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On L_2-directed topological spaces in directed graphs theory

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Abstract. Here we give the notion of L_2-directed topological spaces of directed graphs, and some results about this notion such as Alexandroff property. Next, we study the form of L_2-directed topological space on E-generated subdirected graphs and their relation with the relative topologies. The relations between some fundamental properties in topological spaces with their corresponding properties in graphs such as the isomorphically and connectedness are introduced.

1. Introduction

Recall that Leonhard Euler, in 1736 [9], introduced the graph theory for giving solutions of some problems in discrete mathematics. This theory is considered as good concept in discrete mathematics such that the graphs are mathematically elegant which used in representing the mathematical combinations [15] like topological spaces. Many researchers introduced some topological structure. Graph theory is one of these structures, that is, studying graph theory by means of topology. The notion of creating topologies on the set of vertices or the set of edges in graphs is taken from the notion of the digital image and a graph model. For example, in 2013, Amiri [8] introduced a topology, called graphic topology, on the set $\mathcal{V}$ of vertices of simple graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ by giving the subbasis family $S_G = \{A_x : x \in V\}$ such that A_x is the set of all adjacent vertices of x. In 2018, Abdu and Kiliciman [1] introduced the topologies on the set of edges in directed graphs, called incompatible edge and compatible topologies. In 2020, Sari and Kopuzlu [14], Othman and Alzubaidi [12] and Zomam, Othman and Dammak [10] introduced the topology in simple undirected graphs on the set of vertices. In 2022, Othman, Al-Shamiri, Saif, Acharjee, Lamoudan and Ismail [11] have introduced interesting results in directed graphs. The directed graphs have some interesting applications in physics, communication and electronically engineering. So, the results in this work will open a wide window new for further research projects in those areas.

In this work we present the role of topological spaces in graph theory such as the giving the relation between the connectedness in topological spaces and the connectedness in graph theory. We give the projection in directed graphs of compatible topological spaces on the set of vertices. In Section 2 we define

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the notion of L_2-directed topological spaces of directed graphs and we give some results about this new topology such as Alexandroff property. In Section 3, we study the form of L_2-directed topological space =on E-generated subdirected graphs and its relation with the relative topologies. In Section 4, we present relations between some fundamental properties in topological spaces with their corresponding properties in graphs such as the isomorphism and connectedness.

A directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ (simply, dirgraph) consists of a non-empty vertices set $\mathcal{V}$ and a set $\mathcal{E}$ of directed edges $K_{v_1v_2}$ where $v_1 \in \mathcal{V}$ is called the initial vertex of a directed edge $K_{v_1v_2}$ and $v_2 \in \mathcal{V}$ is called the terminal vertex of a directed edge $K_{v_1v_2}$. For any directed edge $K_{v_1v_2}$, $end(K_{v_1v_2}) = \{v_1, v_2\}$ is called the set of ends of $K_{v_1v_2}$. The adjacent edges are distinct edges that have a common vertex. Two directed edges $K_{v_1v_2}$ and $K'_{v'_1v'_2}$ are said to have the same direction (or adjacent directed edges) if $v_2 = v'_1$ or $v'_2 = v_1$. For $v \in \mathcal{V}$, the directed edge K_{vv} is called a loop. An alternating sequence of directed edge of the form $\{K^1_{v_1v_2}, K^2_{v_2v_3}, K^3_{v_3v_4}, ...\}$ is called directed path. The parallel edges are directed edges which have the same started vertex and the same end vertex. Thedigraph which has no parallel edges or no loops is called simple.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a dirgraph. Recall [1] that the compatible edges topological space is a topological space $(\mathcal{E}, T_{CE})$ which has a subbasis S_{CE}, where S_{CE} is a collection of subsets $B \subseteq \mathcal{E}$ such that

1. $|B| \le 2$;
2. If $E \in B$ and E' an edge that has the same direction to E, then $E' \in B$.

The Alexandroff space [10], is a topological space such that arbitrary intersection of open sets is an open set. Recall [1] that the compatible edges topological space $(\mathcal{E}, T_{CE})$ of adigraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is an Alexandroff space.

2. L_2-directed topological spaces

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a dirgraph. A set $\mathcal{H} \subseteq \mathcal{V}$ is called C-set in $\mathcal{V}$ if $|\mathcal{H}| \ge 2$ and for every $u \in \mathcal{H}$ there is at last one vertex $v \in \mathcal{H}$ such that there is directed edge between u and v. For any dirgraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and for any C-set $\mathcal{H} \subseteq \mathcal{V}$,

$$\mathcal{E}(\mathcal{H}) = \{K_{v_1v_2} \in \mathcal{E} : v_1, v_2 \in \mathcal{H}\}$$

and $|\mathcal{H}|_{\mathcal{E}}$ denotes the number of adjacent directed edges in $\mathcal{E}(\mathcal{H})$. If $|\mathcal{E}(\mathcal{H})| = 1$ then we consider $|\mathcal{H}|_{\mathcal{E}} = 1$. For any directed edge $K_{v_1v_2} \in \mathcal{E}$, $\mathcal{E}(K_{v_1v_2})$ denotes the set of all adjacent directed edges with $K_{v_1v_2}$. $|K_{v_1v_2}|_{\mathcal{E}}$ denotes the number of elements $\mathcal{E}(K_{v_1v_2})$.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a dirgraph. A subset E of $\mathcal{E}$ is called closed under directed edge if $K \in E$ and K' is adjacent directed edge with K implies $K' \in \mathcal{E}(\mathcal{H})$.

