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Result on Controllability of Hilfer fractional integro-differential equations of Sobolev-type with Non-instantaneous Impulses

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Abstract. This paper is concerned with the existence and controllability results for a class of Hilfer fractional differential equations of Sobolev-type with non-instantaneous impulse in Banach space. In order to bring off the main results, the author used the theory of propagation family $\{\mathscr{P}(\tau)\}_{\tau\geq 0}$ (generated by the operator pair $(\mathscr{A},\mathscr{R})$), measure of non-compactness, and the fixed point methods. The primary goal of this study is to determine the controllability of a dynamical system without assuming that $\mathscr{R}^{-1}$ is a bounded operator, and no relationship between the domain of the operators $\mathscr{A}$ and $\mathscr{R}$. At the end, we provide an example to illustrate the main results.

1. Introduction

Numerous evolutionary processes that are subject to sudden changes in state occur at certain time instant, that sudden changes can be well approximated as being in the form of impulses. These processes are characterised by impulsive differential equations, (see for instance, [8, 10, 24, 40]). Natural disasters such as tsunamis, earthquakes, volcano eruptions, shocks etc., are the processes that involve negligible time instants, sudden changes in their states and these short-term disturbances are predicted as instantaneous impulses. Furthermore, drug distribution in the bloodstream and subsequent body absorption are moderate and consequential evolution processes, whose dynamics are non-interpretable by instantaneous impulsive models. These circumstance might be explained by an impulsive action that begins suddenly and lasts for a limited amount of time, such impulsive effects that stays for finite time interval are known as non-instantaneous impulse [25–27]. On the other hand, fractional calculus is a growing area of research which deals with the derivatives and integrals of non-integer order. Nowdays, it became a major branch of mathematics. In several scientific fields, the possibility of fractional calculus has been successfully applied. Fractional order models are better than integer order models for several sorts of realistic application, [4, 6, 20, 22, 29, 35–38]. In twentieth century, Hilfer [21] introduced the generalised Riemann-Liouville fractional derivative known as Hilfer fractional derivative. Hilfer fractional derivative have a large number of applications in the field of fractional calculus. This operator appeared in the theoretical simulations of dielectric relaxation in the

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glass forming materials [15, 18, 23]. Furati et al. [17] discussed the existence and uniqueness for Hilfer fractional derivative with initial value problem. Gu and Trujillo [18] investigated the existence of the mild solution for a class of evolution equations with Hilfer fractional derivative using Laplace transform and probability density function. Controllability is the elementary concept in the mathematical control theory which plays an important role in the development of modern mathematical control theory. Controllability is a qualitative property of dynamical control systems which means that various dynamical systems can be steer from an initial state to any arbitrary final state with the help of some admissible controls. Many authors have investigated controllability results extensively in both finite and infinite dimensional spaces, for more details on it, readers refer to [2, 3, 10, 16, 18, 23, 28, 39, 41] and reference cited therein. Moreover, Sobolev-type equation appears in a variety of physical problems such as the flow of fluid through fissured rocks, thermodynamics, propagation of long waves of small amplitude, and so on [7, 13, 33]. Liang and Xiao [31], studied the existence results for the mild solutions of fractional integro-differential equations of Sobolev-type with nonlocal initial conditions in a separable Banach space using the theory of propagation family the theory of measure of non-compactness [30]. Brill [11] established the existence results of the mild solution for a semi linear Sobolev-type differential equation in a Banach space. Recently, many authors [1, 5, 9] investigated qualitative behaviours including existence, controllability and stability results. Dineshkumar [16] et al. are formulated necessary and sufficient condition which guarantees the approximate controllability of Sobolev-type Hilfer neutral fractional stochastic differential inclusion. Controllability problem for various kind of systems are described by fractional differential Sobolev-type Hilfer equations still have a lack of contributions. In [42] Sousa et al. extended the existence of mild solutions of the Hilfer fractional differential system with non-instantaneous impulses using the results of equicontinuous, (α, ς)-resolvent operator function and the Kuratowski measure of non-compactness in the Banach space. Kumar et al. [23] investigated the controllability of Hilfer fractional integro-differential equation of Sobolev-type with non-local condition in Banach space. To the best of authors knowledge, existence and controllability results for a class of Hilfer fractional integro-differential equations of Sobolev-type with non-instantaneous impulses has not been yet reported in the literature.

Motivated by the above discussion, in this study we consider the Hilfer fractional integro-differential system of Sobolev-type with non-instantaneous impulses as follows:

$$\begin{cases} \mathscr{D}_{0^+}^{\varsigma,\alpha}\mathscr{R}\varkappa(\tau) = \mathscr{A}\varkappa(\tau) + \mathscr{R}\mathscr{F}\left(\tau, \varkappa(\tau), \int_0^\tau \mathfrak{G}(\tau, \mathfrak{r}, \varkappa(\mathfrak{r}))\,d\mathfrak{r}\right) + \mathscr{R}\mathcal{B}w(\tau), & \tau \in \bigcup_{\mathrm{j}=0}^{q}(\omega_{\mathrm{j}}, \tau_{\mathrm{j}+1}], \\ \varkappa(\tau) = \zeta_{\mathrm{j}}(\tau, \varkappa(\tau)), & \tau \in \bigcup_{\mathrm{j}=1}^{q}(\tau_{\mathrm{j}}, \omega_{\mathrm{j}}], \\ I_{0^+}^{\Upsilon}\mathscr{R}\varkappa(0) = \mathscr{R}\varkappa_0, \end{cases} \tag{1}$$

where $\Upsilon = \varsigma + \alpha - \varsigma\alpha$, $\mathscr{D}_{0^+}^{\varsigma,\alpha}$ is Hilfer fractional derivative of type α $(0 \leq \alpha \leq 1)$, and order ς $(0 < \varsigma < 1)$, $\varkappa(\cdot)$ is a state function in Banach space $\mathcal{X}$; $I_{0^+}^{\Upsilon}$ is the Riemann-Liouville fractional integral of order $\Upsilon > 0$; $\mathscr{A}$ and $\mathscr{R}$ are closed linear operators (not necessarily bounded) with domains contained in $\mathcal{X}$; $w(\cdot) \in \mathcal{L}^2(\mathfrak{J}, \mathscr{W})$, denotes the control function and $\mathscr{W}$ is an another Banach space; $\mathcal{B}$ is a bounded linear operator from $\mathscr{W}$ into $\mathscr{D}(\mathscr{R})$; the nonlinear functions $\mathscr{F}\colon \mathfrak{J} \times \mathcal{X} \times \mathcal{X} \to \mathscr{D}(\mathscr{R})$ and $\mathfrak{G}\colon \Sigma \times \mathcal{X} \to \mathcal{X}$ are to be specified later, in addition, $0 = \omega_0 < \tau_1 < \omega_1 < \tau_2 < \omega_2 \cdots < \tau_q < \omega_q < \tau_{q+1} = \mathfrak{b}$ and $\zeta_{\mathrm{j}} : [\tau_{\mathrm{j}}, \omega_{\mathrm{j}}] \times \mathcal{X} \to \mathcal{X}$ is non-instantaneous impulsive function for all $\mathrm{j} = 1, 2, 3, \cdots, q$. $\Sigma = \{(\tau, \mathfrak{r}) \mid 0 \leq \mathfrak{r} \leq \tau \leq \mathfrak{b}\}$; $\varkappa_0 \in \mathscr{D}(\mathscr{R})$, here $\mathcal{X} = \{\varkappa \in \mathscr{C}(\mathfrak{J}', \mathcal{X}) : \lim_{\tau \to 0} \tau^{(1-\alpha)(1-\varsigma)}\varkappa(\tau)$ exists and finite$\}$ and $\mathfrak{J} = [0, \mathfrak{b}]$.

Without putting any limitations on the operator $\mathscr{R}$ (i.e., the operator $\mathscr{R}$ does not necessarily have a bounded inverse), as well as without considering any relationship between $\mathscr{D}(\mathscr{A})$ and $\mathscr{D}(\mathscr{R})$, we illustrate our main result using the theory of propagation family $\{\mathscr{P}(\tau),\ \tau \geq 0\}$ generated by the operators $\mathscr{A}$ and $\mathscr{R}$. The definition of the propagation family of operators $\mathscr{A}$ and $\mathscr{R}$ will be given in the next section. We also use fractional calculus theory, measure of non-compactness and Sadovskii's fixed point theorem in our analysis. The exact controllability of Sobolev-type fractional integro-differential system (1) with non-instantaneous impulses in a Banach space $\mathcal{X}$ has not yet been studied in any of the research paper. The rest of the manuscript is structured as follows: Section 2 provides some fundamental definitions and preliminary results, which are useful in the later sections. In section 3, we take some assumptions which help us to prove the main results. In the same section, we prove fractional differential system (1) is controllable with the help of measure of non-compactness and Sadovaskii's fixed point theorm. Finally in section 4, we discuss a concrete example to demonstrate the application of the developed theory.

2. Preliminaries

Let $\mathscr{C}(\mathfrak{J},\mathcal{X})(\mathfrak{J}=[0,\mathfrak{b}])$ be the space of all continuous function on $\mathfrak{J}$ with the norm given by

$$\|\varkappa\|=\sup_{\tau\in\mathfrak{J}}|\varkappa(\tau)|,$$

and

$$\mathscr{C}_{1-\gamma}(\mathfrak{J},\mathcal{X})=\{\varkappa\in\mathscr{C}(\mathfrak{J}',\mathcal{X}):\tau^{1-\Upsilon}\varkappa(\tau)\in\mathscr{C}(\mathfrak{J},\mathcal{X})\},$$

be the weighted space of function on $\mathfrak{J}'=(0,\mathfrak{b}]$, where $0\leq\Upsilon\leq1$ with the norm $\|\varkappa\|_{\mathscr{C}_{1-\Upsilon}}=\sup_{\tau\in\mathfrak{J}'}\|\tau^{1-\Upsilon}\varkappa(\tau)\|$ is a Banach space. On the other hand,

$$\mathscr{P}\mathscr{C}_{1-\Upsilon}(\mathfrak{J},\mathcal{X})=\{(\tau-\tau_{\mathfrak{j}})^{1-\Upsilon}\varkappa(\tau)\in\mathscr{C}_{1-\Upsilon}((\tau_{\mathfrak{j}},\tau_{\mathfrak{j}+1}],\mathscr{R}):\lim_{\tau\to\tau_{\mathfrak{j}}}(\tau-\tau_{\mathfrak{j}})^{1-\gamma}\varkappa(\tau)\}$$

exist and finite for $\mathfrak{j}=1,2,\cdots q$, denotes the space of piecewise continuous function with the norm is given by

$$\|\varkappa(\tau)\|_{\mathscr{P}\mathscr{C}_{1-\Upsilon}}=\max\left\{\sup_{\tau\in\mathfrak{J}}\|\tau^{1-\Upsilon}\varkappa(\tau^{+})\|,\sup_{\tau\in\mathfrak{J}}\|\tau^{1-\Upsilon}\varkappa(\tau^{-})\|\right\},$$

where $\varkappa(\tau^{+})$ and $\varkappa(\tau^{-})$ represent, the right and left limits of $\varkappa(\tau)$ at $\tau\in\mathfrak{J}$ respectively.

Definition 2.1 ([36]). *The Riemann-Liouville fractional integral of order $\varsigma>0$ for a function $\mathscr{F}\in\mathcal{L}^{1}(\mathfrak{J},\mathcal{X})$ can be written as*

$$\mathcal{I}^{\varsigma}_{0^{+}}\mathscr{F}(\tau)=\frac{1}{\Gamma(\varsigma)}\int_{0}^{\tau}(\tau-\nu)^{\varsigma-1}\mathscr{F}(\nu)d\nu,\quad\tau>0,$$

where Γ is the gamma function.

Definition 2.2 ([36]). *The Riemann-Liouville fractional derivative of order ς, $0\leq\mathfrak{n}-1<\varsigma<\mathfrak{n}$, is defined as*

$$\mathscr{D}^{\varsigma}_{0^{+}}\mathscr{F}(\tau)=\frac{1}{\Gamma(\mathfrak{n}-\varsigma)}\frac{d^{\mathfrak{n}}}{d\tau^{\mathfrak{n}}}\int_{0}^{\tau}\frac{\mathscr{F}(\nu)}{(\tau-\nu)^{\varsigma+1-\mathfrak{n}}}d\nu,\quad\tau>0,$$

where $\mathscr{F}$ is an $\mathfrak{n}$-times continuous differentiable function.

Definition 2.3 ([21]). *The Hilfer fractional derivative of the type α $(0\leq\alpha\leq1)$ and order ς $(0<\varsigma<1)$ for a function $\mathscr{F}$ can be written as*

$$\mathscr{D}^{\varsigma,\alpha}_{0^{+}}\mathscr{F}(\tau)=\left(\mathcal{I}^{\alpha(1-\varsigma)}_{0^{+}}\frac{d}{d\tau}(\mathcal{I}^{(1-\Upsilon)}_{0^{+}}\mathscr{F})\right)(\tau)=\left(\mathcal{I}^{\alpha(1-\varsigma)}_{0^{+}}(\mathscr{D}^{\Upsilon}_{0^{+}}\mathscr{F})\right)(\tau),$$

provided that the right hand side exists, where $0\leq\alpha\leq1,0<\varsigma<1,\Upsilon=\varsigma+\alpha-\varsigma\alpha$.

Consider the abstract degenerate Cauchy problem:

$$\begin{cases}\frac{d}{d\tau}\mathscr{R}\varkappa(\tau)=\mathscr{A}\varkappa(\tau),\quad\tau\in\mathfrak{J},\\ \mathscr{R}\varkappa(0)=\mathscr{R}\varkappa_{0}.\end{cases}\tag{2}$$

Definition 2.4 ([31]). *A strongly continuous operators family $\{\mathscr{P}(\tau)\}_{\tau\geq 0}$ is said to be exponentially bounded propagation family for* (2) *if there exists constant $c > 0$, $\mathscr{M} > 0$ such that*

$$\|\mathscr{P}(\tau)\varkappa\| \leq \mathscr{M}e^{c\tau}\|\varkappa\|, \quad \tau \geq 0 \text{ and } \varkappa \in \mathscr{D}(\mathscr{R}).$$

The exponentially bounded propagation family $\{\mathscr{P}(\tau)\}_{\tau\geq 0}$ is generated by ordered pair $(\mathscr{A}, \mathscr{R})$ if

$$(\lambda\mathscr{R} - \mathscr{A})^{-1}\mathscr{R}\varkappa = \int_0^\infty e^{-\lambda\tau}\mathscr{P}(\tau)\varkappa\, d\tau, \quad \text{for } \lambda > c \text{ and } \varkappa \in \mathscr{D}(\mathscr{R}),$$

holds.

