the reference element $[-1,1]$ to the subinterval $[x_i, x_{i+1}]$. The piecewise Hermite polynomial interpolation function is defined as following

$$\mathcal{H}_3(x) = \sum_{j=i}^{i+1} f\left(x_j\right)\alpha_j(x) + \sum_{j=i}^{i+1} f'\left(x_j\right)\beta_j(x), \quad x \in [x_i, x_{i+1}], \tag{7}$$

where $\alpha_i(x)$,$\alpha_{i+1}(x)$,$\beta_i(x)$,$\beta_{i+1}(x)$ are the Hermite interpolation basis functions [14] for the grid point x_i and x_{i+1}, and have

$$\alpha_i(x) = \left(1 - 2\frac{x - x_i}{h}\right)\frac{(x - x_{i+1})^2}{h^2}, \tag{8a}$$

$$\alpha_{i+1}(x) = \left(1 - 2\frac{x - x_{i+1}}{h}\right)\frac{(x - x_i)^2}{h^2}, \tag{8b}$$

$$\beta_i(x) = (x - x_i)\,\frac{(x - x_{i+1})^2}{h^2}, \tag{8c}$$

$$\beta_{i+1}(x) = (x - x_{i+1})\,\frac{(x - x_i)^2}{h^2}. \tag{8d}$$

Replacing $f(x)$ in equation (2) with $\mathcal{H}_3(x)$ gives the general composite Hermite rule

$$\begin{aligned} Q_n(s,f) :&= \fint_a^b \frac{\mathcal{H}_3(x)}{(x-s)^3}\,\mathrm{d}x \\ &= \sum_{i=0}^{n-1} \fint_{x_i}^{x_{i+1}} \frac{\mathcal{H}_3(x)}{(x-s)^3}\,\mathrm{d}x = \mathcal{I}(f,s) - \mathcal{E}_n(f,s), \end{aligned} \tag{9}$$

where $\mathcal{E}_n(f,s)$ denotes the error functional, for $x \in [x_i, x_{i+1}]$, it has the following integral formula

$$Q_n(f,s) = \sum_{i=0}^{n-1} \fint_{x_i}^{x_{i+1}} \frac{\mathcal{H}_3(x)}{(x-s)^3}\,\mathrm{d}x = \sum_{i=0}^{n-1}(a_i f_i + b_i f_{i+1} + c_i f_i' + d_i f_{i+1}'), \tag{10}$$

where we get

$$a_i(x) = \frac{6}{h^2} + \frac{1}{2(x_i - s)^2} + 3\frac{2s - 2x_i - h}{h^3}\ln\left|\frac{s - x_{i+1}}{s - x_i}\right|, \tag{11a}$$

$$b_i(x) = -\frac{6}{h^2} - \frac{1}{2(x_{i+1} - s)^2} + 3\frac{-2s + 2x_{i+1} - h}{h^3}\ln\left|\frac{s - x_{i+1}}{s - x_i}\right|, \tag{11b}$$

$$c_i(x) = \frac{-6s + 6x_i + h}{2h(x_i - s)} + \frac{3s - 3x_i - 2h}{h^2}\ln\left|\frac{s - x_{i+1}}{s - x_i}\right|, \tag{11c}$$

$$d_i(x) = \frac{-6s + 6x_{i+1} - h}{2h(x_{i+1} - s)} + \frac{3s - 3x_{i+1} + 2h}{h^2}\ln\left|\frac{s - x_{i+1}}{s - x_i}\right|. \tag{11d}$$

Let

$$\gamma(\tau) = \min_{0\le i\le n}\frac{|s - x_i|}{h} = \frac{1 - |\tau|}{2}, \tau \in (-1, 1), \tag{12}$$

where τ is the local coordinate of the singular point s.

In this paper, C is represented as a constant, which is independent of h and s, C in different formulas means different values.

Theorem 2.1. *Assume $f(x) \in C^{(4)}[a,b]$, and $s \neq x_i, i = 0,1,\cdots,n$, for the general composite Hermite rule $Q_n(s,f)$ defined in equation(10), there are the following error estimates*

$$\left|\mathcal{E}_n(f,s)\right| \leq C\gamma^{-2}(\tau)h^2. \tag{13}$$

Proof. Assume $R(x) = f(x) - \mathcal{H}_3(x)$, then we have $|R(x)| \leq Ch^4$, and

$$\begin{aligned}
\mathcal{E}_n(f,s) &= \int_a^b \frac{f(x)-\mathcal{H}_3(x)}{(x-s)^3}dx \\
&= \int_a^b \frac{R(x)}{(x-s)^3}dx \\
&= \sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}} \frac{R(x)}{(x-s)^3}\,\mathrm{d}x + \int_{x_m}^{x_{m+1}} \frac{R(x)}{(x-s)^3}\,\mathrm{d}x.
\end{aligned} \tag{14}$$