Definition 2.1. For any directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, the L_2-directed topological space of $\mathcal{G}$ is a pair $(\mathcal{V}, T_{\mathcal{G}})$ where $T_{\mathcal{G}}$ is a topology on $\mathcal{V}$ induced by a subbasis $\beta_{\mathcal{G}}$ which is a collection of $\emptyset$ and all C-sets in $\mathcal{V}$ such that $|\mathcal{H}|_{\mathcal{E}} \le 2$ and $\mathcal{E}(\mathcal{H})$ is closed under directed edge.

Example 2.2. In Fig.[1] or the digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$,

$$\mathcal{V} = \{v_1, v_2, v_3, v_4, v_5, v_6\},\ \mathcal{E} = \{v_2v_1, v_2v_3, v_6v_5, v_5v_4\}$$

and the subbasis

$$\beta_{\mathcal{G}} = \{\emptyset, \{v_1, v_2\}, \{v_2, v_3\}, \{v_4, v_5, v_6\}\}.$$

That is, the L_2-directed topology

$$T_{\mathcal{G}} = \{\emptyset, \mathcal{V}, \{v_2\}, \{v_1, v_2\}, \{v_2, v_3\}, \{v_4, v_5, v_6\}, \{v_1, v_2, v_3\}, \{v_2, v_4, v_5, v_6\},$$
$$\{v_1, v_2, v_4, v_5, v_6\}, \{v_2, v_3, v_4, v_5, v_6\}\}$$

.

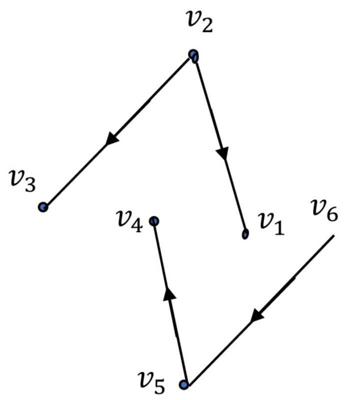

Figure 1:

Example 2.3. In Fig.2 [1-A], or the digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$,

$$\mathcal{V} = \{v_1, v_2, v_3, v_4, v_5\},\ \mathcal{E} = \{v_1v_2, v_2v_3, v_4v_5\}$$

and the subbasis

$$\beta_{\mathcal{G}} = \{\emptyset, \{v_1, v_2, v_3\}, \{v_4, v_5\}\}.$$

That is, the L_2-directed topology $T_{\mathcal{G}} = \{\emptyset, \mathcal{V}, \{v_1, v_2, v_3\}, \{v_4, v_5\}\}$.

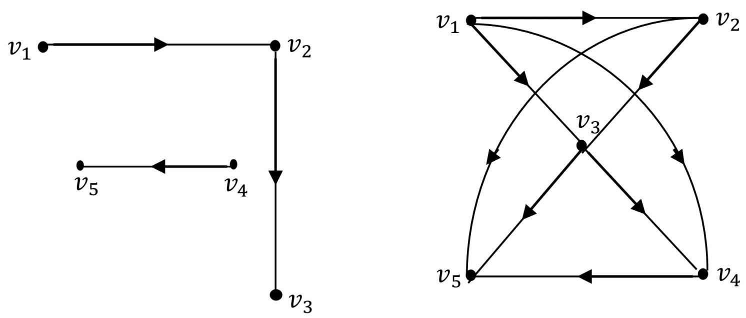

Figure 2: digraph [1-A] and [1-B]

Example 2.4. For the digraph in Fig.2 [1-B], the subbasis

$$\beta_{\mathcal{G}} = \{\emptyset, \{v_1, v_2, v_3\}, \{v_2, v_3, v_5\}, \{v_1, v_3, v_4\}, \{v_3, v_4, v_5\}, \{v_1, v_4, v_5\}, \{v_1, v_2, v_5\},$$
$$\{v_2, v_3, v_4\}, \{v_1, v_3, v_5\}\}.$$

includes the L_2- directed topology is a discrete topology.

Theorem 2.5. *If $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a directed graph then $end(K_{v_1v_2})$ is an open set in $(\mathcal{V}, T_{\mathcal{G}})$ for all $K_{v_1v_2} \in \mathcal{E}$ with $|K_{v_1v_2}|_{\mathcal{E}} = 0$ or $|K_{v_1v_2}|_{\mathcal{E}} > 2$.*

Proof. It is clear by definition of $\beta_{\mathcal{G}}$, if $|K_{v_1v_2}|_{\mathcal{E}} = 0$, then $end(K_{v_1v_2}) \in \beta_{\mathcal{G}}$, that is, the set $end(K_{v_1v_2})$ is an open set in $(\mathcal{V}, T_{\mathcal{G}})$. Let $|K_{v_1v_2}|_{\mathcal{E}} > 2$. Then, there are at least three directed edges $K^1_{u_1u_2}, K^2_{w_1w_2}, K^3_{z_1z_2} \in \mathcal{E}(K_{v_1v_2})$ such that one of the following holds:

1. $v_1 = u_2 = w_2$ and $v_2 = z_1$;
2. $v_1 = z_2 = w_2$ and $v_2 = u_1$;
3. $v_1 = z_2 = u_2$ and $v_2 = w_1$;
4. $v_2 = u_1 = w_1$ and $v_1 = z_2$;
5. $v_2 = z_1 = w_1$ and $v_1 = u_2$;
6. $v_2 = z_1 = u_1$ and $v_1 = w_2$.