Lemma 2.5 ([17, 42]). *The fractional non-linear differential equation (1) is equivalent to the integral equation*

$$\varkappa(\tau) = \begin{cases} \frac{\tau^{(\Upsilon-1)}}{\Gamma\alpha(1-\varsigma)+\varsigma}\varkappa_0 \\ \quad + \frac{1}{\Gamma\varsigma}\int_0^\tau (\tau-\nu)^{\varsigma-1}\Big[\mathscr{F}(\nu,\varkappa(\nu),\int_0^\tau \mathfrak{G}(\nu,\mathfrak{r},\varkappa(\mathfrak{r}))d\mathfrak{r}) + \mathcal{B}w(\tau) + \mathscr{A}\varkappa(\tau)\Big]d\nu, & \tau \in [0,\tau_1], \\ \zeta_{\mathrm{j}}(\tau,\varkappa(\tau)), & \tau \in (\tau_{\mathrm{j}},\omega_{\mathrm{j}}], \\ \zeta_{\mathrm{j}}(\tau,\varkappa(\tau)) + \frac{1}{\Gamma\varsigma}\int_0^\tau (\tau-\nu)^{\varsigma-1} \\ \quad \times\Big[\mathscr{F}(\nu,\varkappa(\nu),\int_0^\tau \mathfrak{G}(\nu,\mathfrak{r},\varkappa(\mathfrak{r}))d\mathfrak{r}) + \mathcal{B}w(\tau) + \mathscr{A}\varkappa(\tau)\Big]d\nu, & \tau \in (\omega_{\mathrm{j}},\tau_{\mathrm{j}+1}]. \end{cases} \tag{3}$$

Definition 2.6 ([18, 19, 42]). *A function $\varkappa \in \mathscr{PC}_{1-\Upsilon}(\mathfrak{J},\varkappa)$ is said to be a mild solution of the system* (1) *if integral equation* (3) *holds*

$$\varkappa(\tau) = \begin{cases} \mathcal{S}_{\varsigma,\alpha}(\tau)\varkappa_0 + \int_0^\tau \mathcal{K}_\varsigma(\tau-\nu)\Big[\mathscr{F}(\nu,\varkappa(\nu),\int_0^\tau \mathfrak{G}(\nu,\mathfrak{r},\varkappa(\mathfrak{r}))d\mathfrak{r}) + \mathcal{B}w(\tau)\Big]d\nu, & \tau \in [0,\tau_1], \\ \zeta_{\mathrm{j}}(\tau,\varkappa(\tau)), & \tau \in (\tau_{\mathrm{j}},\omega_{\mathrm{j}}], \\ \mathcal{S}_{\varsigma,\alpha}(\tau)\zeta_{\mathrm{j}}(\tau,\varkappa(\tau)) \\ \quad + \int_{\omega_{\mathrm{j}}}^\tau \mathcal{K}_\varsigma(\tau-\nu)\Big[\mathscr{F}(\nu,\varkappa(\nu),\int_0^\tau \mathfrak{G}(\nu,\mathfrak{r},\varkappa(\mathfrak{r}))d\mathfrak{r}) + \mathcal{B}w(\tau)\Big]d\nu, & \tau \in (\omega_{\mathrm{j}},\tau_{\mathrm{j}+1}], \end{cases} \tag{4}$$

where

$$\mathcal{S}_{\varsigma,\alpha}(\tau) = \mathcal{I}^{\alpha(1-\varsigma)}\mathcal{K}_\varsigma(\tau), \quad \mathcal{K}_\varsigma(\tau) = \tau^{\varsigma-1}\mathcal{Q}_\varsigma(\tau), \quad \mathcal{Q}_\varsigma(\tau) = \varsigma\int_0^\infty \vartheta\psi_\varsigma(\vartheta)\mathscr{P}(\tau^\varsigma\vartheta)\,d\vartheta,$$

$$\psi_\varsigma(\vartheta) = \frac{1}{\varsigma}\vartheta^{-1-1/\varsigma}\xi_\varsigma(\vartheta^{-1/\varsigma}), \quad \xi_\varsigma(\vartheta) = \frac{1}{\pi}\sum_{\mathfrak{n}=1}^\infty (-1)^{\mathfrak{n}-1}\vartheta^{-\mathfrak{n}\varsigma-1}\frac{\Gamma(\mathfrak{n}\varsigma+1)}{\mathfrak{n}!}\sin(\mathfrak{n}\pi\varsigma),\ \vartheta \in (0,\infty).$$

Lemma 2.7 ([19]). *If $\{\mathscr{P}(\tau)\}_{\tau\geq 0}$ is norm continuous and $\|\mathscr{P}(\tau)\| \leq \mathscr{M}$ for some $\mathscr{M} \geq 1$ and any $\tau > 0$, then $\{\mathcal{K}_\varsigma(\tau)\}_{\tau>0}$ and $\{\mathcal{S}_{\varsigma,\alpha}(\tau)\}_{\tau>0}$ are strongly continuous linear operators, and for any $\varkappa \in \mathcal{X}$ and $\tau > 0$, we have*

$$\|\mathcal{K}_\varsigma(\tau)\varkappa\| \leq \frac{\mathscr{M}\tau^{\varsigma-1}}{\Gamma(\varsigma)}\|\varkappa\|,$$

and

$$\|\mathcal{S}_{\alpha,\varsigma}(\tau)\varkappa\| \leq \frac{\mathscr{M}\tau^{(\alpha-1)(1-\varsigma)}}{\Gamma(\alpha(1-\varsigma)+\varsigma)}\|\varkappa\|.$$

Definition 2.8 ([14]). *Let $\mathscr{B}(\mathcal{X})$ be a collection of bounded subsets of $\mathcal{X}$. The function $\Delta\colon \mathscr{B}(\mathcal{X}) \to \mathscr{R}^+$ defined by*

$$\Delta(\mathfrak{D}) = \inf\Big\{\varepsilon > 0 \ :\ \mathfrak{D} \subset \bigcup_{\mathrm{j}=1}^{\mathfrak{n}}\mathfrak{D}_{\mathrm{j}},\ diam(\mathfrak{D}_{\mathrm{j}}) < \varepsilon\ \ (\mathrm{j} = 1,2,\cdots,\mathfrak{n} \in \mathbf{N})\Big\}, \quad \mathfrak{D} \in \mathscr{B}(\mathcal{X}),$$

is called the Kuratowski's measure of non-compactness.

Lemma 2.9 ([14]). *If $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}$ are bounded subsets of a Banach space $\mathcal{X}$, then the following statements are true:*

(i) $\mathcal{E}$ *is relatively compact set in* $\mathcal{X}$ *if and only if* $\Delta(\mathcal{E}) = 0$,

(ii) $\Delta(\mathcal{E}_1) \leq \Delta(\mathcal{E}_2)$ *if* $\mathcal{E}_1 \subset \mathcal{E}_2$,

(iii) $\Delta(\mathcal{E}_1 + \mathcal{E}_2) \leq \Delta(\mathcal{E}_1) + \Delta(\mathcal{E}_2)$,

(iv) $\Delta(c\mathcal{E}) \leq |c|\Delta(\mathcal{E})$ *for any* $c \in \mathscr{R}$.

Lemma 2.10 ([32]). *If $S \subset \mathcal{X}$ is bounded, there is a countable set $\mathcal{E} \subset S$ such that $\Delta(S) \leq 2\Delta(\mathcal{E})$.*

Lemma 2.11 ([14]). *If $\mathcal{E}$ is a bounded subset in $C([\mathfrak{b}_1, \mathfrak{b}_2], \mathcal{X})$, then $\mathcal{E}(\tau)$ is bounded in $\mathcal{X}$, and $\Delta(\mathcal{E}(\tau)) \leq \Delta(\mathcal{E})$. Further, if $\mathcal{E}$ is also equi-continuous on $[\mathfrak{b}_1, \mathfrak{b}_2]$, then $\Delta(\mathcal{E}(\tau))$ is continuous for $\tau \in [\mathfrak{b}_1, \mathfrak{b}_2]$ and*

$$\Delta(\mathcal{E}) = \sup\{\Delta(\mathcal{E}(\tau)), \tau \in [\mathfrak{b}_1, \mathfrak{b}_2]\}, \quad \textit{where } \mathcal{E}(\tau) = \{\varkappa(\tau) \colon \varkappa \in \mathcal{E}\} \subseteq \mathcal{E} \textit{ and } \mathfrak{b}_1, \mathfrak{b}_2 \geq 0.$$

Lemma 2.12 ([34]). *Let $\{\mathfrak{G}_q\}_{q=1}^{\infty}$ be a sequence of functions in $\mathcal{L}^1([0,\sigma], \mathscr{R}^+)$, and suppose that there are $\phi_1, \phi_2 \in \mathcal{L}^1([0,\sigma], \mathbb{R}^+)$ satisfying $\sup_{q\geq 1} \|\mathfrak{G}_q(\mathfrak{r}_1)\| \leq \phi_1(\mathfrak{r}_1)$ and $\Delta(\{\mathfrak{G}_q\}_{q=1}^{\infty}) \leq \phi_2(\mathfrak{r}_1)$ a.e. $\mathfrak{r}_1 \in [0,\sigma]$, then for each $\mathfrak{r}_1 \in [0,\sigma]$, we get*

$$\Delta\left(\int_0^{\mathfrak{r}_1} \mathfrak{G}_q(\mathfrak{r})d\mathfrak{r} : q \geq 1\right) \leq 2\int_0^{\mathfrak{r}_1} \phi_2(\mathfrak{r})d\mathfrak{r}.$$

Definition 2.13 ([14]). *A mapping $\mathfrak{F} : \mathcal{H} \subset \mathcal{X} \to \mathcal{Y}$ is said to be condensing map if $\mathfrak{F}$ is continuous, maps bounded sets into bounded sets and $\Delta(\mathfrak{F}(\mathcal{S})) < \Delta(\mathcal{S})$, for all bounded sets $\mathcal{S} \subset \mathcal{H}$ with $\Delta(\mathcal{S}) \neq 0$.*

Lemma 2.14 ([14]). *Let $\mathcal{H} \subseteq \mathcal{X}$ be closed, convex, bounded in Banach space $\mathcal{X}$. If $\mathfrak{F} : \mathcal{H} \to \mathcal{H}$ is a condensing map. Then $\mathfrak{F}$ has a fixed point in $\mathcal{H}$.*

3. Main result

To prove the existence and controllability results of Hilfer fractional impulsive differential equation of Sobolev-type (1), we take the following hypotheses:

(A0) $\{\mathscr{P}(\tau)\}_{\tau\geq 0}$ is norm continuous and uniformly bounded, i.e. $\|\mathscr{P}(\tau)\| \leq \mathscr{M}$ for some $\mathscr{M} \geq 1$ and any $\tau \geq 0$.

(A1) The function $\mathscr{F} : \mathfrak{J} \times \mathcal{X} \times \mathcal{X} \to \mathscr{D}(\mathscr{R})$ satisfies the Caratheodary condition, i.e. the function $\mathscr{F}(\tau, \cdot, \cdot) : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is continuous for each $\tau \in \mathfrak{J}$ and $\mathscr{F}(\cdot, \varkappa_1, \varkappa_2) \colon \mathfrak{J} \to \mathcal{X}$ is strongly measurable for any $\varkappa_1, \varkappa_2 \in \mathcal{X}$. For each $\tau \in \mathfrak{J}$ and $\kappa > 0$,
there are $\frac{\mathfrak{l}_{\kappa,\mathfrak{j}}(\cdot)}{(\tau-\cdot)^{1-\varsigma}} \in \mathcal{L}^1([0,\tau], \mathbb{R}^+)$, $\mathfrak{j} = 1, 2$ such that

$$\sup\{\|\mathscr{F}(\tau, \varkappa_1, \varkappa_2)\| \colon \|\tau^{(1-\alpha)(1-\varsigma)}\varkappa_1\| \leq \kappa\} \leq \mathfrak{l}_{\kappa,1}(\tau) + \mathfrak{l}_{\kappa,2}(\tau)\|\varkappa_2\|, \quad \text{for a.e. } \tau \in \mathfrak{J},$$

$$\lim_{\kappa\to\infty} \inf \frac{1}{\kappa} \int_0^{\tau} \frac{\mathfrak{l}_{\kappa,1}(\nu)}{(\tau-\nu)^{1-\varsigma}} d\nu = \mathfrak{l}_1 < +\infty,$$

and

$$\lim_{\kappa\to\infty} \int_0^{\tau} \frac{\mathfrak{l}_{\kappa,2}(\nu)}{(\tau-\nu)^{1-\varsigma}} d\nu = \mathfrak{l}_2 < +\infty.$$

(A2) The function $\mathfrak{G}\colon \Sigma \times \mathcal{X} \to \mathcal{X}$ satisfies that the function $\mathfrak{G}(\tau, \nu, \cdot) : \mathcal{X} \to \mathcal{X}$ is continuous for each $(\tau, \nu) \in \Sigma$ and the function $\mathfrak{G}(\cdot, \cdot, \mathfrak{y})\colon \Sigma \to \mathcal{X}$ is strongly measurable for each $\mathfrak{y} \in \mathcal{X}$. There exists a function $\mathfrak{m} \in \mathcal{L}^1(\Sigma, \mathbb{R}^+)$ such that

$$\|\mathfrak{G}(\tau, \nu, \mathfrak{y})\| \leq \mathfrak{m}(\tau, \nu)\|\nu^{(1-\alpha)(1-\varsigma)}\mathfrak{y}\|.$$

Take $\mathfrak{m}^* = \max_{\tau\in\mathfrak{J}} \int_0^\tau \mathfrak{m}(\tau, \nu)d\nu.$

(A3) There exist a function $\varrho\colon [0, \mathfrak{b}] \to \mathbb{R}^+$ with the condition $(\tau - \cdot)^{1-\varsigma}\varrho(\cdot) \in \mathcal{L}^1([0, \tau], \mathbb{R}^+)$ and an integrable function $\rho\colon \Sigma \to [0, \infty)$ such that

$$\Delta(\mathscr{F}(\tau, \mathcal{E}, \mathcal{S})) \leq \varrho(\tau)\left[\Delta(\tau^{(1-\alpha)(1-\varsigma)}\mathcal{E}) + \Delta(\mathcal{S})\right],$$

and

$$\Delta(\mathfrak{G}(\tau, \nu, \mathcal{E})) \leq \rho(\tau, \nu)\Delta(\nu^{(1-\alpha)(1-\varsigma)}\mathcal{E}), \quad \text{for a.e.}\tau \in \mathfrak{J}, \text{ and } \mathcal{E}, \mathcal{S} \subset \mathcal{X}.$$

Also suppose that $\varrho^* = \max_{\tau\in\mathfrak{J}} \int_0^\tau (\tau - \nu)^{1-\varsigma}\varrho(\nu)\, d\nu$ and $\rho^* = \max_{\tau\in\mathfrak{J}} \int_0^\tau \rho(\tau, \nu)d\nu.$

(A4) $\|\zeta_{\mathfrak{j}}(\tau, u) - \zeta_{\mathfrak{j}}(\tau, v)\|_{\mathscr{C}_{1-\Upsilon}} \leq \mathcal{K}_{\zeta_{\mathfrak{j}}}\|u - v\|_{\mathscr{C}_{1-\Upsilon}}, \quad \forall\tau \in (\tau_{\mathfrak{j}}, \omega_{\mathfrak{j}}],$

and $\sup \|\zeta_{\mathfrak{j}}(\omega_{\mathfrak{j}}, \varkappa(\omega_{\mathfrak{j}}))\|_{\mathscr{C}_{1-\Upsilon}} \leq \mathcal{N}_4.$

(A5) The linear operator $\Gamma_{\omega_{\mathfrak{j}}}^{\tau_{\mathfrak{j}+1}}\colon \mathcal{L}^2(\mathfrak{J}, \mathscr{W}) \to \mathcal{X}$ defined by

$$\Gamma_{\omega_{\mathfrak{j}}}^{\tau_{\mathfrak{j}+1}}w = \int_{\omega_{\mathfrak{j}}}^{\tau_{\mathfrak{j}+1}} \mathcal{K}_\varsigma(\tau_{\mathfrak{j}+1} - \nu)\mathcal{B}w(\nu)d\nu, \;\; \mathfrak{j} = 0, 1, \cdots, q,$$

has an induced inverse operator $(\Gamma_{\omega_{\mathfrak{j}}}^{\tau_{\mathfrak{j}+1}})^{-1}$ which takes values in $\mathcal{L}^2(\mathfrak{J}, \mathcal{X})/\ker(\Gamma_{\omega_{\mathfrak{j}}}^{\tau_{\mathfrak{j}+1}})$ and

$$\|(\Gamma_{\omega_{\mathfrak{j}}}^{\tau_{\mathfrak{j}+1}})^{-1}\|_{\mathcal{L}^2} \leq \mathscr{C}_T, \quad \text{where} \quad \mathscr{C}_T > 0,$$

and for every bounded set $\mathcal{E} \subset \mathcal{X}$, there is an integrable function $\Phi\colon J \to \mathbb{R}^+$ such that

$$\Delta(\Gamma_{\omega_{\mathfrak{j}}}^{\tau_{\mathfrak{j}+1}\,-1}\mathcal{E}(\tau)) \leq \Phi(\tau)\Delta(\mathcal{E}),$$

and

$$\Phi^* = \max_{\tau\in\mathfrak{J}} \int_0^\tau (\tau - \nu)^{1-\varsigma}\Phi(\nu)\, d\nu.$$