For the first part of equation (14), we have

$$\begin{aligned}
&\left|\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}} \frac{f(x)-\mathcal{H}_3(x)}{(x-s)^3}\,\mathrm{d}x\right| \\
&\leq Ch^4\left[\int_a^{x_m}\frac{1}{(x-s)^3}\,\mathrm{d}x + \int_{x_{m+1}}^b \frac{1}{(x-s)^3}\,\mathrm{d}x\right] \\
&\leq Ch^4\left[\left(\frac{1}{(a-s)^2}-\frac{1}{(b-s)^2}\right)-\left(\frac{1}{(x_m-s)^2}-\frac{1}{(x_{m+1}-s)^2}\right)\right] \\
&\leq C\gamma^{-2}(\tau)h^2.
\end{aligned}$$

For the second part of equation (14), taking the Taylor expansion for $R(x)$ at s, we have

$$\begin{aligned}
\int_{x_m}^{x_{m+1}}\frac{R(x)}{(x-s)^3}\,\mathrm{d}x =& \frac{(x_{m+1}+x_m-2s)h}{2(x_{m+1}-s)^2(x_m-s)^2}\cdot R(s) + \frac{h}{(x_m-s)(x_{m+1}-s)}\cdot R'(s) \\
&+ \frac{R''(s)}{2}\ln\frac{x_{m+1}-s}{s-x_m} + \int_{x_m}^{x_{m+1}}\frac{R^{(3)}(\theta(x))}{3!}\,\mathrm{d}x, \quad \theta(x)\in(x_m,x_{m+1}),
\end{aligned} \tag{15}$$

the errors of each part of the equation(15) are estimated respectively, we have

$$\begin{aligned}
&\left|\frac{(x_{m+1}+x_m-2s)h}{2(x_{m+1}-s)^2(x_m-s)^2}\cdot R(s)\right| \\
=&\left|-\frac{R(s)}{2}\left[\frac{1}{(x_{m+1}-s)^2}-\frac{1}{(x_m-s)^2}\right]\right| \leq C\gamma^{-2}(\tau)h^2,
\end{aligned} \tag{16}$$

$$\left|\frac{h}{(x_m-s)(x_{m+1}-s)}\cdot R'(s)\right| \leq C\gamma^{-1}(\tau)h^2, \tag{17}$$

$$\left|\frac{R''(s)}{2}\ln\frac{x_{m+1}-s}{s-x_m}\right| \leq C\left|\ln\gamma(\tau)\right|h^2, \tag{18}$$

$$\left|\int_{x_m}^{x_{m+1}}\frac{R^{(3)}(\theta(x))}{3!}\,\mathrm{d}x\right| \leq Ch^2. \tag{19}$$

The theorem 2.1 is proved. □

The supersingular integral, hypersingular integral and the Cauchy principal value integral have the following relationship [16]

$$\oint_a^b \frac{f(x)}{(x-s)^3}\,\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left(\oint_a^b \frac{f(x)}{(x-s)^2}\,\mathrm{d}x\right) = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}s^2}\left(\int\!\!\!\!-_a^b \frac{f(x)}{x-s}\,\mathrm{d}x\right). \tag{20}$$

We can write

$$\psi'(t) = \frac{d}{dt}\psi(t), \tag{21}$$

$$\psi''(t) = \frac{d}{dt}\psi'(t) = \frac{d^2}{dt^2}\psi(t), \tag{22}$$

we have

$$\psi''(t) = \begin{cases} -\oint_{-1}^{1} \dfrac{\phi(\tau)}{(\tau-t)^3}\,\mathrm{d}\tau, & |t|<1, \\ -\displaystyle\int_{-1}^{1} \dfrac{\phi(\tau)}{(\tau-t)^3}\,\mathrm{d}\tau, & |t|>1. \end{cases} \tag{23}$$

Let $\mathcal{J} := (-\infty,-1)\cup(-1,1)\cup(1,+\infty)$, define the operator $\mathcal{W} : C(\mathcal{J}) \to C(-1,1)$ as

$$\mathcal{W}f(\tau) := f(\tau) + \sum_{i=1}^{\infty}[f(2i+\tau) + f(-2i+\tau)], \quad \tau\in(-1,1), \tag{24}$$

it is evident that $\mathcal{W}$ is a linear operator, so we can write $\mathcal{S}'(\tau)$ by equation (24)

$$\mathcal{S}'(\tau) = \mathcal{W}\psi''(\tau) = \psi''(\tau) + \sum_{i=1}^{\infty}[\psi''(2i+\tau) + \psi''(-2i+\tau)], \quad \tau\in(-1,1). \tag{25}$$