Then six cases may happen

1. $A := \{u_1, v_1, v_2\}, B := \{w_1, v_1, v_2\}, C := \{v_1, v_2, z_2\} \in \beta_{\mathcal{G}}$;
2. $A := \{z_1, v_1, v_2\}, B := \{w_1, v_1, v_2\}, C := \{v_1, v_2, u_2\} \in \beta_{\mathcal{G}}$;
3. $A := \{u_1, v_1, v_2\}, B := \{z_1, v_1, v_2\}, C := \{v_1, v_2, w_2\} \in \beta_{\mathcal{G}}$;
4. $A := \{u_2, v_1, v_2\}, B := \{w_2, v_1, v_2\}, C := \{v_1, v_2, z_1\} \in \beta_{\mathcal{G}}$;
5. $A := \{z_2, v_1, v_2\}, B := \{w_2, v_1, v_2\}, C := \{v_1, v_2, u_1\} \in \beta_{\mathcal{G}}$;
6. $A := \{u_2, v_1, v_2\}, B := \{z_2, v_1, v_2\}, C := \{v_1, v_2, w_1\} \in \beta_{\mathcal{G}}$;

respectively. Note that for all the pervious cases, $A \cap B \cap C = end(K_{v_1v_2})$ is an open set in $(\mathcal{V}, T_{\mathcal{G}})$. □

Corollary 2.6. *If $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a directed graph then $\{v\}$ is an open set in $(\mathcal{V}, T_{\mathcal{G}})$ for all distinct directed edges $K_{vu_1}, K'_{vu_2} \in \mathcal{E}$ in following conditions:*

1. $|K_{vu_1}|_{\mathcal{E}} = 0$ *and* $|K'_{vu_2}|_{\mathcal{E}} = 0$.
2. $|K_{vu_1}|_{\mathcal{E}} = 0$ *and* $|K'_{vu_2}|_{\mathcal{E}} > 2$.
3. $|K_{vu_1}|_{\mathcal{E}} \geq 2$ *and* $|K'_{vu_2}|_{\mathcal{E}} = 0$.
4. $|K_{vu_1}|_{\mathcal{E}} \geq 2$ *and* $|K'_{vu_2}|_{\mathcal{E}} > 2$.

Proof. It is sufficient to notice that for each of the six cases, we find that the sets $end(K_{vu_1})$ and $end(K'_{vu_2})$ are open sets in $(\mathcal{V}, T_{\mathcal{G}})$. □

Theorem 2.7. *If $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a directed graph then the L_2- directed topological space of $\mathcal{G}$ is an Alexandroff space.*

Proof. It is enough to prove that arbitrary intersection of elements of $\beta_{\mathcal{G}}$ is an open set in $(\mathcal{V}, T_{\mathcal{G}})$. Let $\{A_\lambda : \lambda \in \Delta\}$ be the collection of elements of $\beta_{\mathcal{G}}$. Then it clear that by the definition of $\beta_{\mathcal{G}}$, $|A_\lambda| = 2$ or $|A_\lambda| = 3$ for all $\lambda \in \Delta$. Then, one of the following holds: $\cap_{\lambda\in\Delta} A_\lambda = \emptyset$ or $\cap_{\lambda\in\Delta} A_\lambda = \{u, v\}$ or $\cap_{\lambda\in\Delta} A_\lambda = \{u\}$ for some $u, v \in \mathcal{V}$. If $\cap_{\lambda\in\Delta} A_\lambda = \emptyset$, then $\cap_{\lambda\in\Delta} A_\lambda$ is an open set in $(\mathcal{V}, T_{\mathcal{G}})$. If $\cap_{\lambda\in\Delta} A_\lambda = \{u, v\}$ for some $u, v \in \mathcal{V}$ then by Theorem(2.5), $\cap_{\lambda\in\Delta} A_\lambda$ is an open set in $(\mathcal{V}, T_{\mathcal{G}})$. If $\cap_{\lambda\in\Delta} A_\lambda = \{u\}$ for some $u \in \mathcal{V}$, then one of the following three cases may occur:

$$\cap_{\lambda\in\Delta} A_\lambda = \{u\} = A \cap B$$

where $|A| = |B| = 3$ or $|A| = 2$ and $|B| = 3$ or $|A| = 2$ and $|B| = 2$.
Case 1: If $|A| = |B| = 3$, then $A, B \in \beta_{\mathcal{G}}$. Hence, $\cap_{\lambda\in\Delta} A_\lambda$ is an open set in $(\mathcal{V}, T_{\mathcal{G}})$.
Case 2: If $|A| = 2$ and $|B| = 3$, then $B \in \beta_{\mathcal{G}}$ and, hence B is an open set in $(\mathcal{V}, T_{\mathcal{G}})$. For $|A| = 2$, we have one of the following: $A \in \beta_{\mathcal{G}}$ or $A = D_1 \cap D_2$ where $|D_1| = |D_2| = 3$, that is, $D_1, D_2 \in \beta_{\mathcal{G}}$. This implies in two cases, A is an open set in $(\mathcal{V}, T_{\mathcal{G}})$. Therefore $\cap_{\lambda\in\Delta} A_\lambda$ is an open set in $(\mathcal{V}, T_{\mathcal{G}})$.
Case 3: If $|A| = 2$ and $|B| = 2$, then we have one of the following: $A, B \in \beta_{\mathcal{G}}$ or $B \in \beta_{\mathcal{G}}$ and $A = D_1 \cap D_2$ where $|D_1| = |D_2| = 3$, that is, $D_1, D_2 \in \beta_{\mathcal{G}}$ or $A \in \beta_{\mathcal{G}}$ and $B = D_1 \cap D_2$ where $|D_1| = |D_2| = 3$, that is, $D_1, D_2 \in \beta_{\mathcal{G}}$ or $A = D_1 \cap D_2$ and $B = D'_1 \cap D'_2$ where

