Filomat 37:28 (2023), 9769–9775
https://doi.org/10.2298/FIL2328769B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Cesàro convergence of sequences of bi-complex numbers using BC-Orlicz function

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Abstract. In this article we have introduced the concept of Cesàro convergence, Cesàro null and Cesàro bounded sequences of bi-complex numbers defined by BC-Orlicz function having hyperbolic norm. we have investigated some of their algebraic and topological properties by defining a D-norm on these spaces. Also inclusion results involving these sequence spaces have been established.

1. Introduction

Bi-complex numbers are being studied for quite a long time now. Probably Italian school of Segre [12] introduced the bi-complex numbers. For more details on bi-complex numbers and bi-complex functional analysis see ([14], [16], [11]). The hyperbolic numbers studied by Cockle [2], Lie and Scheffers [7]. Hyperbolic number system has been studied for various reasons. Many research developed the hyperbolic numbers.

The sequence space has been investigated by different researchers from different aspects, such as Buck [1], Fast[5], Schoenberg [13], Fridy [6], Rath and Tripathy [10], Tripathy and Nath[15].
A real sequence $x = (x_k)$ is said to be Cesàro convergent to l if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} x_k = l.$$

Definition 1.1. *An Orlicz function is a function $\mathcal{M} : [0,\infty) \to [0,\infty)$, which is continuous, non-decreasing and convex with $\mathcal{M}(0) = 0$, $\mathcal{M}(x) > 0$, for $x > 0$ and $\mathcal{M}(x) \to \infty$, as $x \to \infty$.*
Lindendstrauss and Tzafriri [8] used the idea of Orlicz function to construct the sequence space

$$\ell_M := \left\{ x \in \omega : \sum_{k=1}^{\infty} \mathcal{M}\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}.$$

The sequence space ℓ_M is Banach space with the norm

$$\|x\| := \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} \mathcal{M}\left(\frac{|x_k|}{\rho}\right) < 1 \right\}.$$

2020 *Mathematics Subject Classification.* 46A45; 46E30, 40A35
Keywords. Orlicz Function, Cesàro Convergence, Bi-complex
Received: 09 February 2023; Revised: 27 April 2023; Accepted: 29 April 2023
Communicated by Miodrag Spalević
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The concept of Orlicz function has been applied for studying different classes of sequences by Datta and Tripathy[3], Nath and Tripathy[9] and many more. In this article we developed the Cesàro convergence using BC-Orlicz function. Throughout the article we denote C_0, C_1 and C_2 by set of real, complex and bi-complex numbers respectively also we denote by w^*, the sequences of all bi-complex numbers.

2. Definition and Preliminaries

2.1. Bi-complex Numbers

A bi-complex number ξ is of the form

$$\xi = z_1 + i_2 z_2 = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4,$$

where $z_1, z_2 \in C_1$ and $x_1, x_2, x_3, x_4 \in C_0$ and the independent units i_1, i_2 are such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$, The set of bi-complex numbers C_2 is defined as:

$$C_2 = \{\xi : \xi = z_1 + i_2 z_2; z_1, z_2 \in C_1(i_1)\},$$

where $C_1(i_1) = \{x_1 + i_1 x_2 : x_1, x_2 \in C_0\}$. C_2 is a vector space over $C_1(i_1)$. Other than 0 and 1, there are two more idempotent elements in C_2 given by $e_1 = \frac{1+i_1 i_2}{2}$ and $e_2 = \frac{1-i_1 i_2}{2}$ such that $e_1 + e_2 = 1$ and $e_1 e_2 = 0$.
Every bi-complex number $\xi = z_1 + i_2 z_2$ can be uniquely expressed as the combination of e_1 and e_2, namely

$$\xi = z_1 + i_2 z_2 = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2 = \mu_1 e_1 + \mu_2 e_2,$$

where $\mu_1 = (z_1 - i_1 z_2)$ and $\mu_2 = (z_1 + i_1 z_2)$.
For $\xi = z_1 + i_2 z_2 \in C_2$, the norm is defined as

$$||\xi||_{C_2} = \sqrt{|z_1|^2 + |z_2|^2}.$$

The product of two bi-complex numbers is connected by the following inequality:

$$||\xi \cdot \eta||_{C_2} \leq \sqrt{2}||\xi||_{C_2}.||\eta||_{C_2}.$$

C_2 together with the norm defined above form a generalized algebra. Since $C_2 \simeq C_0^4$ and C_0^4 is complete with respect to usual metric, it follows that C_2 forms a generalized Banach algebra.
The bi-complex number $\xi = z_1 + i_2 z_2$ is called singular if $|z_1^2 + z_2^2| = 0$.
The set of all singular numbers is denoted by $\mathbb{O}_2$.