(A6) Take

$$\mathcal{N}_1 = \max\left\{\left[\frac{2\mathscr{M}\,\mathfrak{b}^{(1-\alpha)(1-\varsigma)}}{\Gamma(\varsigma)}\left[1 + \frac{2\mathscr{M}\|\mathcal{B}\|\Phi^*}{\Gamma(\varsigma)}\right](1 + 2\rho^*)\varrho^*\right], \left[\frac{\mathscr{M}\mathfrak{b}^{(\alpha-1)(1-\varsigma)}}{\Gamma(\alpha(1 - \varsigma) + \varsigma)}\mathcal{K}_{\zeta_\kappa}\right.\right.$$
$$\left.\left.\times\left(1 + \frac{2\mathscr{M}\phi^*\|\mathcal{B}\|}{\Gamma\varsigma}\right) + \frac{2\mathscr{M}}{\Gamma(\varsigma)}(1 + 2\rho^*)\varrho^*\left(\mathfrak{b}^{(1-\alpha)(1-\varsigma)} + \frac{2\mathscr{M}\phi^*\|\mathcal{B}\|}{\Gamma\varsigma}\right)\right]\right\} < \frac{1}{2}.$$

Define

$$\mathcal{E}_\kappa = \{\mathfrak{y}(\cdot) \in \mathscr{C}(\mathfrak{J}, \mathcal{X})\colon \|\mathfrak{y}\|_\infty \leq \kappa\},$$

and

$$\mathcal{E}_\kappa^{\mathscr{P}\mathscr{C}_{1-\Upsilon}} = \{\varkappa \in \mathscr{P}\mathscr{C}_{1-\gamma}\colon \|\varkappa\|_{\mathscr{P}\mathscr{C}_{1-\Upsilon}} \leq \kappa\}.$$

Theorem 3.1. *Let conditions (A0)-(A6) hold, then controllability of system* (1) *is assured on* $\mathfrak{J}$ *if*

$$\mathscr{M}\left\{1+\frac{\mathscr{M}\mathfrak{b}^{\varsigma}}{\Gamma(\varsigma+1)}\|\mathcal{B}\|\mathscr{C}_T\right\}\left\{\frac{\mathfrak{b}^{(1-\alpha)(1-\varsigma)}}{\Gamma(\varsigma)}\left\{l_1+l_2\mathfrak{m}^*\right\}\right\}<1, \tag{5}$$

is satisfied.

Proof. The control function for $\tau \in [0,\tau_1]$ is defined as:

$$w_{\varkappa}(\tau)=(\Gamma_0^{\tau_1})^{-1}\left[\varkappa_{\tau_1}-\mathscr{M}_{\varsigma,\alpha}(\tau_1)\varkappa_0-\int_0^{\tau_1}\mathcal{K}_{\varsigma}(\tau_1-\nu)\mathscr{F}\left(\nu,\varkappa(\nu),\int_0^{\nu}\mathfrak{G}(\nu,\mathfrak{r},\varkappa(\mathfrak{r}))\,d\mathfrak{r}\right)d\nu\right](\tau).$$

For $\tau\in(\omega_{\mathrm{j}},\tau_{\mathrm{j}+1}]$; $\mathrm{j}=1,2,3,\cdots,q$, it is defined as:

$$\begin{aligned}w_{\varkappa}(\tau)=(\Gamma_{\omega_{\mathrm{j}}}^{\tau_{\mathrm{j}+1}})^{-1}\Big[&\varkappa_{\tau_{\mathrm{j}+1}}-\mathscr{M}_{\varsigma,\alpha}(\tau_{\mathrm{j}+1}-\nu)\zeta_{\mathrm{j}}(\omega_{\mathrm{j}},\varkappa_{\mathrm{j}}(\tau))\\&-\int_{\omega_{\mathrm{j}}}^{\tau_{\mathrm{j}+1}}\mathcal{K}_{\varsigma}(\tau_{\mathrm{j}+1}-\nu)\mathscr{F}\left(\nu,\varkappa(\nu),\int_0^{\nu}\mathfrak{G}(\nu,\mathfrak{r},\varkappa(\mathfrak{r}))\,d\mathfrak{r}\right)d\nu\Big](\tau).\end{aligned} \tag{6}$$

Put $\tau=\tau_{\mathrm{j}+1}$ in the mild solution of (4), we have:

$$\begin{aligned}\varkappa(\tau_{\mathrm{j}+1})=\;&\mathcal{S}_{\varsigma,\alpha}(\tau_{\mathrm{j}+1})\zeta_{\mathrm{j}}(\omega_{\mathrm{j}},\varkappa(\omega_{\mathrm{j}}))\\&+\int_{\omega_{\mathrm{j}}}^{\tau_{\mathrm{j}+1}}\mathcal{K}_{\varsigma}(\tau_{\mathrm{j}+1}-\nu)\left[\mathscr{F}(\nu,\varkappa(\nu),\int_0^{\nu}\mathfrak{G}(\nu,\mathfrak{r},\varkappa(\mathfrak{r}))d\mathfrak{r})+\mathcal{B}w(\tau)\right]d\nu;\ \mathrm{j}=0,1,2,\cdots,q,\end{aligned}$$

$$\begin{aligned}\varkappa(\tau_{\mathrm{j}+1})=\;&\mathcal{S}_{\varsigma,\alpha}(\tau_{\mathrm{j}+1})\zeta_{\mathrm{j}}(\omega_{\mathrm{j}},\varkappa(\omega_{\mathrm{j}}))+\int_{\omega_{\mathrm{j}}}^{\tau_{\mathrm{j}+1}}\mathcal{K}_{\varsigma}(\tau_{\mathrm{j}+1}-\nu)\left[\mathscr{F}(\nu,\varkappa(\nu),\int_0^{\nu}\mathfrak{G}(\nu,\mathfrak{r},\varkappa(\mathfrak{r}))d\mathfrak{r})\right]d\nu\\&+(\Gamma_{\omega_{\mathrm{j}}}^{\tau_{\mathrm{j}+1}})(\Gamma_{\omega_{\mathrm{j}}}^{\tau_{\mathrm{j}+1}})^{-1}\Big[\varkappa(\tau_{\mathrm{j}+1})-\mathcal{S}_{\varsigma,\alpha}(\tau_{\mathrm{j}+1}-\nu)\zeta_{\mathrm{j}}(\omega_{\mathrm{j}},\varkappa(\omega_{\mathrm{j}}))\\&-\int_{\omega_{\mathrm{j}}}^{\tau_{\mathrm{j}+1}}\mathcal{K}_{\varsigma}(\tau_{\mathrm{j}+1}-\nu)[\mathscr{F}(\nu,\varkappa(\nu),\int_0^{\nu}\mathfrak{G}(\nu,\mathfrak{r},\varkappa(\mathfrak{r}))d\mathfrak{r})]d\nu\Big]\\\varkappa(\tau_{\mathrm{j}+1})=\;&\varkappa_{\tau_{\mathrm{j}+1}}.\end{aligned}$$

Hence, the control function is well-defined for $\tau\in(\omega_{\mathrm{j}},\tau_{\mathrm{j}+1}]$ and $[0,\tau_1]$.

Using the control (6), we define an operator $\mathfrak{F}\colon\mathscr{PC}_{1-\gamma}[\mathfrak{J},\mathcal{X}]\to\mathscr{PC}_{1-\gamma}[\mathfrak{J},\mathcal{X}]$ as

$$(\mathfrak{F}\varkappa)(\tau)=\begin{cases}\mathcal{S}_{\varsigma,\alpha}(\tau)\varkappa_0+\int_0^{\tau}\mathcal{K}_{\varsigma}(\tau-\nu)\left[\mathscr{F}(\nu,\varkappa(\nu),\int_0^{\tau}\mathfrak{G}(\nu,\mathfrak{r},\varkappa(\mathfrak{r}))d\mathfrak{r})+\mathcal{B}w(\tau)\right]d\nu, & \tau\in[0,\tau_1],\\ \zeta_{\mathrm{j}}(\tau,\varkappa(\tau)), & \tau\in(\tau_{\mathrm{j}},\omega_{\mathrm{j}}],\\ \mathcal{S}_{\varsigma,\alpha}(\tau)\zeta_{\mathrm{j}}(\tau,\varkappa(\tau))\\ \quad+\int_{\omega_{\mathrm{j}}}^{\tau}\mathcal{K}_{\varsigma}(\tau-\nu)\left[\mathscr{F}(\nu,\varkappa(\nu),\int_0^{\tau}\mathfrak{G}(\nu,\mathfrak{r},\varkappa(\mathfrak{r}))d\mathfrak{r})+\mathcal{B}w(\tau)\right]d\nu, & \tau\in(\omega_{\mathrm{j}},\tau_{\mathrm{j}+1}].\end{cases} \tag{7}$$

For any $\mathfrak{y} \in \mathscr{C}(\mathfrak{J}, \mathcal{X})$, $\varkappa(\tau) = \tau^{(\alpha-1)(1-\varsigma)}\mathfrak{y}(\tau)$, $\tau \in \mathfrak{J}'$. We define a map $\mathfrak{X}$ as

$$(\mathfrak{X}\mathfrak{y})(\tau) = \begin{cases} \tau^{(1-\alpha)(1-\varsigma)}\Big(\mathcal{S}_{\varsigma,\alpha}(\tau)\varkappa_0 \\ \quad + \int_0^\tau \mathcal{K}_\varsigma(\tau-\nu)\Big[\mathscr{F}(\nu, \varkappa(\nu), \int_0^\tau \mathfrak{G}(\nu, \mathfrak{r}, \varkappa(\mathfrak{r}))d\mathfrak{r}) + \mathcal{B}w(\tau)\Big]d\nu\Big), & \tau \in (0, \tau_1], \\ \tau^{(1-\alpha)(1-\varsigma)}\zeta_{\mathrm{j}}(\tau, \varkappa(\tau)), & \tau \in (\tau_{\mathrm{j}}, \omega_{\mathrm{j}}], \\ \tau^{(1-\alpha)(1-\varsigma)}\Big(\mathcal{S}_{\varsigma,\alpha}(\tau)\zeta_{\mathrm{j}}(\tau, \varkappa(\tau)) \\ \quad + \int_{\omega_{\mathrm{j}}}^\tau \mathcal{K}_\varsigma(\tau-\nu)\Big[\mathscr{F}(\nu, \varkappa(\nu), \int_0^\tau \mathfrak{G}(\nu, \mathfrak{r}, \varkappa(\mathfrak{r}))d\mathfrak{r}) + \mathcal{B}w(\tau)\Big]d\nu\Big), & \tau \in (\omega_{\mathrm{j}}, \tau_{\mathrm{j}+1}], \\ \frac{\varkappa_0}{\Gamma[\alpha(1-\varsigma)+\varsigma]}, & \tau = 0. \end{cases} \tag{8}$$

From equations (4) and (7), we conclude that any fixed point of $\mathfrak{F}$ is equivalent to the mild solution of (1). If $\varkappa$ is the mild solution of (1) with the control (6), then $\varkappa(\mathfrak{b}) = \varkappa_{\mathfrak{b}}$. For $\tau \in [0, \tau_1]$, let $\mathfrak{y} \in \mathcal{E}_\kappa$ and $\varkappa(\tau) = \tau^{(\alpha-1)(1-\varsigma)}\mathfrak{y}(\tau)$, $\tau \in \mathfrak{J}' = (0, \mathfrak{J}]$. Therefore, $\varkappa \in \mathcal{E}_\kappa{}^{\mathscr{P}\mathscr{C}_{1-\Upsilon}}$. From (6) and Lemma (2.7), we get

$$\begin{aligned} \|\mathcal{B}w_\varkappa(\tau)\| &\le \|\mathcal{B}\|\mathscr{C}_T\Bigg[\|\varkappa_{\tau_1}\| + \frac{\mathscr{M}}{\Gamma(\alpha(1-\varsigma)+\varsigma)}\tau^{(\alpha-1)(1-\varsigma)}\|\varkappa_0\| \\ &\quad + \frac{\mathscr{M}}{\Gamma(\varsigma)}\int_0^{\tau_1}(\tau_1-\nu)^{\varsigma-1}\left\|\mathscr{F}\left(\nu, \varkappa(\nu), \int_0^\nu \mathfrak{G}(\nu, \mathfrak{r}, \varkappa(\mathfrak{r}))\,d\mathfrak{r}\right)\right\|d\nu\Bigg] \\ &\le \|\mathcal{B}\|\mathscr{C}_T\Bigg[\|\varkappa_{\tau_1}\| + \frac{\mathscr{C}}{\Gamma(\alpha(1-\varsigma)+\varsigma)}\tau^{(\alpha-1)(1-\varsigma)}\|\varkappa_0\| \\ &\quad + \frac{\mathscr{M}}{\Gamma(\varsigma)}\int_0^{\tau_1}(\tau_1-\nu)^{\varsigma-1}\left\{\mathfrak{l}_{\kappa,1}(\nu) + \mathfrak{l}_{\kappa,2}(\nu)\left\|\int_0^\nu \mathfrak{G}(\nu, \mathfrak{r}, \varkappa(\mathfrak{r}))\,d\mathfrak{r}\right\|\right\}d\nu\Bigg] \\ &\le \|\mathcal{B}\|\mathscr{C}_T\Bigg[\|\varkappa_{\tau_1}\| + \frac{\mathscr{M}}{\Gamma(\alpha(1-\varsigma)+\varsigma)}\tau_1^{(\alpha-1)(1-\varsigma)}\|\varkappa_0\| \\ &\quad + \frac{\mathscr{M}}{\Gamma(\varsigma)}\int_0^{\tau_1}(\tau_1-\nu)^{\varsigma-1}\left\{\mathfrak{l}_{\kappa,1}(\nu) + \mathfrak{m}^*\mathfrak{l}_{\kappa,2}(\nu)\|\varkappa\|_{\mathscr{P}\mathscr{C}_{1-\gamma}}\right\}d\nu\Bigg] \\ &\le \|\mathcal{B}\|\mathscr{C}_T\Bigg[\|\varkappa_{\tau_1}\| + \frac{\mathscr{M}}{\Gamma(\alpha(1-\varsigma)+\varsigma)}\tau_1^{(\alpha-1)(1-\varsigma)}\|\varkappa_0\| \\ &\quad + \frac{\mathscr{M}}{\Gamma(\varsigma)}\int_0^{\tau_1}(\tau_1-\nu)^{\varsigma-1}\left\{\mathfrak{l}_{\kappa,1}(\nu) + \kappa\mathfrak{m}^*\mathfrak{l}_{\kappa,2}(\nu)\right\}d\nu\Bigg] \\ &= \mathscr{M}^*. \end{aligned} \tag{9}$$

For $\tau \in [\tau_{\mathrm{j}}, \omega_{\mathrm{j}}]$, we have

$$\|\mathcal{B}w_\varkappa(\tau)\| = \|\zeta_{\mathrm{j}}(\tau, \varkappa(\tau))\| \le \mathcal{N}_4. \tag{10}$$