$$|D_1| = |D_2| = |D'_1| = |D'_2| = 3,$$

that is, $D_1, D_2, D'_1, D'_2 \in \beta_{\mathcal{G}}$. This implies in four cases, $\cap_{\lambda\in\Delta} A_\lambda$ is an open set in $(\mathcal{V}, T_{\mathcal{G}})$. □

Remark 2.8. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a directed graph. For $\mathcal{H} \subseteq \mathcal{V}$, $O_{\mathcal{G}}(\mathcal{H})$ denotes the intersection of all open sets in $(\mathcal{V}, T_{\mathcal{G}})$ containing $\mathcal{H}$. From the above theorem, $(\mathcal{V}, T_{\mathcal{G}})$ is Alexandroff space, then it is clear that $O_{\mathcal{G}}(\mathcal{H})$ is the smallest open set in $(\mathcal{V}, T_{\mathcal{G}})$ containing $\mathcal{H}$. For $v \in \mathcal{V}$, we write $O_{\mathcal{G}}(v)$ replaced of $O_{\mathcal{G}}(\{v\})$. The collection $\beta_{\mathcal{G}}(\mathcal{V}) := \{O_{\mathcal{G}}(v) : v \in \mathcal{V}\}$ forms a minimal basis of $(\mathcal{V}, T_{\mathcal{G}})$.

Theorem 2.9. *If $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a directed graph then for all $\mathcal{H} \subseteq \mathcal{V}$,*

$$O_{\mathcal{G}}(\mathcal{H}) = \cap\{A \in \beta_{\mathcal{G}} : \mathcal{H} \subseteq A\}.$$

Proof. It is clear that B is an open set in $(\mathcal{V}, T_{\mathcal{G}})$ for all $B \in \beta_{\mathcal{G}}$. So by Theorem 2.7, $\cap\{A \in \beta_{\mathcal{G}} : \mathcal{H} \subseteq A\}$ is an open set in $(\mathcal{V}, T_{\mathcal{G}})$. Since $\mathcal{H} \subseteq A$ for all $A \in \{A \in \beta_{\mathcal{G}} : \mathcal{H} \subseteq A\}$, then $\mathcal{H} \subseteq \cap\{A \in \beta_{\mathcal{G}} : \mathcal{H} \subseteq A\}$ and so

$$O_{\mathcal{G}}(\mathcal{H}) \subseteq \cap\{A \in \beta_{\mathcal{G}} : \mathcal{H} \subseteq A\}.$$

Since the collection of all intersections of members of $\beta_{\mathcal{G}}$ forms a basis for $(\mathcal{V}, T_{\mathcal{G}})$, then

$$\cap\{A \in \beta_{\mathcal{G}} : \mathcal{H} \subseteq A\} \subseteq O_{\mathcal{G}}(\mathcal{H}).$$

Hence $O_{\mathcal{G}}(\mathcal{H}) = \cap\{A \in \beta_{\mathcal{G}} : \mathcal{H} \subseteq A\}$. $\square$

Corollary 2.10. *If $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a digraph then for all $v \in \mathcal{V}$,*

$$O_{\mathcal{G}}(v) = \cap\{A \in \beta_{\mathcal{G}} : v \in A\}.$$

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a digraph and $K \in \mathcal{E}$. If $|K|_{\mathcal{E}} = 0$ then $end(K) \in \beta_{\mathcal{G}}$ is an open set in $(\mathcal{V}, T_{\mathcal{G}})$. So $O_{\mathcal{G}}(end(K)) = end(K)$. If $|K|_{\mathcal{E}} = 1$, then there is $K' \in \mathcal{E}(K)$ such that $end(K) \cup end(K') \in \beta_{\mathcal{G}}$. So

$$O_{\mathcal{G}}(end(K)) = end(K) \cup end(K').$$

If $|K|_{\mathcal{E}} = 2$, then there are $K', K'' \in \mathcal{E}(K)$ such that $end(K) \cup end(K') \cup end(K'') \in \beta_{\mathcal{G}}$ or $end(K) \in \beta_{\mathcal{G}}$. So

$$O_{\mathcal{G}}(end(K)) = end(K) \cup end(K') \cup end(K'') \text{ or } O_{\mathcal{G}}(end(K)) = end(K).$$

If $|K|_{\mathcal{E}} > 2$, then by Theorem 2.5, $end(K)$ is an open set in $(\mathcal{V}, T_{\mathcal{G}})$. So $O_{\mathcal{G}}(end(K)) = end(K)$.