2.2. Hyperbolic Numbers

The hyperbolic number is of the form

$$\alpha = x_1 + i_1 i_2 x_2; x_1, x_2 \in C_0.$$

The idempotent representation of any hyperbolic number $\alpha = x_1 + i_1 i_2 x_2$ is

$$\alpha = v_1 e_1 + v_2 e_2,$$

where $v_1 = x_1 + x_2, v_2 = x_2 - x_1$.
The set of hyperbolic numbers is given by

$$D = \{v_1 e_1 + v_2 e_2 : v_1, v_2 \in C_0\}.$$

The set of positive hyperbolic numbers is given by

$$D_+ = \{v_1 e_1 + v_2 e_2 : v_1, v_2 \geq 0\}.$$

Let $\xi \in C_2$, then hyperbolic norm(D- valued) norm on C_2 is given by

$$|\xi|_D = |\mu_1|e_1 + |\mu_2|e_2 \in D_+.$$

If $\xi, \eta \in C_2$, then

$$|\xi + \eta|_D \leq^{'} |\xi_D + |\eta|_D \text{ and } |\xi\eta|_D = |\xi|_D|\eta|_D.$$

Let S be a subset of D. Consider the two sets $D_1 = \{v_1 : v_1e_1 + v_2e_2 \in S\}$ and $D_2 = \{v_2 : v_1e_1 + v_2e_2 \in S\}$. Then supremum of the set S is given by

$$\sup_D S = e_1 \sup D_1 + e_2 \sup D_2.$$

Similarly, infimum of the set S is given by

$$\inf_D S = e_1 \inf D_1 + e_2 \inf D_2.$$

The partial order relation on D is given by

$$\alpha \leq^{'} \beta \text{ if and only if } \beta - \alpha \in D_+ \forall \alpha, \beta \in D.$$

Remark 2.1. *Denote D_+^*, by the the non negative extended hyperbolic numbers*

$$D_+^* = \{\mu_1e_1 + \mu_2e_2, \mu_1, \mu_2 > 0\} \cup \{\infty\} \cup \{-\infty\} \cup \{\infty e_1 + \mu_2e_2\} \cup \{\mu_1e_1 - \infty e_2\}$$

Throughout the article we denote

$$0_D = 0 + 0i_1i_2.$$

Definition 2.2. *A function $\Upsilon_D : D \to D_+^*$ is called D-valued convex function if for every $\xi, \eta \in D$ with $0 \leq^{'} \alpha \leq^{'} 1$ such that*

$$\Upsilon_D(\alpha\xi + (1 - \alpha)\eta) \leq^{'} \alpha\Upsilon_D(\xi) + (1 - \alpha)\Upsilon_D(\eta).$$

Definition 2.3. *[4] A convex function $\Upsilon_D : D_+ \to D_+^*$ is said to be BC-Orlicz function if it satisfies the following conditions*
(i) $\Upsilon_D(0_D) = 0_D$;
(ii) $\lim_{\xi\to\infty} \Upsilon_D(\xi) = \infty^*$, where $\infty^* = \mu_1e_1 + \infty e_2 = \infty e_1 + \mu_2e_2 = \infty e_1 + \infty e_2$ and $\lim_{\xi\to\infty} \Upsilon_D(\xi)$ must exist along any line in the hyperbolic plane and must be equal.
We denote the BC-Orlicz function by $\mathcal{M}_D$.