3. Some lemmas

Lemma 3.1. [14] *Assume $f(x)\in C^{(5)}[a,b]$, $\mathcal{H}_3(x)$ be defined in equation (7), then for $x\in[x_i,x_{i+1}]$, and $s\in(a,b)$, there holds*

$$f(x) - \mathcal{H}_3(x) - \frac{f^{(4)}(s)}{4!}(x-x_i)^2(x-x_{i+1})^2 = E_i(x), \tag{26}$$

where

$$E_i(x) = \mathcal{N}_{i1}(x) + \mathcal{N}_{i2}(x) = \sum_{j=1}^{4} E_i^{(j)}(x) + \mathcal{N}_{i2}(x), \tag{27}$$

with

$$E_i^{(1)}(x) = \frac{f^{(5)}(\rho_{i1})}{5!}(x_i-x)^5\alpha_i(x),$$

$$E_i^{(2)}(x) = \frac{f^{(5)}(\rho_{i2})}{5!}(x_{i+1}-x)^5\alpha_{i+1}(x),$$

$$E_i^{(3)}(x) = \frac{f^{(5)}(\rho_{i3})}{4!}(x_i-x)^4\beta_i(x),$$

$$E_i^{(4)}(x) = \frac{f^{(5)}(\rho_{i4})}{4!}(x_{i+1}-x)^4\beta_{i+1}(x),$$

$$\mathcal{N}_{i2}(x) = \frac{f^{(5)}(\eta_i)}{4!}(x-s)(x-x_i)^2(x-x_{i+1})^2, \tag{28}$$

where $\rho_{ij} \in (x_i, x_{i+1}), \eta_i \in (x, s)$ *or* (s, x).

Lemma 3.2. *Assume* $s \in (x_m, x_{m+1})$, *let* $c_i = 2(s - x_i)/h - 1, 0 \le i \le n-1$, *then we have*

$$\psi''(c_i) = \begin{cases} -\dfrac{2^2}{h^2} \displaystyle\fint_{x_i}^{x_{i+1}} \dfrac{(x-x_i)^2 (x - x_{i+1})^2}{(x-s)^3}\, \mathrm{d}x, & i = m, \\ -\dfrac{2^2}{h^2} \displaystyle\int_{x_i}^{x_{i+1}} \dfrac{(x-x_i)^2 (x - x_{i+1})^2}{(x-s)^3}\, \mathrm{d}x, & i \neq m. \end{cases} \tag{29}$$

Proof. For the case of $i = m$, according to the supersingular integral calculation formula (2), we have

$$\begin{aligned} &\fint_{x_m}^{x_{m+1}} \frac{(x-x_i)^2 (x-x_{i+1})^2}{(x-s)^3}\, \mathrm{d}x \\ &= \lim_{\varepsilon \to 0} \left\{ \left(\int_{x_m}^{s-\varepsilon} + \int_{s+\varepsilon}^{x_{m+1}} \right) \frac{(x-x_i)^2 (x-x_{i+1})^2}{(x-s)^3}\, \mathrm{d}x \right. \\ &\left. - \frac{4(s-x_i)^2 (s - x_{i+1}) + 4(s-x_i)(s-x_{i+1})^2}{\varepsilon} \right\} \\ &= \left(\frac{h}{2}\right)^2 \lim_{\varepsilon \to 0} \left\{ \left(\int_{-1}^{c_m - \frac{2\varepsilon}{h}} + \int_{c_m + \frac{2\varepsilon}{h}}^{1} \right) \frac{\phi(\tau)}{(\tau - c_m)^3}\, \mathrm{d}\tau - \frac{h}{\varepsilon} \phi'(c_m) \right\} \\ &= \left(\frac{h}{2}\right)^2 \int_{-1}^{1} \frac{\phi(\tau)}{(\tau - c_m)^3}\, \mathrm{d}\tau \\ &= -\frac{h^2}{2^2} \psi''(c_m). \end{aligned} \tag{30}$$

The linear transformation $x = \hat{x}_i(\tau)$ is used in the proof, the case $i \neq m$ can be proved by using the same method in the correspondent Riemann integral. □

Lemma 3.3. *Assume* $f(x) \in C^{(5)}[a, b]$, *and* $E_m(x)$ *be defined in equation* (27), *there holds*

$$\left| \fint_{x_m}^{x_{m+1}} \frac{E_m(x)}{(x-s)^3}\, \mathrm{d}x \right| \le C\{\gamma^{-2}(\tau)\} h^3. \tag{31}$$