Proposition 2.11. *Let $K, K' \in \mathcal{E}$ in a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. If $\mathcal{E}(K) = \{K'\}$ then $end(K') \subseteq O_{\mathcal{G}}(end(K))$.*

Corollary 2.12. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a digraph and $K, K' \in \mathcal{E}$. If $\mathcal{E}(K') = \{K\}$ then $end(K') \subseteq \overline{O_{\mathcal{G}}(end(K))}$.*

Proof. Suppose that $\mathcal{E}(K') = \{K\}$. Then, by Proposition 2.11, $end(K) \subseteq O_{\mathcal{G}}(end(K'))$. Then for all open set A containing $end(K')$, $end(K) \subseteq A$ and $A \cap end(K) = end(K) \neq \emptyset$. Since $end(K) \subseteq O_{\mathcal{G}}(end(K))$, then $A \cap O_{\mathcal{G}}(end(K)) \neq \emptyset$. That is, $end(K') \subseteq \overline{O_{\mathcal{G}}(end(K))}$. $\square$

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a digraph. Recall [10] that an Alexandroff space $(\mathcal{V}, T_{\mathcal{G}})$ is T_0-space if and only if $O_{\mathcal{G}}(v) \neq O_{\mathcal{G}}(u)$ for all $u \neq v \in \mathcal{V}$. An Alexandroff space $(\mathcal{V}, T_{\mathcal{G}})$ is T_1-space if and only if $O_{\mathcal{G}}(v) = \{v\}$ for all $u \neq v \in \mathcal{V}$, that is, if and only if $(\mathcal{V}, T_{\mathcal{G}})$ is discrete.

Proposition 2.13. *If $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a digraph then*

$$\cup_{K \in \mathcal{E}}\{end(K) : |K|_{\mathcal{E}} = 0 \textit{ or } |K|_{\mathcal{E}} \geq 2\}$$

is an open set in $(\mathcal{V}, T_{\mathcal{G}})$.

Proof. By Theorem 2.5, for $K \in \mathcal{E}$ with $|K|_{\mathcal{E}} = 0$ or $|K|_{\mathcal{E}} \geq 2$, $end(K)$ is an open set in $(\mathcal{V}, T_{\mathcal{G}})$. So $\cup_{K \in \mathcal{E}}\{end(K) : |K|_{\mathcal{E}} = 0 \text{ or } |K|_{\mathcal{E}} \geq 2\}$ is an open set in $(\mathcal{V}, T_{\mathcal{G}})$. $\square$

Proposition 2.14. *If $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a digraph, then $\cup_{K \in \mathcal{E}}\{end(K) : |K|_{\mathcal{E}} = 1\}$ is a closed set in $(\mathcal{V}, T_{\mathcal{G}})$.*

Proof. Let

$$C = \cup_{K\in\mathcal{E}}\{end(K) : |K|_{\mathcal{E}} = 1\}.$$

It is clear that

$$\overline{C} = \cup_{K\in\mathcal{E}}\{\overline{end(K)} : |K|_{\mathcal{E}} = 1\}.$$

By Corollary 2.12, $\overline{end(K)} \subseteq C$ for all $end(K) \subseteq C$. So $\overline{C} \subseteq C$, that is, C is a closed set in $(\mathcal{V}, T_{\mathcal{G}})$. □

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a digraph. For $v \in \mathcal{V}$, $\mathcal{E}(v)$ denotes the set of all $K \in \mathcal{H}$ such that $v \in end(K)$ and $\mathcal{V}(v)$ denotes the set of all $v' \in \mathcal{V}$ such that v is join with v' by directed edge.

Proposition 2.15. *If $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a digraph then for $v \in \mathcal{V}$,*

$$\cap_{K\in\mathcal{E}(v)} O_{\mathcal{G}}(end(K)) = O_{\mathcal{G}}(v).$$

Proof. It is clear that $O_{\mathcal{G}}(v) \subseteq \cap_{K\in\mathcal{E}(v)} O_{\mathcal{G}}(end(K))$. Since $O_{\mathcal{G}}(v)$ is the intersection of all open sets in $(\mathcal{V}, T_{\mathcal{G}})$ containing v and $\beta_{\mathcal{G}}$ is the subbasis of $(\mathcal{V}, T_{\mathcal{G}})$, the

$$O_{\mathcal{G}}(v) = \cap_{K\in\mathcal{K}'} O_{\mathcal{G}}(end(K))$$

for some subset $\mathcal{K}'$ of $\mathcal{E}$. Then, $v \in O_{\mathcal{G}}(end(K))$ for all $K \in \mathcal{K}'$. Hence, $K \in \mathcal{E}(v)$ for all $K \in \mathcal{K}'$, that is, $\mathcal{K}' \subseteq \mathcal{E}(v)$. So $\cap_{K\in\mathcal{E}(v)} O_{\mathcal{G}}(end(K)) \subseteq O_{\mathcal{G}}(v)$. □

3. On E-generated subdirected graph

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a digraph and $\mathcal{H}$ be any digraph. If the direction function of $\mathcal{H}$ is the restriction of the direction function of $\mathcal{G}$ on $\mathcal{E}(\mathcal{H})$ and all edges and vertices of $\mathcal{H}$ are in $\mathcal{G}$, then $\mathcal{H}$ is called subdirected graph of $\mathcal{G}$. A collection of the edges in a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ together with their terminals is called Edge-generated (or E-generated) subdirected graph of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. For any E-generated subdirected graph $\mathcal{G}_{\mathcal{H}}$ of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $\mathcal{V}_{\mathcal{H}}$ denotes the set of all vertices of $\mathcal{G}_{\mathcal{H}}$, $\mathcal{E}_{\mathcal{H}}$ denotes the set of all edges of $\mathcal{G}_{\mathcal{H}}$, $T_{\mathcal{G}\mathcal{H}}$ denotes the L_2−directed topology of $\mathcal{G}_{\mathcal{H}}$ and $\beta_{\mathcal{G}\mathcal{H}}$ is the subbasis of $(\mathcal{V}_{\mathcal{H}}, T_{\mathcal{G}\mathcal{H}})$.