Definition 2.4. *An BC-Orlicz function $\mathcal{M}_D$ is said to satisfy the Δ_D^2-condition denoted by $\mathcal{M}_D \in \Delta_D^2$ if there exist some hyperbolic constants $K \geq^{'} 0$ and ξ_0(depending upon K) such that*

$$\mathcal{M}_D((2e_1 + 2e_2)\xi) \leq^{'} K\mathcal{M}_D(\xi), \forall\, 0 \leq^{'} \xi \leq^{'} \xi_0.$$

Definition 2.5. *A function $g : C_2 \to D_+^*$ is called D-norm if the following conditions are satisfied;*
$p_1 : g(\xi) \geq^{'} 0_D$, for all $\xi \in C_2$;
$p_2 : g(-\xi) = g(\xi)$, for all $\xi \in C_2$;
$p_3 : g(\xi + \eta) \leq^{'} g(\xi) + g(\eta)$, for all $\xi, \eta \in C_2$;
$p_4 : \alpha_k \to \alpha, |x_k - x|_D \to 0_D$, then $|\alpha_k\xi_k - \alpha\xi|_D \to 0_D$.

3. Main result

In this section we introduce the notion of different types of Cesàro convergence sequences of bi-complex numbers defined by BC-Orlicz function. We investigate their different properties and we define the following sets

$$[b_1^*, \mathcal{M}_D] := \left\{\xi \in \omega^* : \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathcal{M}_D\left(\frac{|\xi_k - \xi^*|_D}{\alpha}\right) = 0_D, \text{ for some hyperbolic number } \alpha >' 0\right\}$$

$$[b_0^*, \mathcal{M}_D] := \left\{\xi \in \omega^* : \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathcal{M}_D\left(\frac{|\xi_k|_D}{\alpha}\right) = 0_D, \text{ for some hyperbolic number } \alpha >' 0\right\}$$

$$[b_\infty^*, \mathcal{M}_D] := \left\{\xi \in \omega^* : \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathcal{M}_D\left(\frac{|\xi_k|_D}{\alpha}\right) <' \infty, \text{ for some hyperbolic number } \alpha >' 0\right\}.$$

Theorem 3.1. *The sets $[b_1^*, \mathcal{M}_D]$, $[b_0^*, \mathcal{M}_D]$ and $[b_\infty^*, \mathcal{M}_D]$ are linear space over $C_2 \setminus \mathbb{O}_2$.*

Proof. Let $\xi, \eta \in [b_\infty^*, \mathcal{M}_D]$, then for some small hyperbolic numbers $\alpha_1, \alpha_2 >' 0$ such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathcal{M}_D\left(\frac{|\xi_k|_D}{\alpha_1}\right) <' \infty$$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathcal{M}_D\left(\frac{|\eta_k|_D}{\alpha_2}\right) <' \infty.$$

Let $k_1, k_2 \in C_2 \setminus \mathbb{O}_2$. and $\alpha = \max\{|k_1|_D\alpha_1, |k_2|_D\alpha_2\}$.
Now

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathcal{M}_D\left(\frac{|k_1\xi_k + k_2\eta_k|_D}{\alpha}\right)\\
&\leq' \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathcal{M}_D\left(\frac{|k_1\xi_k|_D}{\alpha}\right) + \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathcal{M}_D\left(\frac{|k_2\eta_k|_D}{\alpha}\right)\\
&= \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathcal{M}_D\left(\frac{|k_1|_D|\xi_k|_D}{\alpha}\right) + \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathcal{M}_D\left(\frac{|k_2|_D|\eta_k|_D}{\alpha}\right)\\
&= \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathcal{M}_D\left(\frac{|\xi_k|_D}{\alpha_1}\right) + \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathcal{M}_D\left(\frac{|\eta_k|_D}{\alpha_2}\right) <' \infty.
\end{aligned}$$

Therefore, $[b_\infty^*, \mathcal{M}_D]$ is linear space over $C_2 \setminus \mathbb{O}_2$.

Result 3.2. *Let $\mathcal{M}_D$ be BC-Orlicz function then*

$$[b_0^*, \mathcal{M}_D] \subset [b_1^*, \mathcal{M}_D] \subset [b_\infty^*, \mathcal{M}_D].$$

Theorem 3.3. *The spaces $[b_0^*, \mathcal{M}_D]$ and $[b_\infty^*, \mathcal{M}_D]$ are solid.*

Proof. Let $\xi = (\xi_k) \in [b_\infty^*, \mathcal{M}_D]$, then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathcal{M}_D\left(\frac{|\xi_k|_D}{\alpha}\right) <' \infty.$$

Let us consider a sequence of bi-complex scalars (ζ_k) with $|\zeta_k|_D \leq' 1$.
Now

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathcal{M}_D\left(\frac{|\zeta_k \xi_k|_D}{\alpha}\right)$$
$$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathcal{M}_D\left(\frac{|\zeta_k|_D |\xi_k|_D}{\alpha}\right)$$
$$<' \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathcal{M}_D\left(\frac{|\xi_k|_D}{\alpha}\right) <' \infty.$$

Hence, $[b^*_\infty, \mathcal{M}_D]$ is solid.
Similarly other cases can be proved.

