



Function characterizations of some topological spaces

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Abstract. We present some new characterizations of some spaces such as stratifiable spaces, MCP-spaces in terms of real-valued functions.

1. Introduction and preliminaries

A space always means a topological space. The set of all positive integers is denoted by \mathbb{N} . For a space X , denote by C_X the family of all compact subsets of X . τ and τ^c denote the topology of X and the family of all closed subsets of X , respectively. For a subset A of a space, we write \overline{A} for the closure of A . Also, we use χ_A to denote the characteristic function of A .

Let f be a real-valued function on a space X and r a real number, denote $\{f > r\} = \{x \in X : f(x) > r\}$ and analogously for the others. f is called lower (upper) semi-continuous [4] if for any real number r , the set $\{f > r\}$ ($\{f < r\}$) is open. We write $L(X)$ ($U(X)$) for the set of all lower (upper) semi-continuous functions from X into $[0, 1]$. $C(X)$ is the set of all continuous functions from X into $[0, 1]$.

Definition 1.1. A space X is called *stratifiable* [2] (*semi-stratifiable* [3]) if there is a map $\rho : \mathbb{N} \times \tau^c \rightarrow \tau$ such that

- (1) $F = \bigcap_{n \in \mathbb{N}} \rho(n, F) = \bigcap_{n \in \mathbb{N}} \overline{\rho(n, F)}$ ($F = \bigcap_{n \in \mathbb{N}} \rho(n, F)$) for each $F \in \tau^c$;
- (2) if $F, H \in \tau^c$ and $F \subset H$, then $\rho(n, F) \subset \rho(n, H)$ for all $n \in \mathbb{N}$.

X is called *k*-semi-stratifiable [10] if there is a map $\rho : \mathbb{N} \times \tau^c \rightarrow \tau$ satisfies the conditions for a semi-stratifiable space and

- (3) For each $K \in C_X$ and $F \in \tau^c$ with $K \cap F = \emptyset$, there is $m \in \mathbb{N}$ such that $K \cap \rho(m, F) = \emptyset$.

The map ρ is called the stratification (semi-stratification, *k*-semi-stratification) for X .

Definition 1.2. ([5]) A space X is called an MCP-spaces (MCM-spaces) if there is an operator U assigning to each decreasing sequence $\langle F_j \rangle$ of closed subsets of X with empty intersection, a sequence of open sets $\{U(n, \langle F_j \rangle) : n \in \mathbb{N}\}$ such that

- (1) $F_n \subset U(n, \langle F_j \rangle)$ for each $n \in \mathbb{N}$,
- (2) if $\langle F_j \rangle \leq \langle G_j \rangle$, then $U(n, \langle F_j \rangle) \subset U(n, \langle G_j \rangle)$ for all $n \in \mathbb{N}$,
- (3) $\bigcap_{n \in \mathbb{N}} \overline{U(n, \langle F_j \rangle)} = \emptyset$ ($\bigcap_{n \in \mathbb{N}} U(n, \langle F_j \rangle) = \emptyset$).

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Definition 1.3. ([11]) A space X is called a k -MCM space if there is an operator U assigning to each decreasing sequence $\langle F_j \rangle$ of closed subsets of X with empty intersection, a decreasing sequence $\{U(n, \langle F_j \rangle)\}_{n \in \mathbb{N}}$ of open subsets of X such that

- (1) $F_n \subset U(n, \langle F_j \rangle)$ for each $n \in \mathbb{N}$,
- (2) if $\langle F_j \rangle \leq \langle H_j \rangle$, then $U(n, \langle F_j \rangle) \subset U(n, \langle H_j \rangle)$ for each $n \in \mathbb{N}$,
- (3) for each $K \in C_X$, there exists $m \in \mathbb{N}$ such that $K \cap U(m, \langle F_j \rangle) = \emptyset$.

Characterizations of some topological spaces such as stratifiable spaces, MCP-spaces in terms of real-valued functions which are known to be insertion theorems were studied extensively in literature [8, 9, 12–15]. Various conditions formulated with real-valued functions which guarantee a space possesses some corresponding properties were presented. For example.

Theorem 1.4. For a space X , the following are equivalent.

- (a) X is stratifiable.
- (b) ([8]) There exists an order-preserving map $\phi : L(X) \rightarrow C(X)$ such that for each $h \in L(X)$, $\phi(h) \leq h$ and $0 < \phi(h)(x) < h(x)$ whenever $h(x) > 0$.
- (c) ([12]) There exist two order-preserving maps $\phi : L(X) \rightarrow U(X)$ and $\psi : L(X) \rightarrow L(X)$ such that for each $h \in L(X)$, $\psi(h) \leq \phi(h) \leq h$ and $0 < \psi(h)(x) \leq \phi(h)(x) < h(x)$ whenever $h(x) > 0$.