Proof. According to the definition of $E_m(x)$, then we have $|E_m(x)| \le Ch^5$, by performing Taylor expansion of $f(x)$ at the point s, we have

$$\begin{aligned} \fint_a^b \frac{f(x)}{(x-s)^3}\, \mathrm{d}x = &\fint_a^b \frac{f(s)}{(x-s)^3}\, \mathrm{d}x + \fint_a^b \frac{f'(s)}{(x-s)^2}\, \mathrm{d}x \\ &+ \frac{1}{2} \fint_a^b \frac{f''(s)}{(x-s)}\, \mathrm{d}x + \int_a^b \frac{f(x) - f(s) - f'(s)(x-s) - f''(s)(x-s)^2/2}{3!}\, \mathrm{d}x, \end{aligned} \tag{32}$$

we have

$$\begin{aligned} \fint_{x_m}^{x_{m+1}} \frac{E_m(x)}{(x-s)^3}\, \mathrm{d}x = &\frac{(x_{m+1} + x_m - 2s)h}{2(x_{m+1} - s)^2 (x_m - s)^2} \cdot E_m(s) + \frac{h}{(x_m - s)(x_{m+1} - s)} \cdot E'_m(s) \\ &+ \frac{E''_m(s)}{2} \ln \frac{x_{m+1} - s}{s - x_m} + \int_{x_m}^{x_{m+1}} \frac{E_m^{(3)}(\theta(x))}{3!}\, \mathrm{d}x, \quad \theta(x) \in (x_m, x_{m+1}). \end{aligned} \tag{33}$$

The errors of each part of the equation(33) are estimated respectively, we have

$$\begin{aligned}&\left|\frac{(x_{m+1}+x_m-2s)h}{2(x_{m+1}-s)^2(x_m-s)^2}\cdot E_m(s)\right|\\ =&\left|-\frac{E_m(s)}{2}\left[\frac{1}{(x_{m+1}-s)^2}-\frac{1}{(x_m-s)^2}\right]\right|\leq C\gamma^{-2}(\tau)h^3 \qquad (34)\end{aligned}$$

$$\left|\frac{h}{(x_m-s)\,(x_{m+1}-s)}\cdot E'_m(s)\right|\leq C\gamma^{-1}(\tau)h^3 \qquad (35)$$

$$\left|\frac{E''_m(s)}{2}\ln\frac{x_{m+1}-s}{s-x_m}\right|\leq C\left|\ln\gamma(\tau)\right|h^3 \qquad (36)$$

$$\left|\int_{x_m}^{x_{m+1}}\frac{E_m^{(3)}(\theta(x))}{3!}\,\mathrm{d}x\right|\leq Ch^3. \qquad (37)$$

From equation (34) to (37), we can obtain

$$\left|\,⨎_{x_m}^{x_{m+1}}\frac{E_m(x)}{(x-s)^3}\,\mathrm{d}x\right|\leq C\{\gamma^{-2}(\tau)\}h^3. \qquad (38)$$

The lemma 3.3 is proved. □

Now we give the Theorem 3.4 as following.

Theorem 3.4. *Assume $f(x)\in C^{(5)}[a,b]$, for the general composite Hermite rule $Q_n(s,f)$ defined in equation (10), there are the following error estimates*

$$\mathcal{E}_n(f,s)=-\frac{h^2f^{(4)}(s)}{2^2\cdot 4!}\mathcal{S}'(\tau)+R_n(f), \qquad (39)$$

where

$$\left|R_n(f)\right|\leq C\{\gamma^{-2}(\tau)+\eta(s)h\}h^3, \qquad (40)$$

$\gamma(\tau)$ is defined in Equation (12), we define

$$\eta(s)=\max\left\{\frac{1}{(s-a)^2},\frac{1}{(b-s)^2}\right\}. \qquad (41)$$

Proof. As

$$\begin{aligned}⨎_a^b\frac{f(x)-\mathcal{H}_3(x)}{(x-s)^3}\,\mathrm{d}x&=\sum_{i=0}^{n-1}⨎_{x_i}^{x_{i+1}}\frac{f(x)-\mathcal{H}_3(x)}{(x-s)^3}\,\mathrm{d}x\\&=-\frac{h^2f^{(4)}(s)}{2^2\cdot 4!}\mathcal{S}'(\tau)+\mathcal{R}_n(f), \qquad (42)\end{aligned}$$

where

$$\mathcal{R}_n(f)=\mathcal{R}_n^1(s)+\mathcal{R}_n^2(s)+\mathcal{R}_n^3(s), \qquad (43)$$

$$\mathcal{R}_n^1(s)=⨎_{x_m}^{x_{m+1}}\frac{E_m(x)}{(x-s)^3}\,\mathrm{d}x, \qquad (44)$$

$$\mathcal{R}_n^2(s)=\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\frac{\mathcal{N}_{i1}(x)}{(x-s)^3}\,\mathrm{d}x+\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\frac{\mathcal{N}_{i2}(x)}{(x-s)^3}\,\mathrm{d}x, \qquad (45)$$

$$\mathcal{R}_n^3(s)=\frac{f^{(4)}(s)h^2}{2^2\cdot 4!}\left[\sum_{i=m+1}^{\infty}\psi''(2i+\tau)+\sum_{i=n-m}^{\infty}\psi''(-2i+\tau)\right]. \qquad (46)$$