Theorem 3.1. *For any E-generated subdirected graph $\mathcal{G}_{\mathcal{H}}$ of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $T_{\mathcal{G}\mathcal{H}} \subseteq T_{\mathcal{G}}|\mathcal{V}_{\mathcal{H}}$, where $T_{\mathcal{G}}|\mathcal{V}_{\mathcal{H}}$ is the relative topology of $T_{\mathcal{G}}$ on $\mathcal{V}_{\mathcal{H}}$.*

Proof. Let $G \in T_{\mathcal{G}\mathcal{H}}$. We will prove that $G = F \cap \mathcal{V}_{\mathcal{H}}$ for some open set F in $(\mathcal{V}, T_{\mathcal{G}})$. Let

$$F' = \cap\{D \in T_{\mathcal{G}} : G \subseteq D\}.$$

Then, by Theorem 2.7, F' is an open set in $(\mathcal{V}, T_{\mathcal{G}})$ and $F' \cap \mathcal{V}_{\mathcal{H}} = G$. That is, $G \in T_{\mathcal{G}}|\mathcal{V}_{\mathcal{H}}$. □

In the theorem above, note that for any E-generated subdirected graph $\mathcal{G}_{\mathcal{H}}$ of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $T_{\mathcal{G}\mathcal{H}} \neq T_{\mathcal{G}}|\mathcal{V}_{\mathcal{H}}$. For example in the Fig. 3,

$$T_{\mathcal{G}} = \{\emptyset, \mathcal{V}, \{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}, \{v_2, v_3\}\}.$$

Take $\mathcal{E}_{\mathcal{H}} = \{K_1, K_2\}$, so $\mathcal{V}_{\mathcal{H}} = \{v_1, v_2, v_3\}$. Note that

$$T_{\mathcal{G}\mathcal{H}} = \{\emptyset, \mathcal{V}_{\mathcal{H}}\} \text{ and } T_{\mathcal{G}}|\mathcal{V}_{\mathcal{H}} = \{\emptyset, \mathcal{V}_{\mathcal{H}}, \{v_2, v_3\}\}.$$

An E-generated subdirected graph $\mathcal{G}_{\mathcal{H}} = (\mathcal{V}_{\mathcal{H}}, \mathcal{E}_{\mathcal{H}})$ of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is called an adjacent with $\mathcal{G}$ if $|K|_{\mathcal{E}} > 2$ in $\mathcal{G}$ implies $|K|_{\mathcal{E}_{\mathcal{H}}} > 2$ in $\mathcal{G}_{\mathcal{H}}$ for all $K \in \mathcal{E}_{\mathcal{H}}$.

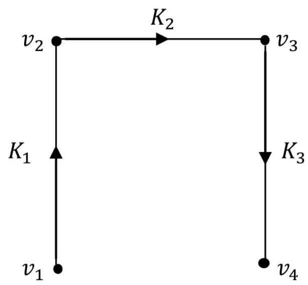

Figure 3:

Theorem 3.2. *Let $\mathcal{G}_{\mathcal{H}} = (\mathcal{V}_{\mathcal{H}}, \mathcal{E}_{\mathcal{H}})$ be an E-generated subdirected graph of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Then, $\mathcal{G}_{\mathcal{H}}$ is an adjacent with $\mathcal{G}$ if and only if $T_{\mathcal{G}\mathcal{H}} = T_{\mathcal{G}}|\mathcal{V}_{\mathcal{H}}$.*