In this paper, we shall present some new characterizations of some spaces such as stratifiable spaces, MCP-spaces in terms of real-valued functions. Conditions in these results are simplified and are easier to be verified.

The following lemma will be frequently used in the proof of the main results.

Lemma 1.5. Let $\{A_n : n \in \mathbb{N}\}$ be a decreasing sequence of subsets of a space X . For each $x \notin \bigcap_{n \in \mathbb{N}} A_n$, let $n_x = \min\{n \in \mathbb{N} : x \notin A_n\}$. Define a map $h : X \rightarrow [0, 1]$ by letting for each $x \in X$, $h(x) = 0$ whenever $x \in \bigcap_{n \in \mathbb{N}} A_n$ and $h(x) = \frac{1}{n_x}$ whenever $x \notin \bigcap_{n \in \mathbb{N}} A_n$.

- (1) If A_n is open for each $n \in \mathbb{N}$ then $h \in U(X)$.
- (2) If A_n is closed for each $n \in \mathbb{N}$ then $h \in L(X)$.

Proof. (1) Let $r \in \mathbb{R}$ and $h(x) < r$.

Case 1. $x \in \bigcap_{n \in \mathbb{N}} A_n$. Then $0 = h(x) < r$. There exists $m \in \mathbb{N}$ such that $\frac{1}{m} < r$. Then A_m is an open neighborhood of x . For each $y \in A_m$, if $y \in \bigcap_{n \in \mathbb{N}} A_n$, then $h(y) = 0 < r$. If $y \notin \bigcap_{n \in \mathbb{N}} A_n$, then $m < n_y$ and thus $h(y) = \frac{1}{n_y} < \frac{1}{m} < r$.

Case 2. $x \notin \bigcap_{n \in \mathbb{N}} A_n$. If $n_x = 1$, then $1 = h(x) < r$. For each $y \in X$, $h(y) \leq 1 < r$. If $n_x > 1$, then A_{n_x-1} is an open neighborhood of x . For each $y \in A_{n_x-1}$, if $y \in \bigcap_{n \in \mathbb{N}} A_n$, then $h(y) = 0 < \frac{1}{n_x} = h(x) < r$. If $y \notin \bigcap_{n \in \mathbb{N}} A_n$, then $n_x - 1 < n_y$ and thus $h(y) = \frac{1}{n_y} \leq \frac{1}{n_x} = h(x) < r$.

The above argument shows that h is upper semi-continuous.

- (2) Let $r \in \mathbb{R}$ and $h(x) > r$.

Case 1. $x \in \bigcap_{n \in \mathbb{N}} A_n$. Then $r < h(x) = 0$. For each $y \in X$, $h(y) \geq 0 > r$.

Case 2. $x \notin \bigcap_{n \in \mathbb{N}} A_n$. Then $W = X \setminus A_{n_x}$ is an open neighborhood of x . For each $y \in W$, $n_y \leq n_x$ from which it follows that $h(x) = \frac{1}{n_x} \leq \frac{1}{n_y} = h(y)$. Thus $h(y) > r$.

The above argument shows that h is lower semi-continuous. \square

2. On stratifiable spaces

This section is devoted to the characterizations of spaces having stratifications, such as stratifiable spaces and k -semi-stratifiable spaces.

Theorem 2.1. For a space X , the following are equivalent.

- (a) X is a stratifiable space.

(b) There exist two order-preserving maps $\phi : L(X) \rightarrow U(X)$ and $\psi : L(X) \rightarrow L(X)$ such that for each $h \in L(X)$, $\psi(h) \leq \phi(h)$ and $h(x) = 0$ if and only if $\phi(h)(x) = 0 = \psi(h)(x)$.

(c) There exists an order-preserving map $\phi : L(X) \rightarrow U(X)$ such that for each $h \in L(X)$ and $x \in X$, if $h(x) = 0$ then $\phi(h)(x) = 0$, and if $h(x) \neq 0$ then there exists an open neighborhood V_x of x such that $\inf\{\phi(h)(y) : y \in V_x\} > 0$.