As for $\mathcal{R}_n^1(s)$, the proof has been completed in Lemma 3.3, which means

$$| \mathcal{R}_n^1(s) |\leq C\gamma^{-2}(\tau)h^3. \tag{47}$$

As for $\mathcal{R}_n^2(s)$, we know that

$$\mathcal{R}_n^2(s) = \sum_{i=0,i\neq m}^{n-1} \int_{x_i}^{x_{i+1}} \frac{\mathcal{N}_{i1}(x)}{(x-s)^3}\,\mathrm{d}x + \sum_{i=0,i\neq m}^{n-1} \int_{x_i}^{x_{i+1}} \frac{\mathcal{N}_{i2}(x)}{(x-s)^3}\,\mathrm{d}x, \tag{48}$$

where

$$\mathcal{N}_{i1}(x) = \sum_{j=1}^{4} E_i^{(j)}(x),$$
$$\mathcal{N}_{i2}(x) = \frac{f^{(5)}(\eta_i)}{4!}(x-s)(x-x_i)^2(x-x_{i+1})^2,$$

we can calculate it

$$\left|\sum_{i=0,i\neq m}^{n-1} \int_{x_i}^{x_{i+1}} \frac{\mathcal{N}_{i1}(x)}{(x-s)^3}\,\mathrm{d}x\right| \leq Ch^5 \sum_{i=0,i\neq m}^{n-1} \int_{x_i}^{x_{i+1}} \frac{1}{(x-s)^3}\,\mathrm{d}x \leq C\gamma^{-2}(\tau)h^3,$$
$$\left|\sum_{i=0,i\neq m}^{n-1} \int_{x_i}^{x_{i+1}} \frac{\mathcal{N}_{i2}(x)}{(x-s)^3}\,\mathrm{d}x\right| \leq Ch^4 \sum_{i=0,i\neq m}^{n-1} \int_{x_i}^{x_{i+1}} \frac{1}{(x-s)^2}\mathrm{d}x \leq C\gamma^{-1}(\tau)h^3,$$

then we get

$$| \mathcal{R}_n^2(s) |\leq C\gamma^{-2}(\tau)h^3. \tag{49}$$

Where

$$|\mathcal{N}_{i1}(x)| \leq Ch^5, \quad \left|\frac{\mathcal{N}_{i2}(x)}{x-s}\right| \leq Ch^4.$$

has been used.

As for $\mathcal{R}_n^3(s)$, the function $\phi''(\tau)$, defined in equation (23), $s = x_m + \frac{\tau+1}{2}h = a + \left(m + \frac{\tau+1}{2}\right)h$ is already known, then we get $2(s-a)/h = \tau + 2m + 1$, and we have

$$\left|\sum_{i=m+1}^{\infty} \psi''(2i+\tau)\right| \leq C \sum_{i=m+1}^{\infty} \int_{-1}^{1} \frac{dt}{|t-2i-\tau|^3}$$
$$= C\int_{-2m-3-\tau}^{\infty} \frac{dx}{x^3} = \frac{C}{[-(\tau+2m+1)-2]^2} \leq \frac{Ch^2}{(s-a)^2}. \tag{50}$$

For the other part, we know $b = a + nh$, then we get $2(b-s)/h = 2(n-m) - 1 - \tau$, and

$$\left|\sum_{i=n-m}^{\infty} \psi''(-2i+\tau)\right| \leq C \sum_{i=n-m}^{\infty} \int_{-1}^{1} \frac{dt}{|t-(-2i+\tau)^3|}$$
$$= C\int_{2(n-m)-1-\tau}^{\infty} \frac{dx}{x^3} = \frac{C}{[2(n-m)-1-\tau]^2} = \frac{Ch^2}{(b-s)^2}. \tag{51}$$

Combining equation (50)and equation (51), we have

$$\left|\sum_{i=m+1}^{\infty}\psi''(2i+\tau)+\sum_{i=n-m}^{\infty}\psi''(-2i+\tau)\right|\le C\eta(s)h^2. \tag{52}$$