Proof. Suppose that $\mathcal{G}_{\mathcal{H}}$ is an adjacent with $\mathcal{G}$. Let $G \in T_{\mathcal{G}}|\mathcal{V}_{\mathcal{H}}$ and $G \notin T_{\mathcal{G}\mathcal{H}}$. Since $\mathcal{V}_{\mathcal{H}} \in T_{\mathcal{G}\mathcal{H}} \cap T_{\mathcal{G}}|\mathcal{V}_{\mathcal{H}}$ then $G = end(K)$ for some $K \in \mathcal{E}_{\mathcal{H}}$ such that $|K|_{\mathcal{E}} > 2$ in $\mathcal{G}$ and $|K|_{\mathcal{E}_{\mathcal{H}}} = 1$ in $\mathcal{G}_{\mathcal{H}}$. This is a contradiction with the hypothesis, that is, $T_{\mathcal{G}}|\mathcal{V}_{\mathcal{H}} \subseteq T_{\mathcal{G}\mathcal{H}}$. For the other hand, $T_{\mathcal{G}\mathcal{H}} \subseteq T_{\mathcal{G}}|\mathcal{V}_{\mathcal{H}}$ by Theorem 3.1. That is, $T_{\mathcal{G}\mathcal{H}} = T_{\mathcal{G}}|\mathcal{V}_{\mathcal{H}}$. Conversely, let $T_{\mathcal{G}\mathcal{H}} = T_{\mathcal{G}}|\mathcal{V}_{\mathcal{H}}$ and there is $K \in \mathcal{E}_{\mathcal{H}}$ such that $|K|_{\mathcal{E}} > 2$ in $\mathcal{G}$ and $|K|_{\mathcal{E}_{\mathcal{H}}} < 2$ in $\mathcal{G}_{\mathcal{H}}$. Then by Theorem 2.5, $end(K)$ is an open set in $(\mathcal{V}, T_{\mathcal{G}})$. Hence $end(K) \cap \mathcal{V}_{\mathcal{H}} = end(K)$ is an open set in $(\mathcal{V}_{\mathcal{H}}, T_{\mathcal{G}}|\mathcal{V}_{\mathcal{H}})$ but $end(K)$ is not open set in $(\mathcal{V}_{\mathcal{H}}, T_{\mathcal{G}\mathcal{H}})$. This is a contradiction with the hypothesis, that is, $\mathcal{G}_{\mathcal{H}}$ is an adjacent with $\mathcal{G}$. □

4. On isomorphisms and connected graphs

For two dirgraphs $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and $\mathcal{G}' = (\mathcal{V}', \mathcal{K}')$, by p-function of $\mathcal{G}$ into $\mathcal{G}'$ we mean a pair $(\Phi_{\mathcal{V}\mathcal{V}'}, \Phi_{\mathcal{E}\mathcal{K}'}) : \mathcal{G} \to \mathcal{G}'$ of two functions $\Phi_{\mathcal{V}\mathcal{V}'} : \mathcal{V} \to \mathcal{V}'$ and $\Phi_{\mathcal{E}\mathcal{K}'} : \mathcal{E} \to \mathcal{K}'$. Recall [15] that the isomorphism of $\mathcal{G}$ onto $\mathcal{G}'$ is a p-function $(\Phi_{\mathcal{V}\mathcal{V}'}, \Phi_{\mathcal{E}\mathcal{K}'}) : \mathcal{G} \to \mathcal{G}'$ of two bijective functions $\Phi_{\mathcal{V}\mathcal{V}'} : \mathcal{V} \to \mathcal{V}'$ and $\Phi_{\mathcal{E}\mathcal{K}'} : \mathcal{E} \to \mathcal{K}'$ such that

$$\Phi_{\mathcal{E}\mathcal{K}'}(K_{v_1v_2}) = \Phi_{\mathcal{E}\mathcal{K}'}(K)_{\Phi_{\mathcal{V}\mathcal{V}'}(v_1)\Phi_{\mathcal{V}\mathcal{V}'}(v_2)}$$

for all $K_{v_1v_2} \in \mathcal{E}$ and $v_1, v_2 \in \mathcal{V}$, that is, such that $K_{v_1v_2} \in \mathcal{E}$ is an edge directed from v_1 into v_2 in $\mathcal{G}$ if and only if $\Phi_{\mathcal{E}\mathcal{K}'}(K_{v_1v_2})$ is an edge directed from $\Phi_{\mathcal{V}\mathcal{V}'}(v_1)$ into $\Phi_{\mathcal{V}\mathcal{V}'}(v_2)$ in $\mathcal{G}'$. If there exists isomorphism of $\mathcal{G}$ onto $\mathcal{G}'$ then we say that $\mathcal{G}$ and $\mathcal{G}'$ are isomorphic and write $\mathcal{G} \cong \mathcal{G}'$.

Remark 4.1. *It is clear that if two digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and $\mathcal{G}' = (\mathcal{V}', \mathcal{K}')$ are isomorphic then the two L_2−directed topological spaces $(\mathcal{V}, T_{\mathcal{G}})$ and $(\mathcal{V}', T_{\mathcal{G}'})$ are homeomorphic but the converse no need to be true. In Fig.4,*

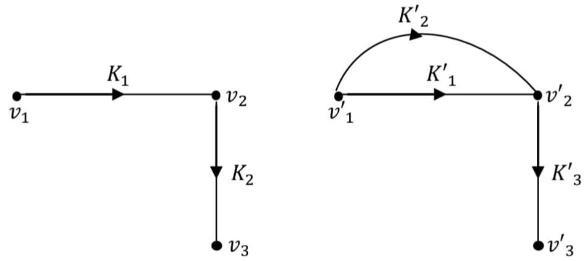

Figure 4:

the graphs $\mathcal{G}$ and $\mathcal{G}'$ are not isomorphic but the two related L_2-directed topological spaces $(\mathcal{V}, T_{\mathcal{G}})$ and $(\mathcal{V}', T_{\mathcal{G}'})$ are obviously homeomorphic because of $T_{\mathcal{G}} = \{\emptyset, \mathcal{V}\}$ and $T_{\mathcal{G}'} = \{\emptyset, \mathcal{V}'\}$.