Proof. (a) \Rightarrow (b) Let ρ be the stratification for X which is decreasing with respect to n . For each $h \in L(X)$ and $n \in \mathbb{N}$, let $F(n, h) = \{h \leq \frac{1}{n}\}$. Then $\{F(n, h) : n \in \mathbb{N}\}$ is a decreasing sequence of closed subsets of X such that $\bigcap_{n \in \mathbb{N}} F(n, h) = h^{-1}(0)$. For each $x \in X$, if $x \notin \bigcap_{n \in \mathbb{N}} F(n, h)$ then there exists $i \in \mathbb{N}$ such that $x \notin \overline{\rho(i, F(i, h))}$. Let

$$n_x(h) = \min\{n \in \mathbb{N} : x \notin \rho(n, F(n, h))\}, \quad m_x(h) = \min\{n \in \mathbb{N} : x \notin \overline{\rho(n, F(n, h))}\}.$$

For each $x \in X$, let $\phi(h)(x) = 0 = \psi(h)(x)$ whenever $x \in \bigcap_{n \in \mathbb{N}} F(n, h)$ and $\phi(h)(x) = \frac{1}{n_x(h)}$, $\psi(h)(x) = \frac{1}{m_x(h)}$ whenever $x \notin \bigcap_{n \in \mathbb{N}} F(n, h)$. Since

$$\bigcap_{n \in \mathbb{N}} F(n, h) = \bigcap_{n \in \mathbb{N}} \rho(n, F(n, h)) = \bigcap_{n \in \mathbb{N}} \overline{\rho(n, F(n, h))},$$

by Lemma 1.5, $\phi(h) \in U(X)$ and $\psi(h) \in L(X)$.

Let $x \in X$. If $x \in \bigcap_{n \in \mathbb{N}} F(n, h)$ then $\phi(h)(x) = 0 = \psi(h)(x)$. If $x \notin \bigcap_{n \in \mathbb{N}} F(n, h)$, since $x \notin \overline{\rho(m_x(h), F(m_x(h), h))} \supset \rho(m_x(h), F(m_x(h), h))$, we have that $n_x(h) \leq m_x(h)$. It follows that $\psi(h)(x) = \frac{1}{m_x(h)} \leq \frac{1}{n_x(h)} = \phi(h)(x)$. Therefore, $\psi(h) \leq \phi(h)$.

Let $x \in X$. Then $h(x) = 0$ if and only if $x \in \bigcap_{n \in \mathbb{N}} F(n, h)$ if and only if $\phi(h)(x) = 0 = \psi(h)(x)$.

Suppose that $h_1 \leq h_2$. For each $x \in X$, if $\phi(h_1)(x) = 0$ then $\phi(h_1)(x) \leq \phi(h_2)(x)$. If $\phi(h_1)(x) = \frac{1}{n_x(h_1)}$ then $x \notin \bigcap_{n \in \mathbb{N}} F(n, h_1)$. From $h_1 \leq h_2$ it follows that $F(n, h_2) \subset F(n, h_1)$ for each $n \in \mathbb{N}$ and thus $x \notin \bigcap_{n \in \mathbb{N}} F(n, h_2)$. Since $x \notin \overline{\rho(n_x(h_1), F(n_x(h_1), h_1))} \supset \overline{\rho(n_x(h_1), F(n_x(h_1), h_2))}$, we have $n_x(h_2) \leq n_x(h_1)$ and thus $\phi(h_1)(x) = \frac{1}{n_x(h_1)} \leq \frac{1}{n_x(h_2)} = \phi(h_2)(x)$. Similarly, $\psi(h_1) \leq \psi(h_2)$.

(b) \Rightarrow (c) Assume (b). If $h(x) \neq 0$ then $\psi(h)(x) \neq 0$ and thus there exists $r > 0$ such that $\psi(h)(x) > r$. Let $V_x = \{\psi(h) > r\}$. Then V_x is an open neighborhood of x . For each $y \in V_x$, $\phi(h)(y) \geq \psi(h)(y) > r$ which implies that $\inf\{\phi(h)(y) : y \in V_x\} > 0$.

(c) \Rightarrow (a) Assume (c). For each $F \in \tau^c$, let $h_F = 1 - \chi_F$. Then $h_F \in L(X)$.

For each $F \in \tau^c$ and $n \in \mathbb{N}$, let $\rho(n, F) = \{\phi(h_F) < \frac{1}{n}\}$. Then $\rho(n, F) \in \tau$.

If $x \in F$ then $h_F(x) = 0$. It follows that $\phi(h_F)(x) = 0$ and thus $x \in \rho(n, F)$ for each $n \in \mathbb{N}$.