For $\tau \in (\omega_{\mathrm{j}}, \tau_{\mathrm{j}+1}]$, we have

$$
\begin{aligned}
\|\mathcal{B}w_{\varkappa}(\tau)\| \leq & \Big\|\mathcal{B}(\Gamma_{\omega_{\mathrm{j}}}^{\tau_{\mathrm{j}+1}})^{-1}\Big[\varkappa_{\mathrm{j}+1} - \mathcal{S}_{\varsigma,\alpha}(\tau_{\mathrm{j}+1} - \nu)\zeta_{\mathrm{j}}(\omega_{\mathrm{j}}, \varkappa_{\mathrm{j}}(\tau)) \\
& - \int_{\omega_{\mathrm{j}}}^{\tau_{\mathrm{j}+1}} \mathcal{K}_{\varsigma}(\tau_{\mathrm{j}+1} - \nu)\mathscr{F}\Big(\nu, \varkappa(\nu), \int_0^{\nu} \mathfrak{G}(\nu, \mathfrak{r}, \varkappa(\mathfrak{r}))\, d\mathfrak{r}\Big)d\nu\Big]\Big\|. \\
\|\mathcal{B}w_{\varkappa}(\tau)\| \leq & \|\mathcal{B}\|\mathscr{C}_T \Bigg[\|\varkappa_{\mathrm{j}+1}\| + \frac{\mathscr{C}}{\Gamma(\alpha(1-\varsigma)+\varsigma)}(\tau_{\mathrm{j}+1} - \nu)^{(\alpha-1)(1-\varsigma)}\|\zeta_{\mathrm{j}}(\tau, \varkappa(\tau))\| \\
& + \frac{\mathscr{M}}{\Gamma(\varsigma)}\int_{\omega_{\mathrm{j}}}^{\tau_{\mathrm{j}+1}} (\tau_{\mathrm{j}+1} - \nu)^{\varsigma-1}\Big\|\mathscr{F}\Big(\nu, \varkappa(\nu), \int_0^{\nu} \mathfrak{G}(\nu, \mathfrak{r}, \varkappa(\mathfrak{r}))\, d\mathfrak{r}\Big)\Big\| d\nu\Bigg] \\
\leq & \|\mathcal{B}\|\mathscr{C}_T \Bigg[\|\varkappa_{\tau_{\mathrm{j}+1}}\| + \frac{\mathscr{M}}{\Gamma(\alpha(1-\varsigma)+\varsigma)}(\tau_{\mathrm{j}+1} - \nu)^{(\alpha-1)(1-\varsigma)}\|\nu_{\mathrm{j}}(\omega_{\mathrm{j}}, \varkappa_{\mathrm{j}}(\tau))\| \\
& + \frac{\mathscr{M}}{\Gamma(\varsigma)}\int_{\omega_{\mathrm{j}}}^{\tau_{\mathrm{j}+1}} (\tau_{\mathrm{j}+1} - \nu)^{\varsigma-1}\Big\{\mathfrak{l}_{\kappa,1}(\nu) + \mathfrak{l}_{\kappa,2}(\nu)\Big\|\int_0^{\nu} \mathfrak{G}(\nu, \mathfrak{r}, \varkappa(\mathfrak{r}))\, d\mathfrak{r}\Big\|\Big\} d\nu\Bigg] \\
\leq & \|\mathcal{B}\|\mathscr{C}_T \Bigg[\|\varkappa_{\tau_{\mathrm{j}+1}}\| + \frac{\mathscr{M}}{\Gamma(\alpha(1-\varsigma)+\varsigma)}(\tau_{\mathrm{j}+1} - \nu)^{(\alpha-1)(1-\varsigma)}\|\nu_{\mathrm{j}}(\omega_{\mathrm{j}}, \varkappa_{\mathrm{j}}(\tau))\| \\
& + \frac{\mathscr{M}}{\Gamma(\varsigma)}\int_{\omega_{\mathrm{j}}}^{\tau_{\mathrm{j}+1}} (\tau_{\mathrm{j}+1} - \nu)^{\varsigma-1}\{\mathfrak{l}_{\kappa,1}(\nu) + \mathfrak{m}^*\mathfrak{l}_{\kappa,2}(\nu)\}\, d\nu\Bigg] \\
\leq & \|\mathcal{B}\|\mathscr{C}_T \Bigg[\|\varkappa_{\tau_{\mathrm{j}+1}}\| + \frac{\mathscr{M}}{\Gamma(\alpha(1-\varsigma)+\varsigma)}(\tau_{\mathrm{j}+1} - \nu)^{(\alpha-1)(1-\varsigma)}\|\zeta_{\mathrm{j}}(\omega_{\mathrm{j}}, \varkappa_{\mathrm{j}}(\tau))\| \\
& + \frac{\mathscr{M}}{\Gamma(\varsigma)}\int_{\omega_{\mathrm{j}}}^{\tau_{\mathrm{j}+1}} (\tau_{\mathrm{j}+1} - \nu)^{\varsigma-1}\{\mathfrak{l}_{\kappa,1}(\nu) + \kappa m^*\mathfrak{l}_{\kappa,2}(\nu)\}\, d\nu\Bigg] \\
= & \mathscr{M}_1^*. \qquad (11)
\end{aligned}
$$

We show that for some $\kappa > 0$, $\mathfrak{X}$ is self map, i.e $\mathfrak{X}(\mathcal{E}_\kappa) \subseteq \mathcal{E}_\kappa$. Suppose that $\mathfrak{X}(\mathcal{E}_\kappa) \nsubseteq \mathcal{E}_\kappa$ for each $\kappa > 0$, i.e for each $\kappa > 0$, $\exists\, \mathfrak{y}_\kappa \in \mathcal{E}_\kappa$ such that $\|\mathfrak{X}(\mathfrak{y}_\kappa)(\tau_\kappa)\| > \kappa$ for some $\tau_k \in \bigcup_{\mathrm{j}=0}^{\mathfrak{m}}(\omega_{\mathrm{j}}, \tau_{\mathrm{j}+1}]$. Let $\varkappa_\kappa(\tau_\kappa) = \tau_\kappa^{(\alpha-1)(1-\varsigma)}\mathfrak{y}_\kappa(\tau_\kappa)$, $\tau_\kappa \in [0, \tau_1]$. Using Lemma (2.7) and assumptions (A0)-(A5), we get

$$
\begin{aligned}
\kappa < \|\mathfrak{X}\mathfrak{y}_\kappa(\tau_\kappa)\| \leq & \frac{\mathscr{M}}{\Gamma(\alpha(1-\varsigma)+\varsigma)}\|\varkappa_0\| + \frac{\mathscr{M}\tau_1^{(1-\alpha)(1-\varsigma)}}{\Gamma(\varsigma)}\int_0^{\tau_\kappa} (\tau_\kappa - \nu)^{\varsigma-1} \\
& \times \Big\|\mathscr{F}\Big(\nu, \varkappa(\nu), \int_0^{\nu} \mathfrak{G}(\nu, \mathfrak{r}, \varkappa(\mathfrak{r}))\, d\mathfrak{r}\Big) + \mathcal{B}w_{\varkappa}(\nu)\Big\| d\nu \\
\leq & \frac{\mathscr{M}}{\Gamma(\alpha(1-\varsigma)+\varsigma)}\|\varkappa_0\| + \frac{\mathscr{M}\tau_1^{(1-\alpha)(1-\varsigma)}}{\Gamma(\varsigma)}\int_0^{\tau_\kappa} (\tau_\kappa - \nu)^{\varsigma-1} \\
& \times \Big[\mathfrak{l}_{\kappa,1}(\nu) + \mathfrak{l}_{\kappa,2}(\nu)\Big\|\int_0^{\nu} \mathfrak{G}(\nu, \mathfrak{r}, \varkappa(\mathfrak{r}))\, d\mathfrak{r}\Big\| + \|\mathcal{B}_{\varkappa}w(\nu)\|\Big] d\nu \\
\leq & \frac{\mathscr{M}}{\Gamma(\alpha(1-\varsigma)+\varsigma)}\|\varkappa_0\| + \frac{\mathscr{M}\tau_1^{(1-\alpha)(1-\varsigma)}}{\Gamma(\varsigma)}\int_0^{\tau_\kappa} (\tau_\kappa - \nu)^{\varsigma-1} \\
& \times \Big[\mathfrak{l}_{\kappa,1}(\nu) + \kappa\mathfrak{m}^*\mathfrak{l}_{\kappa,2}(\nu) + \|\mathcal{B}w_{\varkappa}(\nu)\|\Big] d\nu. \qquad (12)
\end{aligned}
$$

Equation (12) is divided by κ and taking $\kappa \to \infty$,

$$1 < \mathscr{M}\left\{1 + \frac{\mathscr{M}\tau_1^{\varsigma}}{\Gamma(\varsigma+1)}\|\mathcal{B}\|\mathscr{C}_T\right\}\left\{\frac{\tau_1^{(1-\alpha)(1-\varsigma)}}{\Gamma(\varsigma)}\{\mathfrak{l}_1 + \mathfrak{l}_2\mathfrak{m}^*\}\right\}.$$

For $\tau \in [\tau_{\mathfrak{j}}, \omega_{\mathfrak{j}}]$, we have

$$\kappa < \|\mathfrak{X}\mathfrak{y}_\kappa(\tau_\kappa)\| = \|\tau_\kappa^{(\alpha-1)(1-\varsigma)}\nu_{\mathfrak{j}}(\tau_\kappa, \varkappa_\kappa(\tau_\kappa))\| \le \mathcal{N}_4^*. \tag{13}$$

Equation (13) is divided by κ then taking $\kappa \to \infty$, we get $1 < 0$, which contradicts.
For $\tau_\kappa \in (\omega_{\mathfrak{j}}, \tau_{\mathfrak{j}+1}]$, we obtain

$$\begin{aligned}
\kappa < \|\mathfrak{X}\mathfrak{y}_\kappa(\tau_\kappa)\| &\le \frac{\mathscr{M}}{\Gamma(\alpha(1-\varsigma)+\varsigma)}\mathcal{N}_4 + \frac{\mathscr{M}\tau_{\mathfrak{j}+1}{}^{(1-\alpha)(1-\varsigma)}}{\Gamma(\varsigma)}\int_{\omega_{\mathfrak{j}}}^{\tau_{\mathfrak{j}+1}}(\tau_\kappa - \nu)^{\varsigma-1} \\
&\quad \times \left\|\mathscr{F}\left(\nu, \varkappa(\nu), \int_0^{\nu}\mathfrak{G}(\nu, \mathfrak{r}, \varkappa(\mathfrak{r}))\,d\mathfrak{r}\right) + \mathcal{B}w_{\varkappa}(\nu)\right\| d\nu \\
&\le \frac{M}{\Gamma(\alpha(1-\varsigma)+\varsigma)}\mathcal{N}_4 + \frac{\mathscr{M}\tau_{\mathfrak{j}+1}^{(1-\alpha)(1-\varsigma)}}{\Gamma(\varsigma)}\int_{\omega_{\mathfrak{j}}}^{\tau_{\mathfrak{j}+1}}(\tau_\kappa - \nu)^{\varsigma-1} \\
&\quad \times \left[\mathfrak{l}_{\kappa,1}(\nu) + \mathfrak{l}_{\kappa,2}(\nu)\left\|\int_0^{\nu}\mathfrak{G}(\nu, \mathfrak{r}, \varkappa(\mathfrak{r}))\,d\mathfrak{r}\right\|\right] d\nu \\
&\quad + \frac{\mathscr{M}\tau_{\mathfrak{j}+1}^{(1-\alpha)(1-\varsigma)}}{\Gamma(\varsigma)}\int_{\omega_{\mathfrak{j}}}^{\tau_{\mathfrak{j}+1}}(\tau_\kappa - \nu)^{\varsigma-1}\|\mathcal{B}w(\nu)\| d\nu \\
&\le \frac{\mathscr{M}}{\Gamma(\alpha(1-\varsigma)+\varsigma)}\mathcal{N}_4 + \frac{\mathscr{M}\tau_{\mathfrak{j}+1}^{(1-\alpha)(1-\varsigma)}}{\Gamma(\varsigma)}\int_{\tau_{\mathfrak{j}}}^{\tau_{\mathfrak{j}+1}}(\tau_\kappa - \nu)^{(1-\alpha)(1-\varsigma)} \\
&\quad \times [\mathfrak{l}_{\kappa,1}(\nu) + \kappa\mathfrak{m}^*\mathfrak{l}_{\kappa,2}(\nu)]d\nu + \frac{\mathscr{M}\mathfrak{b}^{(1-\alpha)(1-\varsigma)}}{\Gamma(\varsigma)}\int_{\omega_{\mathfrak{j}}}^{\tau_{\mathfrak{j}+1}}(\tau_\kappa - \nu)^{(1-\alpha)(1-\varsigma)} \\
&\quad \times \Bigg\|\Bigg(\|\mathcal{B}\|\mathscr{C}_T\Bigg[\varkappa_{\tau_{\mathfrak{j}+1}} + \frac{\mathscr{M}}{\Gamma(\alpha(1-\varsigma)+\varsigma)}\tau_{\mathfrak{j}+1}^{(\alpha-1)(1-\varsigma)}\|\zeta_{\mathfrak{j}}(\omega_{\mathfrak{j}}, \varkappa_{\mathfrak{j}}(\tau))\| \\
&\quad + \frac{\mathscr{M}}{\Gamma(\varsigma)}\int_{\omega_{\mathfrak{j}}}^{\tau_{\mathfrak{j}+1}}(\tau_{\mathfrak{j}+1} - \nu)^{\varsigma-1}\{\mathfrak{l}_{\kappa,1}(\nu) + \kappa\mathfrak{m}^*\mathfrak{l}_{\kappa,2}(\nu)\}\,d\nu\Bigg]\Bigg)\Bigg\| d\nu.
\end{aligned} \tag{14}$$

Equation (14) is divided by κ and taking $\kappa \to \infty$,

$$1 < \mathscr{M}\left\{1 + \frac{\mathscr{M}\tau_{\mathfrak{j}+1}^{\varsigma}}{\Gamma(\varsigma+1)}\|\mathcal{B}\|\mathscr{C}_T\right\}\left\{\frac{\tau_{\mathfrak{j}+1}^{(1-\alpha)(1-\varsigma)}}{\Gamma(\varsigma)}\{\mathfrak{l}_1 + \mathfrak{l}_2\mathfrak{m}^*\}\right\}.$$

Again we get contradiction for $\tau_\kappa \in (\omega_{\mathfrak{j}}, \tau_{\mathfrak{j}+1}]$. Hence $\mathfrak{X}$ is self map. Next we show that $\mathfrak{X}$ is continuous on $\mathcal{E}_\kappa$. Let $\{\mathfrak{y}_q\} \subset \mathcal{E}_\kappa$ with $\mathfrak{y}_q \to \mathfrak{y} \in \mathcal{E}_\kappa$ as $q \to \infty$. Take $\varkappa_q(\tau) = \tau^{(\alpha-1)(1-\varsigma)}\mathfrak{y}_q(\tau)$, $\tau \in \mathfrak{J}' = (0, \mathfrak{b}]$. Then from equation (9) and assumptions (A1)-(A3), we get

(i) $\|\mathfrak{G}(\tau, \nu, \varkappa_q(\nu)) - \mathfrak{G}(\tau, \nu, \varkappa(\nu))\| \le \mathfrak{m}(\tau, \nu)[\|\nu^{(1-\alpha)(1-\varsigma)}\varkappa_q(\nu)\| + \|\nu^{(1-\alpha)(1-\varsigma)}\varkappa(\nu)\|] \le 2\kappa\,\mathfrak{m}(\tau, \nu)$, $0 < \nu \le \tau$.

(ii) $\|\mathfrak{G}(\tau, \nu, \varkappa_q(\nu)) - \mathfrak{G}(\tau, \nu, \varkappa(\nu))\| \to 0$ as $q \to \infty$, $0 < \nu \le \tau$.

(iii) $\left\|\mathscr{F}\left(\tau, \varkappa_q(\tau), \int_0^{\tau}\mathfrak{G}(\tau, \nu, \varkappa_q(\nu))\,d\nu\right) - \mathscr{F}\left(\tau, \varkappa(\tau), \int_0^{\tau}\mathfrak{G}(\tau, \nu, \varkappa(\nu))\,d\nu\right)\right\| \le 2[\mathfrak{l}_{\kappa,1}(\tau) + \mathfrak{m}^*\mathfrak{l}_{\kappa,2}(\tau)\kappa]$.