Theorem 4.2. *If $\mathcal{G}$ and $\mathcal{G}'$ are two simple directed graphs and $\phi : \mathcal{V} \to \mathcal{V}'$ is continuous function then*

$$\mathcal{E}(K_{uv}) = \{K_{vw}\} \text{ (resp. } \mathcal{E}(K_{uv}) = \{K_{wu}\})$$

implies

$$\mathcal{E}(K_{\phi(u)\phi(v)}) = \{K_{\phi(v)\phi(w)}\} \text{ (resp.} \mathcal{E}(K_{\phi(u)\phi(v)}) = \{K_{\phi(w)\phi(u)}\})$$

for all $u, v, w \in \mathcal{V}$.

Proof. Suppose that $\phi : \mathcal{V} \to \mathcal{V}'$ is continuous and $u, v, w \in \mathcal{V}$ such that $\mathcal{E}(K_{uv}) = \{K_{vw}\}$. By Corollary 2.12, $end(K_{uv}) \subseteq \overline{O_{\mathcal{G}}(end(K_{vw}))}$. Then

$$\phi[end(K_{uv})] \subseteq \phi[\overline{O_{\mathcal{G}}(end(K_{vw}))}].$$

That is,

$$end(K_{\phi(u)\phi(v)}) \subseteq \phi[\overline{O_{\mathcal{G}}(end(K_{vw}))}].$$

Since ϕ is continuous then

$$\phi[\overline{O_{\mathcal{G}}(end(K_{vw}))}] \subseteq \overline{O_{\mathcal{G}}(end(K_{\phi(v)\phi(w)}))}.$$

Hence

$$end(K_{\phi(u)\phi(v)}) \subseteq \overline{O_{\mathcal{G}}(end(K_{\phi(v)\phi(w)}))}.$$

That is, $\mathcal{E}(K_{\phi(u)\phi(v)}) = \{K_{\phi(v)\phi(w)}\}$ from Corollary 2.12. Similar for the other case. □

The converse of Theorem 4.2 is not true, for example, in Fig. 5, let $\phi : \mathcal{V} \to \mathcal{V}'$ be a function given by

$$\phi(v_1) = \phi(v_4) = v'_1,\ \phi(v_2) = v'_2 \text{ and } \phi(v_3) = v'_3.$$

Note that ϕ is not continuous while $\mathcal{E}(K_{v_1v_2}) = \{K_{v_2v_3}\}$ implies $\mathcal{E}(K_{v'_1v'_2}) = \{K_{v'_2v'_3}\}$ and $\mathcal{E}(K_{v_2v_3}) = \{K_{v_1v_2}\}$ implies $\mathcal{E}(K_{v'_2v'_3}) = \{K_{v'_1v'_2}\}$.

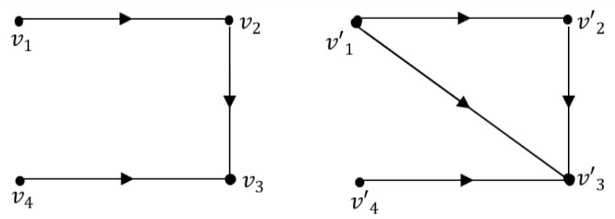

Figure 5:

Theorem 4.3. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a digraphs without isolated points. If $\mathcal{G}$ is a disconnected graph then the L_2-directed topological space $(\mathcal{V}, T_{\mathcal{G}})$ is disconnected space.*

Proof. Let $\{\mathcal{G}_\alpha : \alpha \in \Delta\}$ be the collection of all directed subgraphs of $\mathcal{G}$. Then for every $\alpha \in \Delta$,

$$\mathcal{V}_{\mathcal{G}_\alpha} := \cup\{end(K) : K \in \mathcal{E}(\mathcal{G}_\alpha)\}$$

is an open set in $(\mathcal{V}, T_{\mathcal{G}})$. Since $\mathcal{G}$ has no isolated points then also $\mathcal{V}^c_{\mathcal{G}_\alpha} = \mathcal{V} - \mathcal{V}_{\mathcal{G}_\alpha}$ is an open set in $(\mathcal{V}, T_{\mathcal{G}})$ and $\mathcal{V} = \mathcal{V}^c_{\mathcal{G}_\alpha} \cup \mathcal{V}_{\mathcal{G}_\alpha}$. That is, the L_2-directed topological space $(\mathcal{V}, T_{\mathcal{G}})$ is disconnected space. □

The converse of Theorem is not true, for example, in Fig. 6. the L_2-directed topological space $(\mathcal{V}, T_{\mathcal{G}})$ is disconnected space but the graph $\mathcal{G}$ is connected where

$$T_{\mathcal{G}} = \{\emptyset, \mathcal{V}, \{v_1, v_2\}\{v_3, v_2\}, \{v_3, v_4\}, \{v_2\}, \{v_3\}, \{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}\}.$$

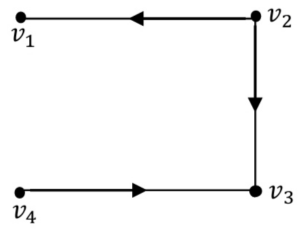

Figure 6:

5. Conclusion

The correlation property in topological spaces is integral to satisfying the correlation property in graph theory, and conversely, the property in graph theory is also important for topological spaces. The L_2- directed topological spaces have been studied in directed graph theory to explore this relationship, as well as the similarities between two directed graphs and their relationships to the similarities of the corresponding L_2- directed topological spaces. Several standard properties exist in both topological space theories and graph theory that require an explanation of their interrelationships, such as interdependence and path continuity. These relationships correlate with results for these references [[7], [3], [2], [5], [6], [4]]

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