If $x \notin F$ then $h_F(x) \neq 0$. By (c), there exists an open neighborhood V_x of x and $m \in \mathbb{N}$ such that $\phi(h_F)(y) > \frac{1}{m}$ for each $y \in V_x$. It follows that $V_x \cap \{\phi(h_F) < \frac{1}{m}\} = \emptyset$ and thus $x \notin \overline{\rho(m, F)}$. This implies that $\bigcap_{n \in \mathbb{N}} \overline{\rho(n, F)} \subset F$.

Suppose that $F_1 \subset F_2$. For each $x \in X$, if $x \in F_2$ then $h_{F_2}(x) = 0$ and thus $h_{F_2}(x) \leq h_{F_1}(x)$. If $x \notin F_2$ then $x \notin F_1$ and thus $h_{F_2}(x) = 1 = h_{F_1}(x)$. This implies that $h_{F_2} \leq h_{F_1}$. Then $\phi(h_{F_2}) \leq \phi(h_{F_1})$ from which it follows that $\rho(n, F_1) \subset \rho(n, F_2)$ for each $n \in \mathbb{N}$.

Therefore, X is a stratifiable space. \square

Theorem 2.2. A space X is k -semi-stratifiable if and only if there exists an order-preserving map $\phi : L(X) \rightarrow U(X)$ such that for each $h \in L(X)$ and $K \in \mathcal{C}_X$, $K \cap h^{-1}(0) = \emptyset$ if and only if $\inf\{\phi(h)(x) : x \in K\} > 0$.

Proof. Let ρ be the k -semi-stratification for X which is decreasing with respect to n . For each $h \in L(X)$ and $n \in \mathbb{N}$, let $F(n, h) = \{h \leq \frac{1}{n}\}$. Then $\{F(n, h) : n \in \mathbb{N}\}$ is a decreasing sequence of closed subsets of X such that $\bigcap_{n \in \mathbb{N}} F(n, h) = h^{-1}(0)$. For each $x \in X$, if $x \notin \bigcap_{n \in \mathbb{N}} F(n, h)$ then there exists $i \in \mathbb{N}$ such that $x \notin \rho(i, F(i, h))$. Let

$$n_x(h) = \min\{n \in \mathbb{N} : x \notin \rho(n, F(n, h))\}.$$

For each $x \in X$, let $\phi(h)(x) = 0$ whenever $x \in \bigcap_{n \in \mathbb{N}} F(n, h)$ and $\phi(h)(x) = \frac{1}{n_x(h)}$ whenever $x \notin \bigcap_{n \in \mathbb{N}} F(n, h)$. Since $\bigcap_{n \in \mathbb{N}} F(n, h) = \bigcap_{n \in \mathbb{N}} \rho(n, F(n, h))$, by Lemma 1.5 (1), $\phi(h) \in U(X)$.

Let $K \in \mathcal{C}_X$. If $K \cap h^{-1}(0) = \emptyset$ then $K \cap \bigcap_{n \in \mathbb{N}} F(n, h) = \emptyset$ and thus $K \cap F(i, h) = \emptyset$ for some $i \in \mathbb{N}$. It follows that $K \cap \rho(m, F(m, h)) = \emptyset$ for some $m \in \mathbb{N}$. Then for each $x \in K$, $n_x(h) \leq m$ and so $\phi(h)(x) \geq \frac{1}{m}$. This implies that $\inf\{\phi(h)(x) : x \in K\} > 0$.

If $\inf\{\phi(h)(x) : x \in K\} > 0$, then for each $x \in K$, $\phi(h)(x) > 0$ which implies that $x \notin \bigcap_{n \in \mathbb{N}} F(n, h) = h^{-1}(0)$. Therefore, $K \cap h^{-1}(0) = \emptyset$.

That ϕ is order-preserving can be shown analogously to the proof of (a) \Rightarrow (b) of Theorem 2.1.

Conversely, for each $F \in \tau^c$, let $h_F = 1 - \chi_F$. Then $h_F \in L(X)$.

For each $F \in \tau^c$ and $n \in \mathbb{N}$, let $\rho(n, F) = \{\phi(h_F) < \frac{1}{n}\}$. Then $\rho(n, F) \in \tau$.

If $x \in F \in \tau^c$, then $h_F(x) = 0$. It follows that $x \in h_F^{-1}(0)$ and thus $\phi(h_F)(x) = 0$. Hence $x \in \rho(n, F)$ for each $n \in \mathbb{N}$.

Let $K \in \mathcal{C}_X$ and $F \in \tau^c$ with $K \cap F = \emptyset$. Then $K \cap h_F^{-1}(0) = \emptyset$ and thus $\inf\{\phi(h_F)(x) : x \in K\} > 0$. It follows that $K \cap \{\phi(h_F) < \frac{1}{m}\} = \emptyset$ for some $m \in \mathbb{N}$. That is, $K \cap \rho(m, F) = \emptyset$.