(iv) $\left\|\mathscr{F}\left(\tau, \varkappa_q(\tau), \int_0^{\tau}\mathfrak{G}(\tau, \nu, \varkappa_q(\nu))\,d\nu\right) - \mathscr{F}\left(\tau, \varkappa(\tau), \int_0^{\tau}\mathfrak{G}(\tau, \nu, \varkappa(\nu))\,d\nu\right)\right\| \to 0$ as $q \to \infty$.

(v) $\|\mathcal{B}w_{\varkappa_q}(\tau) - \mathcal{B}w_{\varkappa}(\tau)\| \leq 2\mathscr{M}^*$.

Combine above condition with Lebesgue dominated convergence theorem, we get for $\tau \in [0, \tau_1]$

$$\begin{aligned}\|\mathcal{B}w_{\varkappa_q}(\tau) - \mathcal{B}w_{\varkappa}(\tau)\| \leq \|\mathcal{B}\|\mathscr{C}_T\Bigg[\frac{\mathscr{M}}{\Gamma(\varsigma)}\int_0^{\tau_1}(\tau-\nu)^{\varsigma-1}\Bigg\|\mathscr{F}\left(\nu, \varkappa_q(\nu), \int_0^{\nu}\mathfrak{G}(\nu, \mathfrak{r}, \varkappa_q(\mathfrak{r}))\,d\mathfrak{r}\right)\\ -\mathscr{F}\left(\nu, \varkappa(\nu), \int_0^{\nu}\mathfrak{G}(\nu, \mathfrak{r}, \varkappa(\mathfrak{r}))\,d\mathfrak{r}\right)\Bigg\|d\nu\Bigg] \to 0 \text{ as } n \to \infty.\end{aligned}$$

For $\tau \in (\omega_{\mathrm{j}}, \tau_{\mathrm{j}+1}]$,

$$\begin{aligned}\|\mathcal{B}w_{\varkappa_q}(\tau) - \mathcal{B}w_{\varkappa}(\tau)\| \leq \|\mathcal{B}\|\mathscr{C}_T\Bigg[&\frac{\mathscr{M}}{\Gamma(\alpha(1-\varsigma)+\varsigma)}\mathfrak{b}^{(\alpha-1)(1-\varsigma)}\mathcal{K}_{\zeta_\kappa}\|\varkappa_q(\tau) - \varkappa(\tau)\|\\ &+\frac{M}{\Gamma(\varsigma)}\int_0^{\tau_1}(\tau-\nu)^{\varsigma-1}\Bigg\|\mathscr{F}\left(\nu, \varkappa_q(\nu), \int_0^{\nu}\mathfrak{G}(\nu, \mathfrak{r}, \varkappa_q(\mathfrak{r}))\,d\mathfrak{r}\right)\\ &-\mathscr{F}\left(\nu, \varkappa(\nu), \int_0^{\nu}\mathfrak{G}(\nu, \mathfrak{r}, \varkappa(\mathfrak{r}))\,d\mathfrak{r}\right)\Bigg\|d\nu\Bigg].\end{aligned}$$

Therefore, we obtain for $\tau \in (\omega_{\mathrm{j}}, \tau_{\mathrm{j}+1}]$; $\mathrm{j} = 0, 1, 2, 3, \cdots, q$.

$$\begin{aligned}\|\mathfrak{X}\mathfrak{y}_q(\tau)) - \mathfrak{X}\mathfrak{y}(\tau)\| \leq &\frac{\mathscr{M}\mathcal{K}_{\zeta_{\mathrm{i}}}\|\varkappa_n(\tau) - \varkappa(\tau)\|}{\Gamma(\alpha(1-\varsigma)+\varsigma)} + \tau_{\mathrm{j}+1}^{(1-\alpha)(1-\varsigma)}\frac{\mathscr{M}}{\Gamma(\varsigma)}\int_0^{\tau_{\mathrm{j}+1}}(\mathfrak{b}-\nu)^{\varsigma-1}\\ &\times\Bigg[\Bigg\|\mathscr{F}\left(\nu, \varkappa_q(\nu), \int_0^{\nu}\mathfrak{G}(\nu, \mathfrak{r}, \varkappa_q(\mathfrak{r}))d\mathfrak{r}\right) - \mathscr{F}\left(\nu, \varkappa(\nu), \int_0^{\nu}\mathfrak{G}(\nu, \mathfrak{r}, \varkappa(\mathfrak{r}))\,d\mathfrak{r}\right)\Bigg\|\\ &+\Big\|\mathcal{B}w_{\varkappa_q}(\nu) - \mathcal{B}w_{\varkappa}(\nu)\Big\|\Bigg]d\nu\\ &\to 0 \text{ as } q \to \infty \text{ independent of } \tau.\end{aligned}$$

Therefore, F is continuous. Now, we aim to prove $\mathfrak{F}(\mathcal{E}_\kappa)$ is equi-continuous. Let $\mathfrak{y} \in \mathcal{E}_\kappa$ and $\varkappa(\tau) = \tau^{(\alpha-1)(1-\varsigma)}\mathfrak{y}(\tau)$, $\tau \in \mathfrak{J}'$.

For any $\lambda_1, \lambda_2 \in [0, \tau_1]$ with $\lambda_1 < \lambda_2$ and $\varkappa \in \mathcal{E}_\kappa$, we have

$$\begin{aligned}\|\mathfrak{X}\mathfrak{y}(\lambda_2)) - \mathfrak{X}\mathfrak{y}(\lambda_1)\| \leq &\left\|\lambda_2^{(1-\alpha)(1-\varsigma)}\mathcal{S}_{\varsigma,\alpha}(\lambda_2)\varkappa_0 - \lambda_1^{(1-\alpha)(1-\varsigma)}\mathcal{S}_{\varsigma,\alpha}(\lambda_1)\varkappa_0\right\|\\ &+\Bigg\|\int_0^{\lambda_1}\left[\lambda_2^{(1-\alpha)(1-\varsigma)}\mathcal{K}_\varsigma(\lambda_2-\nu) - \lambda_1^{(1-\alpha)(1-\varsigma)}\mathcal{K}_\varsigma(\lambda_1-\nu)\right]\\ &\quad\times\left[\mathscr{F}\left(\nu, \varkappa(\nu), \int_0^{\nu}\mathfrak{G}(\nu, \mathfrak{r}, \varkappa(\mathfrak{r}))\,d\mathfrak{r}\right) + \mathcal{B}w_{\varkappa}(\nu)\right]d\nu\Bigg\|\\ &\quad+\frac{\mathscr{M}}{\Gamma(\varsigma)}\lambda_2^{(1-\alpha)(1-\varsigma)}\int_{\lambda_1}^{\lambda_2}(\lambda_2-\nu)^{\varsigma-1}\times\left[\mathfrak{l}_{\kappa,1}(\nu) + \kappa\mathfrak{m}^*\mathfrak{l}_{\kappa,2}(\nu) + \mathscr{M}^*\right]d\nu\\ &= \Lambda_1 + \Lambda_2 + \Lambda_3,\end{aligned}$$

where

$$\begin{aligned}
\Lambda_1 &= \left\| \lambda_2^{(1-\alpha)(1-\varsigma)} \mathcal{S}_{\varsigma,\alpha}(\lambda_2)\varkappa_0 - \lambda_1^{(1-\alpha)(1-\varsigma)} \mathcal{S}_{\varsigma,\alpha}(\lambda_1)\varkappa_0 \right\|, \\
\Lambda_2 &= \left\| \int_0^{\lambda_1} \left[\lambda_2^{(1-\alpha)(1-\varsigma)} \mathcal{K}_\varsigma(\lambda_2-\nu) - \lambda_1^{(1-\alpha)(1-\varsigma)} \mathcal{K}_\varsigma(\lambda_1-\nu) \right] \left[\mathscr{F}\left(\nu, \varkappa(\nu), \int_0^\nu \mathfrak{G}(\nu,\mathfrak{r},\varkappa(\mathfrak{r}))\, d\mathfrak{r}\right) \right.\right. \\
&\qquad \left.\left. + \mathcal{B}w_\varkappa(\nu) \right] d\nu \right\|, \\
\Lambda_3 &= \frac{\mathscr{M}}{\Gamma(\varsigma)} \lambda_2^{(1-\alpha)(1-\varsigma)} \int_{\lambda_1}^{\lambda_2} (\lambda_2-\nu)^{\varsigma-1} \left[\mathfrak{l}_{\kappa,1}(\nu) + \kappa \mathfrak{m}^* \mathfrak{l}_{\kappa,2}(\nu) + \mathscr{M}^* \right] d\nu.
\end{aligned}$$

We have that $\Lambda_3 \to 0$ as $\lambda_2 \to \lambda_1$, independent of $\varkappa$.

$$\begin{aligned}
\Lambda_1 &= \frac{1}{\Gamma(\alpha(1-\varsigma))} \left\| \lambda_2^{(1-\alpha)(1-\varsigma)} \int_0^{\lambda_2} (\lambda_2-\nu)^{\alpha(1-\varsigma)-1} \nu^{\varsigma-1} \mathcal{Q}_\varsigma(\nu)\varkappa_0 \, d\nu \right. \\
&\quad \left. - \lambda_1^{(1-\alpha)(1-\varsigma)} \int_0^{\lambda_1} (\lambda_1-\nu)^{\alpha(1-\varsigma)-1} \nu^{\varsigma-1} \mathcal{Q}_\varsigma(\nu)\varkappa_0 \, d\nu \right\| \\
&\le \frac{\lambda_2^{(1-\alpha)(1-\varsigma)}}{\Gamma(\alpha(1-\varsigma))} \int_{\lambda_1}^{\lambda_2} (\lambda_2-\nu)^{\alpha(1-\varsigma)-1} \nu^{\varsigma-1} \|\mathcal{Q}_\varsigma(\nu)\varkappa_0\| \, d\nu \\
&\quad + \frac{1}{\Gamma(\alpha(1-\varsigma))} \int_0^{\lambda_1} \left| \lambda_2^{(1-\alpha)(1-\varsigma)} (\lambda_2-\nu)^{\alpha(1-\varsigma)-1} - \lambda_1^{(1-\alpha)(1-\varsigma)} (\lambda_1-\nu)^{\alpha(1-\varsigma)-1} \right| \\
&\quad \times \nu^{\varsigma-1} \|\mathcal{Q}_\varsigma(\nu)\varkappa_0\| \, d\nu \\
&\le \frac{\mathscr{M} \lambda_1^{\varsigma-1} \lambda_2^{(1-\alpha)(1-\varsigma)}}{\Gamma(\alpha(1-\varsigma))\Gamma(\varsigma)} \frac{\|\varkappa_0\|}{\alpha(1-\varsigma)} (\lambda_2-\lambda_1)^{\alpha(1-\varsigma)} + \frac{\mathscr{M}\|\varkappa_0\|}{\Gamma(\alpha(1-\varsigma))\Gamma(\varsigma)} \\
&\quad \times \int_0^{\lambda_1} \left| \left[\lambda_2^{(1-\alpha)(1-\varsigma)} (\lambda_2-\nu)^{\alpha(1-\varsigma)-1} - \lambda_1^{(1-\alpha)(1-\varsigma)} (\lambda_1-\nu)^{\alpha(1-\varsigma)-1} \right] \nu^{\varsigma-1} \right| d\nu \\
&\quad \to 0 \text{ as } \lambda_2 \to \lambda_1 \text{ independent of } \varkappa.
\end{aligned}$$

Now,

$$\begin{aligned}
\Lambda_2 &\le \left\| \int_0^{\lambda_1} \left[\lambda_2^{(1-\alpha)(1-\varsigma)} (\lambda_2-\nu)^{\varsigma-1} - \lambda_1^{(1-\alpha)(1-\varsigma)} (\lambda_1-\nu)^{\varsigma-1} \right] \mathcal{Q}_\varsigma(\lambda_2-\nu) \right. \\
&\qquad \left. \times \left[\mathscr{F}\left(\nu, \varkappa(\nu), \int_0^\nu \mathfrak{G}(\nu,\mathfrak{r},\varkappa(\mathfrak{r}))\, d\mathfrak{r}\right) + \mathcal{B}w_\varkappa(\nu) \right] d\nu \right\| + \lambda_1^{(1-\alpha)(1-\varsigma)} \left\| \int_0^{\lambda_1} (\lambda_1-\nu)^{\varsigma-1} \right. \\
&\qquad \left. \times [\mathcal{Q}_\varsigma(\lambda_2-\nu) - \mathcal{Q}_\varsigma(\lambda_1-\nu)] \left[\mathscr{F}\left(\nu, \varkappa(\nu), \int_0^\nu \mathfrak{G}(\nu,\mathfrak{r},\varkappa(\mathfrak{r}))\, d\mathfrak{r}\right) + \mathcal{B}w_\varkappa(\nu) \right] d\nu \right\| \\
&\le \frac{\mathscr{M}}{\Gamma(\varsigma)} \int_0^{\lambda_1} \left| \lambda_2^{(1-\alpha)(1-\varsigma)} (\lambda_2-\nu)^{\varsigma-1} - \lambda_1^{(1-\alpha)(1-\varsigma)} (\lambda_1-\nu)^{\varsigma-1} \right| \left[l_{\kappa,1}(\nu) + \kappa \mathfrak{m}^* l_{\kappa,2}(\nu) + \mathscr{M}^* \right] d\nu \\
&\quad + \lambda_1^{(1-\alpha)(1-\varsigma)} \int_0^{\lambda_1} (\lambda_1-\nu)^{\varsigma-1} \left\| [\mathcal{Q}_\varsigma(\lambda_2-\nu) - \mathcal{Q}_\varsigma(\lambda_1-\nu)] \left[\mathscr{F}\left(\nu, \varkappa(\nu), \int_0^\nu \mathfrak{G}(\nu,\mathfrak{r},\varkappa(\mathfrak{r}))\, d\mathfrak{r}\right) \right.\right. \\
&\qquad \left.\left. + \mathcal{B}w_\varkappa(\nu) \right] \right\| d\nu \\
&= \Lambda_{2,1} + \Lambda_{2,2},
\end{aligned}$$

where

$$\Lambda_{2,1} = \frac{\mathscr{M}}{\Gamma(\varsigma)} \int_0^{\lambda_1} \left| \lambda_2^{(1-\alpha)(1-\varsigma)} (\lambda_2 - \nu)^{\varsigma-1} - \lambda_1^{(1-\alpha)(1-\varsigma)} (\lambda_1 - \nu)^{\varsigma-1} \right| \left[\mathrm{l}_{\kappa,1}(\nu) + \kappa \mathfrak{m}^* \mathrm{l}_{\kappa,2}(\nu) + \mathscr{M}^* \right] d\nu$$

and

$$\Lambda_{2,2} = \lambda_1^{(1-\alpha)(1-\varsigma)} \int_0^{\lambda_1} (\lambda_1 - \nu)^{\varsigma-1} \left\| [\mathcal{Q}_\varsigma(\lambda_2 - \nu) - \mathcal{Q}_\varsigma(\lambda_1 - \nu)] \left[\mathscr{F}\left(\nu, \varkappa(\nu), \int_0^\nu \mathfrak{G}(\nu, \mathfrak{r}, \varkappa(\mathfrak{r}))\, d\mathfrak{r}\right) + \mathcal{B} w_\varkappa(\nu) \right] \right\| d\nu.$$