$\eta(s)$ is defined as equation (41), then we have

$$\mathcal{R}_n^3(s)=\left|\frac{f^{(4)}(s)h^2}{2^2\cdot 4!}\left[\sum_{i=m+1}^{\infty}\psi''(2i+\tau)+\sum_{i=n-m}^{\infty}\psi''(-2i+\tau)\right]\right|\le C\eta(s)h^4. \tag{53}$$

Above all, we get

$$\begin{aligned}\left|\mathcal{R}_n(f)\right| &\le \left|\mathcal{R}_n^1(s)\right|+\left|\mathcal{R}_n^2(s)\right|+\left|\mathcal{R}_n^3(s)\right|\\ &\le C\{\gamma^{-2}(\tau)+\eta(s)h\}h^3.\end{aligned} \tag{54}$$

The theorem 3.4 is proved. □

Remark 3.5. *In the same conditions as theorem 3.4, when τ^* is the zero of $\mathcal{S}'(\tau)=0$, then we have*

$$\mid \mathcal{E}_n(f,s)\mid\le C\{\gamma^{-2}(\tau^*)+\eta(s)h\}h^3. \tag{55}$$

Theorem 3.6. *For the error functional special function, defined as equation $\mathcal{S}'(\tau)$, has least one zero in $(-1,1)$.*

Proof. Let $Q_n(x)$ be the function of the second kind Legendre polynomial, and

$$Q_0(x)=\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|,\quad Q_1(x)=xQ_0(x)-1. \tag{56}$$

We know $\phi(\tau)$ and $\psi(\tau)$ is defined as (3) and (4), we have

$$\left|\psi''(\tau)\right|=8Q_2(\tau),\quad \tau\in(-1,1), \tag{57}$$

where

$$Q_2(\tau)=\frac{3\tau^2-1}{4}\ln\frac{1+\tau}{1-\tau}-\frac{3}{2}\tau,\quad \tau\in(-1,1),$$

if $\tau=0$,we have

$$\psi''(\tau)=0, \tag{58}$$

$$\sum_{i=1}^{\infty}\psi''(-2i)=\sum_{i=1}^{\infty}-\psi''(2i), \tag{59}$$

then we get

$$\mathcal{S}'(0)=0, \tag{60}$$

so $\tau=0$ is the zero of $\mathcal{S}'(\tau)=0$. The theorem 3.6 is proved . □

In Theorem 3.6 we have prove that $\mathcal{S}'(0)=0$, by Figure 1, we can see that there is only one zero in $(-1,1)$.

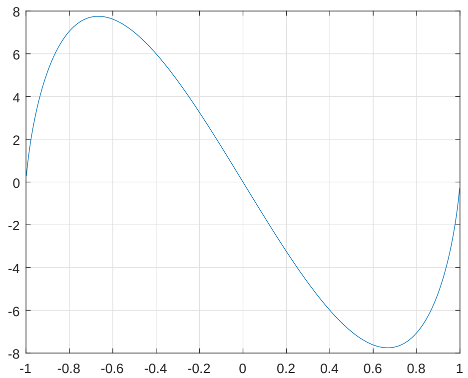

Figure 1: The figure of special function $\mathcal{S}'(\tau) = 0$

4. Numerical Examples

Example 4.1. *Consider the following third order hypersingular integrals*

$$\int_0^1 \frac{x^6}{(x-s)^3}\,dx = \frac{60s^5 - 90s^4 + 20s^3 + 5s^2 + 2s + 1}{4(s-1)^2} + 15s^4 \ln\frac{1-s}{s}, \quad s \in (0,1).$$

Table 1 shows the error of the Hermite integral formula with singular point $s = x_{[n/4]} + (1+\tau)h/2$, the convergence order of the superconvergence point is $O(h^3)$, which is higher than that of the non-superconvergence point. The accuracy of the Hermite integral formula with singular point $s = a + (1+\tau)h/2$ is $O(h^4)$,which is organized in Table 2. Table 3 shows that the accuracy of the Hermite integral formula with singular point $s = b - (1+\tau)h/2$ is $O(h^2)$. The theoretical analysis is justified by calculated results.

Table 1: Error of the Hermite rule with $s = x_{[n/4]} + (1+\tau)h/2$.

n	$\tau = 0$	$\tau = -0.5$	$\tau = 0.5$	$\tau = -0.8$	$\tau = 0.8$
8	3.2798E-04	3.6702E-02	-4.4652E-02	3.1352E-02	-5.2754E-02
16	2.4109E-05	7.6891E-03	-8.6092E-03	7.0666E-03	-9.5004E-03
32	2.1738E-06	1.7557E-03	-1.8656E-03	1.6858E-03	-1.9745E-03
64	2.2586E-07	4.1927E-04	-4.3269E-04	4.1231E-04	-4.4741E-04
128	2.5601E-08	1.0243E-04	-1.0409E-04	1.0199E-04	-1.0632E-04
h^α	3.20	2.10	2.15	2.05	2.20