Suppose that $F_1 \subset F_2$. With a similar argument to the proof of (c) \Rightarrow (a) of Theorem 2.1, one readily shows that $\rho(n, F_1) \subset \rho(n, F_2)$ for each $n \in \mathbb{N}$.

Therefore, X is k -semi-stratifiable. \square

Similarly, we have the following.

Proposition 2.3. *A space X is semi-stratifiable if and only if there exists an order-preserving map $\phi : L(X) \rightarrow U(X)$ such that for each $h \in L(X)$ and $x \in X$, $h(x) = 0$ if and only if $\phi(h)(x) = 0$.*

A space X is perfectly normal [4] if and only if there is a map $\rho : \mathbb{N} \times \tau^c \rightarrow \tau$ such that $F = \bigcap_{n \in \mathbb{N}} \rho(n, F) = \bigcap_{n \in \mathbb{N}} \overline{\rho(n, F)}$ for each $F \in \tau^c$.

Theorem 2.4. *For a space X , the following are equivalent.*

(a) X is perfectly normal.

(b) There exist two maps $\phi : L(X) \rightarrow U(X)$ and $\psi : L(X) \rightarrow L(X)$ such that for each $h \in L(X)$, $\psi(h) \leq \phi(h)$ and $h(x) = 0$ if and only if $\phi(h)(x) = 0 = \psi(h)(x)$.

(c) There exists a map $\phi : L(X) \rightarrow U(X)$ such that for each $h \in L(X)$ and $x \in X$, $h(x) = 0$ if and only if $\phi(h)(x) = 0$, and if $h(x) \neq 0$ then there exists an open neighborhood V_x of x such that $\inf\{\phi(h)(y) : y \in V_x\} > 0$.

Proof. (a) \Rightarrow (b) Let ρ be the map for a perfectly normal space which is decreasing with respect to n . For each $h \in L(X)$ and $n \in \mathbb{N}$, let $F(n, h) = \{h \leq \frac{1}{n}\}$. Then $\{F(n, h) : n \in \mathbb{N}\}$ is a decreasing sequence of closed subsets of X such that $\bigcap_{n \in \mathbb{N}} F(n, h) = h^{-1}(0)$. For each $n \in \mathbb{N}$, let $\sigma(n, F(n, h)) = \bigcap_{j \leq n} \rho(n, F(j, h))$. Then $\sigma(n, F(n, h)) \in \tau$. We show that

$$\bigcap_{n \in \mathbb{N}} F(n, h) = \bigcap_{n \in \mathbb{N}} \sigma(n, F(n, h)) = \bigcap_{n \in \mathbb{N}} \overline{\sigma(n, F(n, h))}.$$

For each $n \in \mathbb{N}$ and $j \leq n$, $F(n, h) \subset F(j, h) \subset \rho(n, F(j, h))$ which implies that $F(n, h) \subset \bigcap_{j \leq n} \rho(n, F(j, h)) = \sigma(n, F(n, h))$. It follows that $\bigcap_{n \in \mathbb{N}} F(n, h) \subset \bigcap_{n \in \mathbb{N}} \sigma(n, F(n, h))$. If $x \notin \bigcap_{n \in \mathbb{N}} F(n, h)$ then $x \notin F(i, h) = \bigcap_{n \in \mathbb{N}} \overline{\rho(n, F(i, h))}$ for some $i \in \mathbb{N}$. Then there exists $m \geq i$ such that

$$x \notin \overline{\rho(m, F(i, h))} \supset \overline{\bigcap_{j \leq m} \rho(m, F(j, h))} = \overline{\sigma(m, F(m, h))}.$$

Hence $\bigcap_{n \in \mathbb{N}} \overline{\sigma(n, F(n, h))} \subset \bigcap_{n \in \mathbb{N}} F(n, h)$.

For each $x \in X$, if $x \notin \bigcap_{n \in \mathbb{N}} F(n, h)$ then there exists $i \in \mathbb{N}$ such that $x \notin \overline{\sigma(i, F(i, h))}$. Let

$$n_x(h) = \min\{n \in \mathbb{N} : x \notin \sigma(n, F(n, h))\}, \quad m_x(h) = \min\{n \in \mathbb{N} : x \notin \overline{\sigma(n, F(n, h))}\}.$$

For each $x \in X$, let $\phi(h)(x) = 0 = \psi(h)(x)$ whenever $x \in \bigcap_{n \in \mathbb{N}} F(n, h)$ and $\phi(h)(x) = \frac{1}{n_x(h)}$, $\psi(h)(x) = \frac{1}{m_x(h)}$ whenever $x \notin \bigcap_{n \in \mathbb{N}} F(n, h)$. Since

$$\bigcap_{n \in \mathbb{N}} F(n, h) = \bigcap_{n \in \mathbb{N}} \sigma(n, F(n, h)) = \bigcap_{n \in \mathbb{N}} \overline{\sigma(n, F(n, h))},$$

by Lemma 1.5, $\phi(h) \in U(X)$ and $\psi(h) \in L(X)$.