If $\epsilon \in (0, \tau_1)$, then we have

$$\begin{aligned}
\Lambda_{2,2} &\le \lambda_1^{(1-\alpha)(1-\varsigma)} \int_0^{\tau_1-\epsilon} (\lambda_1 - \nu)^{\varsigma-1} \|\mathcal{Q}_\varsigma(\lambda_2 - \nu) - \mathcal{Q}_\varsigma(\lambda_1 - \nu)\| \left[\mathrm{l}_{\kappa,1}(\nu) + \kappa \mathfrak{m}^* \mathrm{l}_{\kappa,2}(\nu) + \mathscr{M}^* \right] d\nu \\
&\quad + \frac{2\mathscr{M} \lambda_1^{(1-\alpha)(1-\varsigma)}}{\Gamma(\varsigma)} \int_{\tau_1-\epsilon}^{\lambda_1} (\lambda_1 - \nu)^{\varsigma-1} \left[\mathrm{l}_{\kappa,1}(\nu) + \kappa \mathfrak{m}^* \mathrm{l}_{\kappa,2}(\nu) + \mathscr{M}^* \right] d\nu \\
&\le \lambda_1^{(1-\alpha)(1-\varsigma)} \int_0^{\tau_1-\epsilon} (\lambda_1 - \nu)^{\varsigma-1} \left[\mathrm{l}_{\kappa,1}(\nu) + \kappa \mathfrak{m}^* \mathrm{l}_{\kappa,2}(\nu) + \mathscr{M}^* \right] d\nu \\
&\quad \times \sup_{\nu \in [0, \tau - \epsilon]} \|\mathcal{Q}_\varsigma(\lambda_2 - \nu) - \mathcal{Q}_\varsigma(\lambda_1 - \nu)\| \\
&\quad + \frac{2\mathscr{M} \lambda_1^{(1-\alpha)(1-\varsigma)}}{\Gamma(\varsigma)} \int_{\lambda_1-\epsilon}^{\lambda_1} (\lambda_1 - \nu)^{\varsigma-1} \left[\mathrm{l}_{\kappa,1}(\nu) + \kappa \mathfrak{m}^* \mathrm{l}_{\kappa,2}(\nu) + \mathscr{M}^* \right] d\nu.
\end{aligned}$$

For any $\lambda_1, \lambda_2 \in (\omega_{\mathrm{j}}, \tau_{\mathrm{j}+1}]$ with $\lambda_1 < \lambda_2$, we have

$$\begin{aligned}
\|\mathfrak{X}\mathfrak{y}(\lambda_2)) - \mathfrak{X}\mathfrak{y}(\lambda_1)\| &\le \left\| \lambda_2^{(1-\alpha)(1-\varsigma)} \mathcal{S}_{\varsigma,\alpha}(\lambda_2) \zeta_{\mathrm{j}}(\omega_{\mathrm{j}}, \tau_{\mathrm{j}+1}) - \lambda_1^{(1-\alpha)(1-\varsigma)} \mathcal{S}_{\varsigma,\alpha}(\lambda_1) \zeta_{\mathrm{j}}(\omega_{\mathrm{j}}, \tau_{\mathrm{j}+1}) \right\| \\
&\quad + \left\| \int_0^{\lambda_1} \left[\lambda_2^{(1-\alpha)(1-\varsigma)} \mathcal{K}_\varsigma(\lambda_2 - \nu) - \lambda_1^{(1-\alpha)(1-\varsigma)} \mathcal{K}_\varsigma(\lambda_1 - \nu) \right] \right. \\
&\quad \left. \times \left[\mathscr{F}\left(\nu, \varkappa(\nu), \int_0^\nu \mathfrak{G}(\nu, \mathfrak{r}, \varkappa(\mathfrak{r}))\, d\mathfrak{r}\right) + \mathcal{B} w_\varkappa(\nu) \right] d\nu \right\| \\
&\quad + \frac{\mathscr{M}}{\Gamma(\varsigma)} \lambda_2^{(1-\alpha)(1-\varsigma)} \int_{\lambda_1}^{\lambda_2} (\lambda_2 - \nu)^{\varsigma-1} \times \left[\mathrm{l}_{\kappa,1}(\nu) + \kappa \mathfrak{m}^* \mathrm{l}_{\kappa,2}(\nu) + \mathscr{M}_1{}^* \right] d\nu \\
&= \Lambda_1^* + \Lambda_2^* + \Lambda_3^*,
\end{aligned}$$

where

$$\begin{aligned}
\Lambda_1^* &= \left\| \lambda_2^{(1-\alpha)(1-\varsigma)} \mathcal{S}_{\varsigma,\alpha}(\lambda_2) \varkappa_0 - \lambda_1^{(1-\alpha)(1-\varsigma)} \mathcal{S}_{\varsigma,\alpha}(\lambda_1) \varkappa_0 \right\|, \\
\Lambda_2^* &= \left\| \int_0^{\lambda_1} \left[\lambda_2^{(1-\alpha)(1-\varsigma)} \mathcal{K}_\varsigma(\lambda_2 - \nu) - \lambda_1^{(1-\alpha)(1-\varsigma)} \mathcal{K}_\varsigma(\lambda_1 - \nu) \right] \left[\mathscr{F}\left(\nu, \varkappa(\nu), \int_0^\nu \mathfrak{G}(\nu, \mathfrak{r}, \varkappa(\mathfrak{r}))\, d\mathfrak{r}\right) + \mathcal{B} w_\varkappa(\nu) \right] d\nu \right\|, \\
\Lambda_3^* &= \frac{\mathscr{M}}{\Gamma(\varsigma)} \lambda_2^{(1-\alpha)(1-\varsigma)} \int_{\lambda_1}^{\lambda_2} (\lambda_2 - \nu)^{\varsigma-1} \left[\mathrm{l}_{\kappa,1}(\nu) + \kappa m^* \mathrm{l}_{\kappa,2}(\nu) + \mathscr{M}_1{}^* \right] d\nu.
\end{aligned}$$

We get $\Lambda_3^* \to 0$ as $\lambda_2 \to \lambda_1$ independent of $\varkappa$.

$$\begin{aligned}
\Lambda_1^* &= \frac{1}{\Gamma(\alpha(1-\varsigma))}\bigg\|\lambda_2^{(1-\alpha)(1-\varsigma)}\int_0^{\lambda_2}(\lambda_2-\nu)^{\alpha(1-\varsigma)-1}\nu^{\varsigma-1}\mathcal{Q}_\varsigma(\nu)\varkappa_0\,d\nu \\
&\quad -\lambda_1^{(1-\alpha)(1-\varsigma)}\int_0^{\lambda_1}(\lambda_1-\nu)^{\alpha(1-\varsigma)-1}\nu^{\varsigma-1}\mathcal{Q}_\varsigma(\nu)\varkappa_0\,d\nu\bigg\| \\
&\le \frac{\lambda_2^{(1-\alpha)(1-\varsigma)}}{\Gamma(\alpha(1-\varsigma))}\int_{\lambda_1}^{\lambda_2}(\lambda_2-\nu)^{\alpha(1-\varsigma)-1}\nu^{\varsigma-1}\|\mathcal{Q}_\varsigma(\nu)\varkappa_0\|\,d\nu \\
&\quad +\int_0^{\lambda_1}\left|\lambda_2^{(1-\alpha)(1-\varsigma)}(\lambda_2-\nu)^{\alpha(1-\varsigma)-1}-\lambda_1^{(1-\alpha)(1-\varsigma)}(\lambda_1-\nu)^{\alpha(1-\varsigma)-1}\right|\nu^{\varsigma-1}\|\mathcal{Q}_\varsigma(\nu)\varkappa_0\|\,d\nu \\
&\le \frac{\mathscr{M}\lambda_1^{\varsigma-1}\lambda_2^{(1-\alpha)(1-\varsigma)}}{\Gamma(\alpha(1-\varsigma))\Gamma(\varsigma)}\frac{\|\varkappa_0\|}{\alpha(1-\varsigma)}(\lambda_2-\lambda_1)^{\alpha(1-\varsigma)}+\frac{\mathscr{M}\|\varkappa_0\|}{\Gamma(\alpha(1-\varsigma))\Gamma(\varsigma)} \\
&\quad \times\int_0^{\lambda_1}\left|\left[\lambda_2^{(1-\alpha)(1-\varsigma)}(\lambda_2-\nu)^{\alpha(1-\varsigma)-1}-\lambda_1^{(1-\alpha)(1-\varsigma)}(\lambda_1-\nu)^{\alpha(1-\varsigma)-1}\right]\nu^{\varsigma-1}\right|d\nu \\
&\qquad \to 0 \text{ as } \lambda_2 \to \lambda_1 \text{ independent of } \varkappa.
\end{aligned}$$

Also,

$$\begin{aligned}
\Lambda_2^* &\le \bigg\|\int_0^{\lambda_1}\left[\lambda_2^{(1-\alpha)(1-\varsigma)}(\lambda_2-\nu)^{\varsigma-1}-\lambda_1^{(1-\alpha)(1-\varsigma)}(\lambda_1-\nu)^{\varsigma-1}\right]\mathcal{Q}_\varsigma(\lambda_2-\nu) \\
&\quad \times\left[\mathscr{F}\left(\nu,\varkappa(\nu),\int_0^\nu\mathfrak{G}(\nu,\mathfrak{r},\varkappa(\mathfrak{r}))\,d\mathfrak{r}\right)+\mathcal{B}w_\varkappa(\nu)\right]d\nu\bigg\|+\lambda_1^{(1-\alpha)(1-\varsigma)}\bigg\|\int_0^{\lambda_1}(\lambda_1-\nu)^{\varsigma-1} \\
&\quad \times[\mathcal{Q}_\varsigma(\lambda_2-\nu)-\mathcal{Q}_\varsigma(\lambda_1-\nu)]\left[\mathscr{F}\left(\nu,\varkappa(\nu),\int_0^\nu\mathfrak{G}(\nu,\mathfrak{r},\varkappa(\mathfrak{r}))\,d\mathfrak{r}\right)+\mathcal{B}w_\varkappa(\nu)\right]d\nu\bigg\| \\
&\le \frac{\mathscr{M}}{\Gamma(\varsigma)}\int_0^{\lambda_1}\left|\lambda_2^{(1-\alpha)(1-\varsigma)}(\lambda_2-\nu)^{\varsigma-1}-\lambda_1^{(1-\alpha)(1-\varsigma)}(\lambda_1-\nu)^{\varsigma-1}\right|\left[\mathrm{l}_{\kappa,1}(\nu)+\varkappa\mathfrak{m}^*\mathrm{l}_{\kappa,2}(\nu)+\mathscr{M}^*\right]d\nu \\
&\quad +\lambda_1^{(1-\alpha)(1-\varsigma)}\int_0^{\lambda_1}(\lambda_1-\nu)^{\varsigma-1}\bigg\|[\mathcal{Q}_\varsigma(\lambda_2-\nu)-\mathcal{Q}_\varsigma(\lambda_1-\nu)]\bigg[\mathscr{F}\left(\nu,\varkappa(\nu),\int_0^\nu\mathfrak{G}(\nu,\mathfrak{r},\varkappa(\mathfrak{r}))\,d\mathfrak{r}\right) \\
&\qquad +\mathcal{B}w_\varkappa(\nu)\bigg]\bigg\|\,d\nu \\
&= \Lambda_{2,1}^*+\Lambda_{2,2}^*
\end{aligned}$$

where

$$\Lambda_{2,1}^* = \frac{\mathscr{M}}{\Gamma(\varsigma)}\int_0^{\lambda_1}\left|\lambda_2^{(1-\alpha)(1-\varsigma)}(\lambda_2-\nu)^{\varsigma-1}-\lambda_1^{(1-\alpha)(1-\varsigma)}(\lambda_1-\nu)^{\varsigma-1}\right|\left[\mathrm{l}_{\kappa,1}(\nu)+\kappa\mathfrak{m}^*\mathrm{l}_{\kappa,2}(\nu)+\mathscr{M}_1{}^*\right]d\nu$$

and

$$\begin{aligned}
\Lambda_{2,2}^* &= \lambda_1^{(1-\alpha)(1-\varsigma)}\int_0^{\lambda_1}(\lambda_1-\nu)^{\varsigma-1}\bigg\|[\mathcal{Q}_\varsigma(\lambda_2-\nu)-\mathcal{Q}_\varsigma(\lambda_1-\nu)]\bigg[\mathscr{F}\left(\nu,\varkappa(\nu),\int_0^\nu\mathfrak{G}(\nu,\mathfrak{r},\varkappa(\mathfrak{r}))\,d\mathfrak{r}\right) \\
&\qquad +\mathcal{B}w_\varkappa(\nu)\bigg]\bigg\|\,d\nu.
\end{aligned}$$

If $\epsilon \in (\omega_{\mathrm{j}}, \tau_{\mathrm{j}+1})$, then we have

$$\begin{aligned}\Lambda_{2,2}^* &\leq \lambda_1^{(1-\alpha)(1-\varsigma)} \int_{\omega_{\mathrm{j}}}^{\tau_{\mathrm{j}+1}-\epsilon} (\lambda_1-\nu)^{\varsigma-1} \|\mathcal{Q}_\varsigma(\lambda_2-\nu)-\mathcal{Q}_\varsigma(\lambda_1-\nu)\| \left[\mathrm{l}_{\kappa,1}(\nu)+\kappa\mathfrak{m}^*\mathrm{l}_{\kappa,2}(\nu)+\mathscr{M}_1{}^*\right] d\nu \\ &\quad + \frac{2\mathscr{M}\lambda_1^{(1-\alpha)(1-\varsigma)}}{\Gamma(\varsigma)} \int_{\tau_{\mathrm{j}+1}-\epsilon}^{\tau_{\mathrm{j}+1}} (\lambda_1-\nu)^{\varsigma-1}\left[\mathrm{l}_{\kappa,1}(\nu)+\kappa\mathfrak{m}^*\mathrm{l}_{\kappa,2}(\nu)+\mathscr{M}_1{}^*\right] d\nu \\ &\leq \lambda_1^{(1-\alpha)(1-\varsigma)} \int_{\omega_{\mathrm{j}}}^{\tau_{\mathrm{j}+1}-\epsilon} (\lambda_1-\nu)^{\varsigma-1}\left[\mathrm{l}_{\kappa,1}(\nu)+\kappa\mathfrak{m}^*\mathrm{l}_{\kappa,2}(\nu)+\mathscr{M}_1{}^*\right] d\nu \\ &\quad \times \sup_{\nu\in[\omega_{\mathrm{j}},\tau_{\mathrm{j}+1}-\epsilon]} \|\mathcal{Q}_\varsigma(\lambda_2-\nu)-\mathcal{Q}_\varsigma(\lambda_1-\nu)\| \\ &\qquad + \frac{2\mathscr{M}\lambda_1^{(1-\alpha)(1-\varsigma)}}{\Gamma(\varsigma)} \int_{\tau_{\mathrm{j}+1}-\epsilon}^{\tau_{\mathrm{j}+1}} (\lambda_1-\nu)^{\varsigma-1}\left[\mathrm{l}_{\kappa,1}(\nu)+\kappa\mathfrak{m}^*\mathrm{l}_{\kappa,2}(\nu)+\mathscr{M}_1{}^*\right] d\nu.\end{aligned}$$