Table 2: Error of the Hermite rule with $s = a + (1+\tau)h/2$.

n	$\tau = 0$	$\tau = -0.5$	$\tau = 0.5$	$\tau = -0.8$	$\tau = 0.8$
8	-1.2593E-04	1.1482E-03	-3.1900E-03	6.0626E-04	-5.5738E-03
16	-1.3953E-05	6.6091E-05	-2.0591E-04	3.2452E-05	-3.5520E-04
32	-1.2261E-06	3.7886E-06	-1.3236E-05	1.6931E-06	-2.2574E-05
64	-9.8006E-08	2.1578E-07	-8.4902E-07	8.5020E-08	-1.4329E-06
128	-7.4272E-09	1.2195E-08	-5.4376E-08	4.0295E-09	-9.0874E-08
h^α	3.58	4.12	3.96	4.30	3.98

Table 3: Error of the Hermite rule with $s = b - (1+\tau)h/2$.

n	$\tau = 0$	$\tau = -0.5$	$\tau = 0.5$	$\tau = -0.8$	$\tau = 0.8$
8	1.4547E-02	-3.3611E-01	3.5606E-01	-2.9323E-01	3.3809E-01
16	3.2503E-03	-8.9185E-02	9.6283E-02	-7.6794E-02	9.4122E-02
32	7.6667E-04	-2.2954E-02	2.5018E-02	-1.9641E-02	2.4797E-02
64	1.8615E-04	-5.8213E-03	6.3753E-03	-4.9657E-03	6.3617E-03
128	4.1498E-05	-1.4674E-03	1.5892E-03	-1.2495E-03	1.4792E-03
h^α	2.11	2.09	1.95	2.06	1.96

Example 4.2. *Consider the following third order hypersingular integrals*

$$\int_0^1 \frac{x^5+1}{(x-s)^3}\,\mathrm{d}x = 10s^2 + 5s + \frac{10}{3} + \frac{5s+4}{2s^2} + \frac{s-3}{2s^2(s-1)^2} + 10s^3 \ln\frac{1-s}{s}, \quad s \in (0,1).$$

The error results of the Hermite integral formula with singular point $s = x_{[n/4]} + (1+\tau)h/2$ are organized in Table 4, we can see that the accuracy is $O(h^3)$ at the superconvergence points, the convergence order of the non-superconvergence point is $O(h^2)$. As the singular point is close to a, the results showed that the convergence rate of the Hermite integral formula is $O(h^3)$, which is gathered in Table 5. Furthermore, from Table 6, the quadrature rule reach the convergence rate of $O(h^2)$ when the singular point $s = b - (1+\tau)h/2$. The results are consistent with our theoretical analysis.

Table 4: Error of the Hermite rule with $s = x_{[n/4]} + (1+\tau)h/2$.

n	$\tau = 0$	$\tau = -0.5$	$\tau = 0.5$	$\tau = -0.8$	$\tau = 0.8$
8	2.0159E-04	4.0618E-02	-4.4803E-02	3.7396E-02	-4.8754E-02
16	1.6641E-05	9.3394E-03	-9.8799E-03	8.9727E-03	-1.0410E-02
32	1.5387E-06	2.2346E-03	-2.3033E-03	2.1978E-03	-2.3786E-03
64	1.5792E-07	5.4622E-04	-5.5487E-04	5.4390E-04	-5.6656E-04
128	1.7548E-08	1.3501E-04	-1.3609E-04	1.3529E-04	-1.3812E-04
h^α	3.37	2.07	1.98	2.04	1.74

Table 5: Error of the Hermite rule with $s = a + (1+\tau)h/2$.

n	$\tau = 0$	$\tau = -0.5$	$\tau = 0.5$	$\tau = -0.8$	$\tau = 0.8$
8	4.8623E-04	7.1656E-03	-1.0487E-02	4.8013E-03	-1.4190E-02
16	5.7843E-05	8.9297E-04	-1.3141E-03	5.9755E-04	-1.7771E-03
32	7.0597E-06	1.1146E-04	-1.6444E-04	7.4533E-05	-2.2232E-04
64	8.7216E-07	1.3922E-05	-2.0565E-05	9.3066E-06	-2.7800E-05
128	1.0843E-07	1.7396E-06	-2.5712E-06	1.1621E-06	-3.4757E-06
h^α	3.04	3.00	3.00	3.05	3.00

5. Conclusion

In this paper, we study the Hermite integral formula for numerical evaluation integrals defined on interval with third order hypersingular kernel. The superconvergence phenomenon occurs at the midpoint of the subinterval. The convergence order of the superconvergence point is $O(h^3)$, which is one order higher than general. To sum up, an effective method for calculating third order hypersingular integrals is summarized, and its correctness is proved.