Let $x \in X$. If $x \in \bigcap_{n \in \mathbb{N}} F(n, h)$ then $\phi(h)(x) = 0 = \psi(h)(x)$. If $x \notin \bigcap_{n \in \mathbb{N}} F(n, h)$, since $x \notin \overline{\sigma(m_x(h), F(m_x(h), h))} \supset \sigma(m_x(h), F(m_x(h), h))$, we have that $n_x(h) \leq m_x(h)$. It follows that $\psi(h)(x) = \frac{1}{m_x(h)} \leq \frac{1}{n_x(h)} = \phi(h)(x)$. Therefore, $\psi(h) \leq \phi(h)$.

Let $x \in X$. Then $h(x) = 0$ if and only if $x \in \bigcap_{n \in \mathbb{N}} F(n, h)$ if and only if $\phi(h)(x) = 0 = \psi(h)(x)$.

(b) \Rightarrow (c) Similar to the proof of (b) \Rightarrow (c) of Theorem 2.1.

(c) \Rightarrow (a) Similar to the proof of (c) \Rightarrow (a) of Theorem 2.1. \square

A space X is perfect [4] if and only if there is a map $\rho : \mathbb{N} \times \tau^c \rightarrow \tau$ such that $F = \bigcap_{n \in \mathbb{N}} \rho(n, F)$ for each $F \in \tau^c$.

Proposition 2.5. *A space X is perfect if and only if there exists a map $\phi : L(X) \rightarrow U(X)$ such that for each $h \in L(X)$ and $x \in X$, $h(x) = 0$ if and only if $\phi(h)(x) = 0$.*

3. On MCP-spaces

In this section, we present characterizations of some weak covering properties, such as MCP-spaces and countably paracompact spaces.

Denote by $L^+(X) = \{h \in L(X) : h > 0\}$ and $U^+(X) = \{h \in U(X) : h > 0\}$.

Theorem 3.1. *For a space X , the following are equivalent.*

(a) X is an MCP-space.

(b) There exist two order-preserving maps $\phi : L^+(X) \rightarrow U(X)$ and $\psi : L^+(X) \rightarrow L^+(X)$ such that for each $h \in L^+(X)$, $\psi(h) \leq \phi(h)$ and $\{h \leq \frac{1}{n}\} \subset \{\phi(h) < \frac{1}{n}\}$ for each $n \in \mathbb{N}$.

(c) There exists an order-preserving map $\phi : L^+(X) \rightarrow U(X)$ such that for each $h \in L^+(X)$ and $n \in \mathbb{N}$, $\{h \leq \frac{1}{n}\} \subset \{\phi(h) < \frac{1}{n}\}$ and for each $x \in X$, there exists an open neighborhood V_x of x such that $\inf\{\phi(h)(y) : y \in V_x\} > 0$.

Proof. (a) \Rightarrow (b) Let U be the operator in Definition 1.2 such that $\{U(n, \langle F_j \rangle) : n \in \mathbb{N}\}$ is decreasing with respect to n . For each $h \in L^+(X)$ and $n \in \mathbb{N}$, let $F_n(h) = \{h \leq \frac{1}{n}\}$. Then $\langle F_n(h) \rangle$ is a decreasing sequence of closed subsets of X with an empty intersection. For each $x \in X$, let

$$n_x(h) = \min\{n \in \mathbb{N} : x \notin U(n, \langle F_j(h) \rangle)\}, m_x(h) = \min\{n \in \mathbb{N} : x \notin \overline{U(n, \langle F_j(h) \rangle)}\}.$$

Then let $\phi(h)(x) = \frac{1}{n_x(h)}$ and $\psi(h)(x) = \frac{1}{m_x(h)}$. It is clear that $\psi(h) > 0$. By Lemma 1.5, $\phi(h) \in U(X)$ and $\psi(h) \in L(X)$. Moreover, since $n_x(h) \leq m_x(h)$ for each $x \in X$, we have $\psi(h) \leq \phi(h)$.