Since $\mathcal{Q}_\varsigma(\tau)$ is continuous in uniform operator topology for $\tau > 0$ and $(\tau-\nu)^{\varsigma-1}\mathrm{l}_{\kappa,1}(\nu)$, $(\tau-\nu)^{\varsigma-1}\mathrm{l}_{\kappa,2}(\nu)$ are integrable over $[0,\tau_1]$,$[\omega_{\mathrm{j}},\tau_{\mathrm{j}+1}]$. Clearly, as $\epsilon \to 0$ independent of $\mathfrak{y}$ and $\lambda_1 \to \lambda_2$ the integral $\Lambda_{2,1}$, $\Lambda_{2,2}$ and $\Lambda_{2,1}^*$ and $\Lambda_{2,2}^*$ tend to 0 therefore from above results we concluded that $\mathfrak{X}(\mathcal{E}_\kappa)$ is equi-continuous on $\mathfrak{J}'$. Since $\tau^{(1-\alpha)(1-\varsigma)}S_{\varsigma,\alpha}(\tau)$ is uniformly continuous on $\mathfrak{J}$, therefore $\mathfrak{X}(\mathcal{E}_\kappa)$ is equi-continuous on $\mathfrak{J}$. Let $\mathcal{S} \subset \mathcal{E}$ using Lemma (2.10), we obtain a countable set $\mathcal{S}_1 = \{\mathfrak{y}_q\} \subset \mathcal{S}$ such that

$$\Delta(\mathfrak{X}(\mathcal{S})) \leq 2\Delta(\mathfrak{X}(\mathcal{S}_1)).$$

Let $\varkappa_q(\tau) = \tau^{(\alpha-1)(1-\varsigma)}\mathfrak{y}_q(\tau)$. For $\tau \in [0,\tau_1]$, we have

$$\begin{aligned}\Delta(\mathfrak{X}(\mathcal{S}_1)(\tau)) &\leq \Delta\left(\left\{\tau^{(1-\alpha)(1-\varsigma)}\mathcal{S}_{\varsigma,\alpha}(\tau)\varkappa_0\right\}\right) \\ &\quad + \Delta\left(\left\{\tau^{(1-\alpha)(1-\varsigma)}\int_0^\tau \mathcal{K}_\varsigma(\tau-\nu)\mathscr{F}\left(\nu,\varkappa_q(\nu),\int_0^\nu \mathfrak{G}(\nu,\mathfrak{r},\varkappa_q(\mathfrak{r}))\,d\mathfrak{r}\right)d\nu\right\}\right) \\ &\quad + \Delta\left(\left\{\tau^{(1-\alpha)(1-\varsigma)}\int_0^\tau \mathcal{K}_\varsigma(\tau-\nu)\mathcal{B}w_{\varkappa_q}(\nu)d\nu\right\}\right) \\ &= \Pi_1+\Pi_2+\Pi_3, \end{aligned} \tag{15}$$

where

$$\begin{aligned}\Pi_1 &= \Delta\left(\left\{\tau^{(1-\alpha)(1-\varsigma)}\mathcal{S}_{\varsigma,\alpha}(\tau)\varkappa_0\right\}\right), \\ \Pi_2 &= \Delta\left(\left\{\tau^{(1-\alpha)(1-\varsigma)}\int_0^\tau \mathcal{K}_\varsigma(\tau-\nu)\mathscr{F}\left(\nu,\varkappa_q(\nu),\int_0^\nu \mathfrak{G}(\nu,\mathfrak{r},\varkappa_n(\mathfrak{r}))\,d\mathfrak{r}\right)d\nu\right\}\right), \\ \Pi_3 &= \Delta\left(\left\{\tau^{(1-\alpha)(1-\varsigma)}\int_0^\tau \mathcal{K}_\varsigma(\tau-\nu)\mathcal{B}w_{\varkappa_q}(\nu)d\nu\right\}\right).\end{aligned}$$

Therefore, from Lemma 2.7, we have

$$\Pi_1 \leq \frac{\mathscr{M}}{\Gamma[\alpha(1-\varsigma)+\varsigma]}\Delta(\varkappa_0) \leq 0. \tag{16}$$

By assumption (A4) and Lemma 2.12, we have

$$\begin{aligned}
\Pi_2 &\leq \frac{2\mathscr{M}\mathfrak{b}^{(1-\alpha)(1-\varsigma)}}{\Gamma(\varsigma)} \int_0^\tau (\tau-\nu)^{\varsigma-1}\Delta\Big(\Big\{\mathscr{F}\Big(\nu, \varkappa_n(\nu), \int_0^\nu \mathfrak{G}(\nu,\mathfrak{r},\varkappa_n(\mathfrak{r}))\,d\mathfrak{r}\Big)\Big\}\Big)d\nu \\
&\leq \frac{2\mathscr{M}\mathfrak{b}^{(1-\alpha)(1-\varsigma)}}{\Gamma(\varsigma)} \int_0^\tau (\tau-\nu)^{\varsigma-1}\varrho(\nu)\Big[\Delta\Big(\Big\{\nu^{(1-\alpha)(1-\varsigma)}\varkappa_q(\nu)\Big\}\Big) \\
&\quad + \Delta\Big(\Big\{\int_0^\nu \mathfrak{G}(\nu,\mathfrak{r},\varkappa_q(\mathfrak{r}))\,d\mathfrak{r}\Big\}\Big)\Big]d\nu \\
&\leq \frac{2\mathscr{M}\mathfrak{b}^{(1-\alpha)(1-\varsigma)}}{\Gamma(\varsigma)} \int_0^\tau (\tau-\nu)^{\varsigma-1}\varrho(\nu)\left[\Delta\left(\left\{\nu^{(1-\alpha)(1-\varsigma)}\varkappa_q(\nu)\right\}\right)\right. \\
&\quad \left. +2\int_0^\nu \rho(\nu,\mathfrak{r})\Delta\left(\left\{\mathfrak{r}^{(1-\alpha)(1-\varsigma)}\varkappa_q(\mathfrak{r})\right\}\right)d\mathfrak{r}\right]d\nu \\
&\leq \frac{2\mathscr{M}\mathfrak{b}^{(1-\alpha)(1-\varsigma)}[1+2\rho^*]}{\Gamma(\varsigma)}\varrho^* \sup_{0\leq\nu\leq\mathcal{B}} \Delta\left(\left\{\mathfrak{y}_q(\nu)\right\}\right). \qquad (17)
\end{aligned}$$

Also,

$$\begin{aligned}
\Pi_3 &\leq \frac{2\mathscr{M}\,\|\mathcal{B}\|\mathfrak{b}^{(1-\alpha)(1-\varsigma)}}{\Gamma(\varsigma)} \int_0^\tau (\tau-\nu)^{\varsigma-1}\Delta\left(\left\{w_{\varkappa_q}(\nu)\right\}\right)d\nu \\
&\leq \frac{2\mathscr{M}\,\|\mathcal{B}\|\mathfrak{b}^{(1-\alpha)(1-\varsigma)}}{\Gamma(\varsigma)} \int_0^\tau (\tau-\nu)^{\varsigma-1}\Phi(\nu)\left[\frac{\mathscr{M}\mathfrak{b}^{(\alpha-1)(1-\varsigma)}}{\Gamma(\alpha(1-\varsigma)+\varsigma)}\Delta(\varkappa_0)\right. \\
&\quad \left. +\Delta\left(\left\{\int_0^{\mathfrak{b}} \mathcal{K}_\varsigma(\mathfrak{b}-\nu)\mathscr{F}\left(\nu,\varkappa_q(\nu),\int_0^\nu \mathfrak{G}(\nu,\mathfrak{r},\varkappa_q(\mathfrak{r}))\,d\mathfrak{r}\right)d\nu\right\}\right)\right] \\
&\leq \frac{2\mathscr{M}\,\|\mathcal{B}\|\mathfrak{b}^{(1-\alpha)(1-\varsigma)}}{\Gamma(\varsigma)}\Phi^*\left[\frac{2\mathscr{M}[1+2\rho^*]}{\Gamma(\varsigma)}\varrho^*\right]\sup_{0\leq\nu\leq\mathcal{B}} \Delta\left(\left\{\mathfrak{y}_q(\nu)\right\}\right). \qquad (18)
\end{aligned}$$

Thus, we can evaluate from (15), (16), (17) and (18), and from Lemma 2.11 that

$$\Delta(\mathfrak{X}(\mathcal{S}_1)(\tau)) \leq \mathcal{N}\Delta(\{\mathfrak{y}_q\}). \qquad (19)$$

where

$$\mathcal{N} = \frac{2\mathscr{M}\,\mathfrak{b}^{(1-\alpha)(1-\varsigma)}}{\Gamma(\varsigma)}\left[1+\frac{2\mathscr{M}\|\mathcal{B}\|\Phi^*}{\Gamma(\varsigma)}\right](1+2\rho^*)\varrho^*.$$

For $\tau \in (\omega_{\mathrm{j}}, \tau_{\mathrm{j}+1}]$,

$$\begin{aligned}
\Delta(\mathfrak{X}(\mathcal{S}_1)(\tau)) &\leq \Delta\left(\left\{\tau^{(1-\alpha)(1-\varsigma)}\mathcal{S}_{\varsigma,\alpha}(\tau)\nu_{\mathrm{j}}(\omega_{\mathrm{j}},\varkappa_q(\omega_{\mathrm{j}})\right\}\right) \\
&\quad + \Delta\left(\left\{\tau^{(1-\alpha)(1-\varsigma)}\int_{\omega_{\mathrm{j}}}^{\tau_{\mathrm{j}+1}} \mathcal{K}_\varsigma(\tau_{\mathrm{j}+1}-\nu)\mathscr{F}\left(\nu,\varkappa_q(\nu),\int_0^\nu \mathfrak{G}(\nu,\mathfrak{r},\varkappa_q(\mathfrak{r}))\,d\mathfrak{r}\right)d\nu\right\}\right) \\
&\quad + \Delta\left(\left\{\tau^{(1-\alpha)(1-\varsigma)}\int_{\omega_{\mathrm{j}}}^{\tau_{\mathrm{j}+1}} \mathcal{K}_\varsigma(\tau-\nu)\mathcal{B}w_{\varkappa_q}(\nu)d\nu\right\}\right) \\
&= \Pi_1^* + \Pi_2^* + \Pi_3^*, \qquad (20)
\end{aligned}$$

where

$$\begin{aligned}
\Pi_1^* &= \Delta\left(\left\{\tau^{(1-\alpha)(1-\varsigma)}\mathcal{S}_{\varsigma,\alpha}(\tau)\nu_{\mathrm{j}}(\omega_{\mathrm{j}},\varkappa_q(\omega_{\mathrm{j}})\right\}\right), \\
\Pi_2^* &= \Delta\left(\left\{\tau^{(1-\alpha)(1-\varsigma)}\int_{\omega_{\mathrm{j}}}^{\tau_{\mathrm{j}+1}} \mathcal{K}_\varsigma(\tau-\nu)\mathscr{F}\left(\nu,\varkappa_q(\nu),\int_0^\nu \mathfrak{G}(\nu,\mathfrak{r},\varkappa_q(\mathfrak{r}))\,d\mathfrak{r}\right)d\nu\right\}\right),
\end{aligned}$$

and

$$\Pi_3^* = \Delta\left(\left\{\tau^{(1-\alpha)(1-\varsigma)}\int_{\omega_{\mathfrak{j}}}^{\tau_{\mathfrak{j}+1}}\mathcal{K}_\varsigma(\tau-\nu)\mathcal{B}w_{\varkappa_q}(\nu)d\nu\right\}\right).$$

From Lemma 2.7, we have

$$\Pi_1^* \leq \frac{\mathscr{M}}{\Gamma[\alpha(1-\varsigma)+\varsigma]}\Delta\left(\zeta_{\mathfrak{j}}(\omega_{\mathfrak{j}},\varkappa_q(\omega_{\mathfrak{j}}))\right) \leq \frac{\mathscr{M}\mathcal{K}_{\nu_\kappa}}{\Gamma[\alpha(1-\varsigma)+\varsigma]}\tau^{(\alpha-1)(1-\varsigma)}\Delta(\{\mathfrak{y}_q\}). \tag{21}$$

By assumption (A4) and Lemma 2.12, we have

$$\begin{aligned}
\Pi_2^* &\leq \frac{2\mathscr{M}\mathfrak{b}^{(1-\alpha)(1-\varsigma)}}{\Gamma(\varsigma)}\int_{\omega_{\mathfrak{j}}}^{\tau_{\mathfrak{j}+1}}(\tau-\nu)^{\varsigma-1}\Delta\left(\left\{\mathscr{F}\left(\nu,\varkappa_q(\nu),\int_0^\nu \mathfrak{G}(\nu,\mathfrak{r},\varkappa_q(\mathfrak{r}))\,d\mathfrak{r}\right)\right\}\right)d\nu \\
&\leq \frac{2\mathscr{M}\mathfrak{b}^{(1-\alpha)(1-\varsigma)}}{\Gamma(\varsigma)}\int_{\omega_{\mathfrak{j}}}^{\tau_{\mathfrak{j}+1}}(\tau-\nu)^{\varsigma-1}\varrho(\nu)\left[\Delta\left(\left\{\nu^{(1-\alpha)(1-\varsigma)}\varkappa_q(\nu)\right\}\right)\right. \\
&\quad \left.+\Delta\left(\left\{\int_0^\nu \mathfrak{G}(\nu,\mathfrak{r},\varkappa_q(\mathfrak{r}))\,d\mathfrak{r}\right\}\right)\right]d\nu \\
&\leq \frac{2\mathscr{M}\mathfrak{b}^{(1-\alpha)(1-\varsigma)}}{\Gamma(\varsigma)}\int_{\omega_{\mathfrak{j}}}^{\tau_{\mathfrak{j}+1}}(\tau-\nu)^{\varsigma-1}\varrho(\nu)\left[\Delta\left(\left\{\nu^{(1-\alpha)(1-\varsigma)}\varkappa_q(\nu)\right\}\right)\right. \\
&\quad \left.+2\int_0^\nu \rho(\nu,\mathfrak{r})\Delta\left(\left\{\mathfrak{r}^{(1-\alpha)(1-\varsigma)}\varkappa_q(\mathfrak{r})\right\}\right)d\mathfrak{r}\right]d\nu \\
&\leq \frac{2\mathscr{M}\mathfrak{b}^{(1-\alpha)(1-\varsigma)}[1+2\rho^*]}{\Gamma(\varsigma)}\varrho^*\sup_{\omega_i\leq\nu\leq\tau_{\mathfrak{j}+1}}\Delta\left(\left\{\mathfrak{y}_q(\nu)\right\}\right).
\end{aligned} \tag{22}$$

Also,

$$\begin{aligned}
\Pi_3^* &\leq \frac{2\mathscr{M}\,\|\mathcal{B}\|\mathfrak{b}^{(1-\alpha)(1-\varsigma)}}{\Gamma(\varsigma)}\int_{\omega_{\mathfrak{j}}}^{\tau_{\mathfrak{j}+1}}(\tau_{\mathfrak{j}+1}-\nu)^{\varsigma-1}\Delta\left(\left\{w_{\varkappa_q}(\nu)\right\}\right)d\nu \\
&\leq \frac{2\mathscr{M}\,\|\mathcal{B}\|\mathfrak{b}^{(1-\alpha)(1-\varsigma)}}{\Gamma(\varsigma)}\int_{\omega_{\mathfrak{j}}}^{\tau_{\mathfrak{j}+1}}(\tau-\nu)^{\varsigma-1}\Phi(\nu)\left[\frac{\mathscr{M}\mathfrak{b}^{(\alpha-1)(1-\varsigma)}}{\Gamma(\alpha(1-\varsigma)+\varsigma)}\Delta\left(\zeta_i(\omega_{\mathfrak{j}},\varkappa_q(\omega_{\mathfrak{j}}))\right)\right. \\
&\quad \left.+\Delta\left(\left\{\int_{\omega_{\mathfrak{j}}}^{\tau_{\mathfrak{j}+1}}\mathcal{K}_\varsigma(\tau_{\mathfrak{j}+1}-\nu)\mathscr{F}\left(\nu,\varkappa_q(\nu),\int_0^\nu \mathfrak{G}(\nu,\mathfrak{r},\varkappa_q(\mathfrak{r}))\,d\mathfrak{r}\right)d\nu\right\}\right)\right]d\nu \\
&\leq \frac{2\mathscr{M}\,\|\mathcal{B}\|\mathfrak{b}^{(1-\alpha)(1-\varsigma)}}{\Gamma(\varsigma)}\Phi^*\left[\frac{\mathscr{M}\mathfrak{b}^{(\alpha-1)(1-\varsigma)}}{\Gamma(\alpha(1-\varsigma)+\varsigma)}\mathcal{K}_{\zeta_\kappa}\mathfrak{b}^{(\alpha-1)(1-\varsigma)}+\frac{2\mathscr{M}[1+2\rho^*]}{\Gamma(\varsigma)}\varrho^*\right] \\
&\qquad \times\sup_{0\leq\nu\leq\mathcal{B}}\Delta\left(\{\mathfrak{y}(\nu)\}\right) \\
&\leq \frac{2\mathscr{M}\,\|\mathcal{B}\|}{\Gamma(\varsigma)}\Phi^*\left[\frac{\mathscr{M}\mathcal{B}^{(\alpha-1)(1-\varsigma)}}{\Gamma(\alpha(1-\varsigma)+\varsigma)}\mathcal{K}_{\zeta_\kappa}+\frac{2\mathscr{M}[1+2\rho^*]}{\Gamma(\varsigma)}\varrho^*\right]\sup_{0\leq\nu\leq\mathcal{B}}\Delta\left(\left\{\mathfrak{y}_q(\nu)\right\}\right).
\end{aligned} \tag{23}$$