Table 6: Error of the Hermite rule with $s = b - (1 + \tau)h/2$.

n	$\tau = 0$	$\tau = -0.5$	$\tau = 0.5$	$\tau = -0.8$	$\tau = 0.8$
8	4.3209E-03	-1.1882E-01	1.2811E-01	-1.0234E-01	1.2503E-01
16	1.0209E-03	-3.0599E-02	3.3340E-02	-2.6185E-02	3.3033E-02
32	2.4809E-04	-7.7614E-03	8.4993E-03	-6.6208E-03	8.4804E-03
64	6.1145E-05	-1.9543E-03	2.1454E-03	-1.6645E-03	2.1479E-03
128	1.9155E-05	-4.8854E-04	5.5453E-04	-4.1608E-04	6.3638E-04
h^α	1.95	1.98	1.96	1.99	1.90

References

[1] J.M. Wu, W.W. Sun, *The superconvergence of Newton-Cotes rules for the Hadamard finite-part integral on an interval*, Numer. Math. 109 (2008)143–165.

[2] W.W. Sun,J.M. Wu, *Newton-Cotes formulae for the numerical evaluation of certain hypersingular integral*, Computing, 75 (2005) 297–309.

[3] L.C. Andrews, *Special Functions of Mathematics for Engineers*, 2nd ed., McGraw-Hill, Inc, NewYork, 1992.

[4] C.Y. Hui,D. Shia, *Evaluations of hypersingular integrals using Gaussian quadrature*, Int J Numer Methods Eng, 44 (1999)205–214.

[5] N.I. Ioakimidis, *On the uniform convergence of Gaussian quadrature rules for Cauchy principal value integrals and their derivatives*, Math. Comp. 44 (1985)191–198.

[6] W.W. Sun,J.M. Wu,*Interpolatory quadrature rules for Hadamard finite-part integrals and the superconvergence*, IMA J. Numer.Anal. 28 (2008)580–597.

[7] J.M. Wu,D.H. Yu, *An approximate computation of hypersingular integrals on interval*, Journalon Numerical Methods and Computer Applications, 19(1998) 118–126.

[8] Y. Liu,J. Li,Q.Y. Hu,el,*Some Research Progress on Boundary Element Methods*,Journal of Computational Mathematics,42 (2020)310–348.

[9] D.H. Yu,*Natural boundary element method and its application*,Science Press,Beijing,2017.

[10] D.H. Yu, *Natural boundary integrals method and its applications*,Science Press,Beijing,2002.

[11] M.H. Xia,J. Li,*Extrapolation Method for Cauchy Principal Value Integral with Classical Rectangle Rule on Interval* CMES-Computer Modeling in Engineering & Sciences, 115 (2018)313–326.

[12] J. Li, B. Gong, W. Liu, *The Rectangle Rule for Computing Cauchy Principal Value Integral on Circle*, Mathematical Problems in Engineering, 6 (2016) 98–107.

[13] J. Li, Z.Q. Wang, *Simpson's rule to approximate Hilbert integral and its application*, Applied Mathematics and Computation, 339(2018)398–409.

[14] G.D. Gu, S. An, M.L. Zhao ,*The Cubic Spline Rule for the Hadamard Finite-Part Integral on an Interval*,Numerical Mathematics: Theory, Methods & Applications, 12 (2019)906–922

[15] J. Li, X.P. Zhang,D.H. Yu, *Superconvergence and ultraconvergence of Newton-Cotes rules for supersingular integrals*, Journal of Computational and Applied Mathematics, 233(2010)2841–2854.

[16] J. Li, D.H. Yu, *Numerical calculation and application of hypersingular integrals*, Science Press, 2022.

[17] X.P. Zhang, J.M. Wu,D.H. Yu , *Superconvergence of the composite Simpson's rule for a certain finite-part integral and its applications*, Journal of Computational and Applied Mathematics, 223(2009)598–613.

[18] J.M. Wu, Y. Lv, *A superconvergence result for the second order Newton-Cotes formula for certain finite part integrals*, IMA J. Numer. Anal. 25 (2005)253–263.

[19] F. Calabro, A. Falini, M. L. Sampoli, A. Sestini, *Efficient quadrature rules based on spline quasi-interpolation for application to IGA-BEMs* , Journal of Computational and Applied Mathematics 338 (2018) 153–167.

[20] A. Aimi, F. Calabro, A. Falini, M. L. Sampoli, A. Sestini, *Quadrature formulas based on spline quasi-interpolation for hypersingular integrals arising in IgA-SGBEM*, Computer Methods in Applied Mechanics and Engineering. 372 (2020) 113441.