Let $x \in \{h \leq \frac{1}{n}\}$. Since $x \notin U(n_x(h), \langle F_j(h) \rangle) \supset F_{n_x(h)}(h)$, we have $n < n_x(h)$. It follows that $\phi(h)(x) = \frac{1}{n_x(h)} < \frac{1}{n}$ and thus $x \in \{\phi(h) < \frac{1}{n}\}$.

Suppose that $h_1 \leq h_2$. Then $\langle F_j(h_2) \rangle \leq \langle F_j(h_1) \rangle$ and hence $U(n, \langle F_j(h_2) \rangle) \subset U(n, \langle F_j(h_1) \rangle)$ for each $n \in \mathbb{N}$. Let $x \in X$. Since $x \notin U(n_x(h_1), \langle F_j(h_1) \rangle) \supset U(n_x(h_1), \langle F_j(h_2) \rangle)$, we have that $n_x(h_2) \leq n_x(h_1)$ and thus $\phi(h_1)(x) \leq \phi(h_2)(x)$. Therefore, $\phi(h_1) \leq \phi(h_2)$. Similarly, $\psi(h_1) \leq \psi(h_2)$.

(b) \Rightarrow (c) Since $\psi(h) > 0$, for each $x \in X$, there exists $r > 0$ such that $\psi(h)(x) > r$. Let $V_x = \{\psi(h) > r\}$. Then V_x is an open neighborhood of x . For each $y \in V_x$, $\phi(h)(y) \geq \psi(h)(y) > r$ which implies that $\inf\{\phi(h)(y) : y \in V_x\} > 0$.

(c) \Rightarrow (a) For each decreasing sequence $\langle F_j \rangle$ of closed subsets of X with empty intersection and each $x \in X$, let

$$n_x(\langle F_j \rangle) = \min\{n \in \mathbb{N} : x \notin F_n\}, h_{\langle F_j \rangle}(x) = \frac{1}{n_x(\langle F_j \rangle)}.$$

Then $h_{\langle F_j \rangle} > 0$. By Lemma 1.5 (2), $h_{\langle F_j \rangle} \in L(X)$.

For each $n \in \mathbb{N}$, let

$$U(n, \langle F_j \rangle) = \{\phi(h_{\langle F_j \rangle}) < \frac{1}{n}\}.$$

Then $\langle U(n, \langle F_j \rangle) \rangle$ is a decreasing sequence of open subsets of X . By (c), for each $x \in X$, there exists an open neighborhood V_x of x and $m \in \mathbb{N}$ such that $V_x \cap \{\phi(h_{\langle F_j \rangle}) < \frac{1}{m}\} = \emptyset$. It follows that $x \notin \overline{U(m, \langle F_j \rangle)}$ and thus $\bigcap_{n \in \mathbb{N}} \overline{U(n, \langle F_j \rangle)} = \emptyset$.

Let $n \in \mathbb{N}$ and $x \in F_n$. Then $n < n_x(\langle F_j \rangle)$. It follows that $h_{\langle F_j \rangle}(x) = \frac{1}{n_x(\langle F_j \rangle)} < \frac{1}{n}$. By (c), $x \in \{\phi(h_{\langle F_j \rangle}) < \frac{1}{n}\} = U(n, \langle F_j \rangle)$. Hence $F_n \subset U(n, \langle F_j \rangle)$.

Suppose that $\langle F_j \rangle \leq \langle H_j \rangle$. For each $x \in X$, since $x \notin H_{n_x(\langle H_j \rangle)} \supset F_{n_x(\langle H_j \rangle)}$, we have $n_x(\langle F_j \rangle) \leq n_x(\langle H_j \rangle)$ and thus $h_{\langle H_j \rangle}(x) \leq h_{\langle F_j \rangle}(x)$. This implies that $h_{\langle H_j \rangle} \leq h_{\langle F_j \rangle}$. Hence, $\phi(h_{\langle H_j \rangle}) \leq \phi(h_{\langle F_j \rangle})$ from which it follows that $U(n, \langle F_j \rangle) \subset U(n, \langle H_j \rangle)$ for each $n \in \mathbb{N}$.

By Definition 1.2, X is an MCP-space. \square

Theorem 3.2. X is a k -MCM space if and only if there exists an order-preserving map $\phi : L^+(X) \rightarrow U(X)$ such that for each $h \in L^+(X)$ and $n \in \mathbb{N}$, $\{h \leq \frac{1}{n}\} \subset \{\phi(h) < \frac{1}{n}\}$ and for each $K \in C_X$, $\inf\{\phi(h)(x) : x \in K\} > 0$.