Thus, we can evaluate from (20), (21), (22) and (23), and from Lemma 2.11 that

$$\Delta(\mathfrak{X}(\mathcal{S}_1)(\tau)) \leq \mathcal{N}^*\Delta(\{\mathfrak{y}_q\}). \tag{24}$$

where

$$\mathcal{N}^* = \frac{\mathscr{M}\mathfrak{b}^{(\alpha-1)(1-\varsigma)}}{\Gamma(\alpha(1-\varsigma)+\varsigma)}\mathcal{K}_{\zeta_\kappa}\left(1+\frac{2\mathscr{M}\phi^*\|\mathcal{B}\|}{\Gamma\varsigma}\right)+\frac{2\mathscr{M}}{\Gamma(\varsigma)}(1+2\rho^*)\varrho^*\left(\mathfrak{b}^{(1-\alpha)(1-\varsigma)}+\frac{2\mathscr{M}\phi^*\|\mathcal{B}\|}{\Gamma\varsigma}\right).$$

Let

$$max\{\mathcal{N}, \mathcal{N}^*\} = \mathcal{N}_1.$$

Since $\mathfrak{X}(\mathcal{S}_1) \subset \mathfrak{X}(\mathcal{E}_\kappa)$ is equi-continuous, it follows that from Lemma 2.11

$$\Delta(\mathfrak{X}(\mathcal{S}_1)) = \sup_{\tau\in\mathfrak{J}} \Delta(\mathfrak{X}(\mathcal{S}_1)(\tau)).$$

Thus from the equi-continuity of $\mathfrak{X}(\mathcal{E}_\kappa)$ and the hypothesis (A6), we have

$$\Delta(\mathfrak{X}(\mathcal{S})) \leq 2\Delta(\mathfrak{X}(\mathcal{S}_1)) \leq 2\mathcal{N}_1\Delta((\mathcal{S})).$$

Since $\mathcal{N}_1 < \frac{1}{2}$, therefore the map $\mathfrak{X}$ is a condensing map. With the help of lemma (2.14) $\mathfrak{X}$ has a fixed point. Therefore we deduced that, Hilfer fractional differential equation (1) has a mild solution $\varkappa(\tau) = \tau^{\alpha-1)(1-\varsigma)}\mathfrak{y}(\tau)$ satisfying that $\varkappa(\mathfrak{b}) = \varkappa_\mathfrak{b}$. Hence, Hilfer fractional derivative equations (1) is exactly controllable on $[0, \mathfrak{b}]$. □

4. Application

Let $\mathcal{X} = \mathcal{L}^2([0, \pi], \mathbb{R})$ with norm $\|\cdot\|_2$. Consider the following fractional differential system with non-instantaneous impulses:

$$\begin{cases} \mathscr{D}_{0^+}^{\frac{4}{7},\alpha}\left[\varkappa(\tau,\zeta) - \frac{\partial^2}{\partial\zeta^2}\varkappa(\tau,\zeta)\right] = \frac{\partial^2}{\partial\zeta^2}\varkappa(\tau,\zeta) \\ \qquad + \widehat{\mathscr{F}}\left(\tau, \varkappa(\tau,\zeta), \int_0^\tau \mathfrak{G}(\tau,\mathfrak{r},\varkappa(\mathfrak{r},\zeta))\,d\mathfrak{r}\right) + \widehat{\mathcal{B}}w(\tau,\zeta), & \zeta \in [0,\pi], \tau \in (0, \frac{1}{3}] \cup (\frac{2}{3}, 1], \\ \varkappa(\tau,0) = \varkappa(\tau,\pi) = 0, & \tau \in [0,1], \\ \varkappa(\tau,\zeta) = \frac{\exp(-\tau)}{C_1(1+\exp(-\tau))}\sin\varkappa(\tau,\zeta), & \zeta \in [0,\pi], \text{and } \tau \in (\frac{1}{3}, \frac{2}{3}], \\ \mathcal{I}^{(1-\frac{4}{7})(1-\alpha)}\left[\varkappa(\tau,\zeta) - \frac{\partial}{\partial\zeta}\varkappa(\tau,\zeta)\right] = \widehat{\varkappa}_0(\zeta), & \zeta \in [0,\pi], \end{cases} \tag{25}$$

where $\mathscr{D}_{0^+}^{\frac{4}{7},\alpha}$ is a Hilfer fractional derivative of order $\frac{4}{7}$ and type α, $0 \leq \alpha \leq 1$; $\widehat{\varkappa}_0 \in \mathcal{X}$. Let $\tau_0 = 0 = \omega_0, \tau_1 = \frac{1}{3}, \omega_1 = \frac{2}{3}, \tau_2 = 1$ i.e, $0 = \tau_0 = \omega_0 < \tau_1 = \frac{1}{3} < \omega_1 = \frac{2}{3} < \tau_2 = 1$, where C_1 is positive constant.
Consider

$$\mathscr{R}\varkappa(\tau) = \left[\varkappa(\tau,\zeta) - \frac{\partial^2}{\partial\zeta^2}\varkappa(\tau,\zeta)\right],\ \zeta_\mathfrak{j}(\tau,\varkappa(\tau)) = \frac{\exp(-\tau)}{C_1(1+\exp(-\tau))}\sin\varkappa(\tau,\zeta),\ \tau \in \left(\frac{1}{3}, \frac{2}{3}\right],$$

We define operators $\mathscr{A}: \mathcal{X} \to \mathcal{X}$ and $\mathscr{R}: \mathcal{X} \to \mathcal{X}$ respectively by $\mathscr{A}\mu = \mu''$ and $\mathscr{R}\mu = \mu - \mu''$ with domain

$$\mathscr{D}(\mathscr{A}) = \mathscr{D}(\mathscr{R}) = \{\mu \in \mathcal{X} : \mu' \text{ is absolutely continuous and } \mu'' \in \mathcal{X}\}.$$

Obviously, the operator $\mathscr{A}$ and $\mathscr{R}$ are given by

$$\mathscr{A}\mu = -\sum_{q=1}^{\infty} q^2\langle\mu, e_q\rangle e_q, \quad \mu \in \mathscr{D}(\mathscr{A}),$$

and

$$\mathscr{R}\mu = \sum_{q=1}^{\infty}(q^2+1)\langle\mu, e_q\rangle e_q, \quad \mu \in \mathscr{D}(\mathscr{R}),$$

respectively, where $e_q(\zeta) = \sqrt{\frac{2}{\pi}}\sin(q\zeta), q \in \mathbf{N}$. Clearly the set $\{e_q : q \in \mathbf{N}\}$ consists of eigenfunction of the operator $\mathscr{A}$ and forms an orthonormal basis for $\mathcal{X}$. One can easily obtain that

$$\mathscr{R}^{-1}\mu = \sum_{q=1}^{\infty}(q^2+1)^{-1}\langle\mu, e_q\rangle e_q, \quad \mu \in \mathcal{X}.$$

We can see from paper [31] that the pair of operators $\mathscr{A}$ and $\mathscr{R}$ generates the propagation family $\mathscr{P}(\tau)$ of bounded linear operators is given by

$$\mathscr{P}(\tau)\mu = \sum_{q=1}^{\infty}\exp\left(-\frac{q^2}{q^2+1}\tau\right)\langle\mu, e_q\rangle e_q, \quad \mu \in \mathcal{X}.$$

Obviously, $\mathscr{A}\mathscr{R}^{-1}$ generates a C_0-semigroup $\{\mathscr{P}(\tau)\}_{\tau\geq 0}$ of bounded linear operator operators, which is self-adjoint in Hilbert space $\mathcal{X}$ and $\|\mathscr{P}(\tau)\| \leq 1$. The functions $\widehat{\mathscr{F}}$ and $\mathfrak{G}$ are described below:

$$\widehat{\mathscr{F}}\left(\tau, \varkappa(\tau,\zeta), \int_0^{\tau}\mathfrak{G}(\tau, \mathfrak{r}, \varkappa(\mathfrak{r},\zeta))\,d\mathfrak{r}\right) = \mathscr{R}\mathscr{F}\left(\tau, \varkappa(\tau,\zeta), \int_0^{\tau}\mathfrak{G}(\tau, \mathfrak{r}, \varkappa(\mathfrak{r},\zeta))\,d\mathfrak{r}\right),$$

and

$$\widehat{B}w(\tau,\zeta) = \mathscr{R}Bw(\tau,\zeta).$$

We now take $\mathfrak{G}(\tau,\nu,\varkappa(\nu,\zeta)) = \mathfrak{G}(\tau,\nu,\varkappa(\nu))(\zeta) = \frac{\sigma}{\pi}(\tau-\nu)^{-\frac{1}{2}}\nu^{\frac{1}{2}-\alpha-\varsigma+\alpha\varsigma}\varkappa(\nu,\zeta)$, $\mathscr{F}(\tau,\varkappa(\tau,\zeta),\phi(\zeta)) = \mathscr{F}\left(\tau,\varkappa(\tau),\phi\right)(\zeta) = \mathscr{L}\left[\cos(\varkappa(\tau,\zeta)) + \phi(\zeta)\right]$, here σ, $\mathscr{L}$ are constants. The bounded linear control operator $\mathcal{B} : \mathscr{D}(\mathscr{R}) \to \mathcal{X}$ is defined by $\mathcal{B}\varkappa = \varkappa, \varkappa \in \mathscr{D}(\mathscr{R})$ and $w(\tau)(\zeta) = w(\tau,\zeta)$. We are now able to rewrite the system (25) as in the form of (1). It is easy to verify that functions $\mathscr{F}$, $\mathfrak{G}$ satisfy the conditions (A1) to (A3) with the fact that $\mathfrak{l}_{\kappa,1} = \mathfrak{l}_{\kappa,2} = |\mathscr{L}|$, $\mathfrak{l}_1 = 0$, $\mathfrak{m}(\tau,\nu) = \rho(\tau,\nu) = \frac{|\sigma|}{\pi}(\tau-\nu)^{-\frac{1}{2}}\nu^{-\frac{1}{2}}$, $l_2 = \varrho^* = |\mathscr{L}|\frac{\mathfrak{b}^{\varsigma}}{\varsigma}$, $\mathfrak{m}^* = \rho^* = |\sigma|$, $\|\mathcal{B}\| = 1$. Also, $|\zeta_{\mathrm{j}}(\tau,\varkappa_1(\tau)) - \zeta_{\mathrm{j}}(\tau,\varkappa_2(\tau))| \leq \frac{\exp^{-\tau}}{C_1(1+\exp^{-\tau})}|\sin\varkappa_1(\tau,\zeta) - \sin\varkappa_2(\tau,\zeta)| \leq C_2|\varkappa_1(\tau,\zeta) - \varkappa_2(\tau,\zeta)|$, C_2 is positive constant. Therefore impulsive function satisfies assumption (A4). The linear operator $\Gamma_{\omega_{\mathrm{j}}}^{\tau_{\mathrm{j}+1}} \to \mathcal{X}$ is given by

$$(\Gamma_{\omega_{\mathrm{j}}}^{\tau_{\mathrm{j}+1}}w)(\zeta) = \int_0^{\mathfrak{b}}\mathcal{K}_{\varsigma}(\mathfrak{b}-\nu)w(\nu,\zeta)d\nu.$$

Since $\{\mathscr{P}(\tau)\}_{\tau\geq 0}$ is self-adjoint in Hilbert space $\mathcal{X}$, $\mathcal{K}_{\varsigma}(\tau)$ is also self-adjoint in Hilbert space $\mathcal{X}$. Let $\mu \in \mathcal{X}$ be any element and $\mu_q = \langle\mu, e_q\rangle$. Then $\mu = \sum_{q=1}^{\infty}\mu_q e_q$, and we obtain

$$\begin{aligned}
\|\mathcal{K}_{\varsigma}^*(\tau)\mu\|_{\mathcal{X}}^2 &= \|\mathcal{K}_{\varsigma}(\tau)\mu\|_{\mathcal{X}}^2 = \langle\mathcal{K}_{\varsigma}(\tau)\mu, \mathcal{K}_{\varsigma}(\tau)\mu\rangle \\
&= \langle\tau^{\varsigma-1}\mathcal{Q}_{\varsigma}(\tau)\mu, \tau^{\varsigma-1}\mathcal{Q}_{\varsigma}(\tau)\mu\rangle \\
&= \tau^{2\varsigma-2}\varsigma^2\int_0^{\infty}\int_0^{\infty}\vartheta\theta\psi_{\varsigma}(\vartheta)\psi_{\varsigma}(\theta) \\
&\quad \times\left\langle\sum_{q=1}^{\infty}\exp\left(-\frac{q^2}{q^2+1}\tau\right)\mu_q e_q, \sum_{m=1}^{\infty}\exp\left(-\frac{m^2}{m^2+1}\tau\right)v_m e_m\right\rangle d\vartheta\,d\theta \\
&\geq \tau^{2\varsigma-2}\varsigma^2\int_0^{\infty}\int_0^{\infty}\vartheta\theta\psi_{\varsigma}(\vartheta)\psi_{\varsigma}(\theta)\exp\left(-\mathfrak{b}^{\varsigma}(\vartheta+\theta)\right)d\vartheta\,d\theta\left(\sum_{q=1}^{\infty}|\mu_q|^2\right) \\
&= \lambda_1^2\tau^{2\varsigma-2}\|\mu\|_{\mathcal{X}}^2,
\end{aligned}$$

where $\lambda_1^2 = \varsigma^2 \int_0^\infty \int_0^\infty \vartheta\theta\psi_\varsigma(\vartheta)\psi_\varsigma(\theta)\exp(-\mathfrak{b}^\varsigma(\vartheta+\theta))d\vartheta\, d\theta < \infty$. This implies

$$\int_0^{\mathfrak{b}} \|\mathcal{K}_\varsigma^*(\omega)\mu\|_{\mathcal{X}}^2\, d\omega \geq \lambda^2\|\mu\|_{\mathcal{X}}^2, \quad \text{for all } \mu \in \mathcal{X},$$

where $\lambda^2 = \lambda_1^2\mathfrak{b}^{2\varsigma-1}$. Thus we conclude from [12, Theorem 4.1.7] that controllability map $\Gamma_{\omega_i}^{\tau_{i+1}}$ has induced inverse $(\Gamma_{\omega_i}^{\tau_{i+1}})^{-1}$ in $\mathcal{L}^2(\mathfrak{J}, \mathscr{D}(\mathscr{R}))/\ker(\Gamma_{\omega_i}^{\tau_{i+1}})$ and $\|(\Gamma_{\omega_i}^{\tau_{i+1}})^{-1}\| \leq \frac{1}{\lambda}$. Also $\Gamma_{\omega_i}^{\tau_{i+1}}$ satisfies the assumption (A5). The assumption (A6) along with the condition (5) holds as we take the constants $|\sigma|$, $|\mathscr{L}|$ as small as possible. Thus all the hypothesis of Theorem 3.1 hold, Hence we concluded that the Hilfer fractional differential equation (25) is exactly controllable.

5. Concluding remarks

We investigated the existence of the mild solution and controllability results of Hilfer fractional differential equation of Sobolev-type with non-instantaneous impulses. We proved the main results with the help of propagation family $\{\mathscr{P}(\tau)\}_{\tau\geq 0}$, (generated by the operator pair $(\mathscr{A},\mathscr{R})$), measure of non-compactness Sadovskii fixed point theorem. We analysed (1) without assuming the existence of $\mathscr{R}^{-1}$ as a bounded operator as well as without any assumption on the relation between the domain of the operators $\mathscr{A}$ and $\mathscr{R}$. Further, one can extend the above-obtained results to the various dynamical systems such as stochastic and delay systems.

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