Proof. Let U be the operator in Definition 1.3 such that $\{U(n, \langle F_j \rangle) : n \in \mathbb{N}\}$ is decreasing with respect to n . For each $h \in L^+(X)$, define a map $\phi(h) \in U(X)$ as that in the proof of (a) \Rightarrow (b) of Theorem 3.1. Then we only need to show that for each $K \in C_X$, $\inf\{\phi(h)(x) : x \in K\} > 0$.

Let $K \in C_X$. Then there exists $m \in \mathbb{N}$ such that $K \cap U(m, \langle F_j(h) \rangle) = \emptyset$. For each $x \in K$, $x \notin U(m, \langle F_j(h) \rangle)$ and thus $n_x(h) \leq m$. It follows that $\phi(h)(x) = \frac{1}{n_x(h)} \geq \frac{1}{m}$. Therefore, $\inf\{\phi(h)(x) : x \in K\} > 0$.

Conversely, for each decreasing sequence $\langle F_j \rangle$ of closed subsets of X with empty intersection, define a decreasing sequence $\langle U(n, \langle F_j \rangle) \rangle$ of open subsets of X as that in the proof of (c) \Rightarrow (a) of Theorem 3.1. Then we only need to show that for each $K \in C_X$, there exists $m \in \mathbb{N}$ such that $K \cap U(m, \langle F_j \rangle) = \emptyset$.

Let $K \in C_X$. Then $\inf\{\phi(h_{\langle F_j \rangle})(x) : x \in K\} > 0$. There exists $m \in \mathbb{N}$ such that $\phi(h_{\langle F_j \rangle})(x) > \frac{1}{m}$ for each $x \in K$. It follows that $K \cap U(m, \langle F_j \rangle) = K \cap \{\phi(h_{\langle F_j \rangle}) < \frac{1}{m}\} = \emptyset$. \square

Similarly, we have the following.

Proposition 3.3. X is an MCM-space if and only if there exists an order-preserving map $\phi : L^+(X) \rightarrow U^+(X)$ such that for each $h \in L^+(X)$ and $n \in \mathbb{N}$, $\{h \leq \frac{1}{n}\} \subset \{\phi(h) < \frac{1}{n}\}$.

A space X is countably paracompact [7] (countably metacompact [6], countably mesocompact [1]) if and only if for each decreasing sequence $\{F_n : n \in \mathbb{N}\}$ of closed subsets of X with empty intersection, there is a decreasing sequence $\{U_n : n \in \mathbb{N}\}$ of open subsets of X such that $F_n \subset U_n$ for each $n \in \mathbb{N}$ and $\bigcap_{n \in \mathbb{N}} \overline{U_n} = \emptyset$ ($\bigcap_{n \in \mathbb{N}} U_n = \emptyset$, for each compact subset $K \subset X$, there exists $m \in \mathbb{N}$ such that $K \cap U_m = \emptyset$).

The following three results can be shown with similar arguments to the proof of Theorem 3.1, Theorem 3.2 and Proposition 3.3 respectively.

Theorem 3.4. For a space X , the following are equivalent.

- (a) X is countably paracompact.
- (b) There exist two maps $\phi : L^+(X) \rightarrow U(X)$ and $\psi : L^+(X) \rightarrow L^+(X)$ such that for each $h \in L^+(X)$, $\psi(h) \leq \phi(h)$ and $\{h \leq \frac{1}{n}\} \subset \{\phi(h) < \frac{1}{n}\}$ for each $n \in \mathbb{N}$.
- (c) There exists a map $\phi : L^+(X) \rightarrow U(X)$ such that for each $h \in L^+(X)$ and $n \in \mathbb{N}$, $\{h \leq \frac{1}{n}\} \subset \{\phi(h) < \frac{1}{n}\}$ and for each $x \in X$, there exists an open neighborhood V_x of x such that $\inf\{\phi(h)(y) : y \in V_x\} > 0$.

Theorem 3.5. X is countably mesocompact if and only if there exists a map $\phi : L^+(X) \rightarrow U(X)$ such that for each $h \in L^+(X)$ and $n \in \mathbb{N}$, $\{h \leq \frac{1}{n}\} \subset \{\phi(h) < \frac{1}{n}\}$ and for each $K \in C_X$, $\inf\{\phi(h)(x) : x \in K\} > 0$.

Proposition 3.6. X is countably metacompact if and only if there exists a map $\phi : L^+(X) \rightarrow U^+(X)$ such that for each $h \in L^+(X)$ and $n \in \mathbb{N}$, $\{h \leq \frac{1}{n}\} \subset \{\phi(h) < \frac{1}{n}\}$.

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