Result 3.4. *The spaces $[b^*_1, \mathcal{M}_D]$, $[b^*_0, \mathcal{M}_D]$ and $[b^*_\infty, \mathcal{M}_D]$ are not convergence free.*

Theorem 3.5. *Let $\mathcal{M}^1_D$ and $\mathcal{M}^2_D$ be two BC-Orlicz functions with Δ^2_D-condition, then*

$$[b^*_p, \mathcal{M}^1_D] \cup [b^*_p, \mathcal{M}^2_D] \subseteq [b^*_p, \mathcal{M}^1_D + \mathcal{M}^2_D],$$

where $p = o, 1, \infty$.

Theorem 3.6. *Let $\mathcal{M}^1_D$ and $\mathcal{M}^2_D$-be two BC-Orlicz functions with Δ^2_D-condition, then*

$$[b^*_\infty, \mathcal{M}^2_D] \subset [b^*_\infty, \mathcal{M}^1_D * \mathcal{M}^2_D].$$

Proof. Let $\xi \in [b^*_\infty, \mathcal{M}^2_D]$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathcal{M}^2_D\left(\frac{|\xi_k|_D}{\alpha}\right) <' \infty.$$

Let

$$p = \mathcal{M}^2_D\left(\frac{|\xi_k|_D}{\alpha}\right).$$

Since $\mathcal{M}^1_D$ satisfies Δ^2_D-condition, so there exist $K \geq' 0$ and ξ_0(depending upon K) such that

$$\mathcal{M}^1_D(p) \leq' Kp\mathcal{M}^1_D(2e_1 + 2e_2), \forall\, 0 \leq' \xi \leq' \xi_0.$$

Now,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (\mathcal{M}^1_D * \mathcal{M}^2_D)\left(\frac{|\xi_k|_D}{\alpha}\right)$$
$$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathcal{M}^1_D\left(\mathcal{M}^2_D\left(\frac{|\xi_k|_D}{\alpha}\right)\right)$$
$$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathcal{M}^1_D\,(p)$$
$$\leq' \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} KpM^1_D(2e_1 + 2e_2)$$
$$\leq' \infty.$$

Thus, $\xi \in [b^*_\infty, \mathcal{M}^1_D * \mathcal{M}^2_D]$.
Hence, the theorem.

Theorem 3.7. *Let $\mathcal{M}_D$ be any BC-Orlicz function, the space $[b^*_\infty, \mathcal{M}^2_D]$ is a D-norm space with*

$$g(\xi) = \inf\left\{\alpha : \sum_{k=1}^{n}\left[\mathcal{M}_D\left(\frac{|\xi_k|_D}{\alpha}\right)\right] \leq' 1, \text{ for some hyperbolic number } \alpha >' 0\right\}.$$

Proof. Since $\alpha >' 0$, so $g(\xi) >' 0$ and $g(-\xi) = g(\xi), \forall \xi \in [b^*_\infty, \mathcal{M}^2_D]$.
Let $\xi, \eta \in [b^*_\infty, \mathcal{M}^2_D]$, then for some hyperbolic numbers $\alpha_1, \alpha_2 >' 0$ such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\mathcal{M}_D\left(\frac{|\xi_k|_D}{\alpha_1}\right) <' \infty$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\mathcal{M}_D\left(\frac{|\eta_k|_D}{\alpha_2}\right) <' \infty.$$

Let

$$S = \left\{\alpha : \sum_{k=1}^{n}\left[\mathcal{M}_D\left(\frac{|\xi_k + \eta_k|_D}{\alpha}\right)\right] \leq' 1\right\},$$
$$S_1 = \left\{\alpha_1 : \sum_{k=1}^{n}\left[\mathcal{M}_D\left(\frac{|\xi_k + \eta_k|_D}{\alpha_1}\right)\right] \leq' 1\right\},$$
$$S_2 = \left\{\alpha_2 : \sum_{k=1}^{n}\left[\mathcal{M}_D\left(\frac{|\xi_k + \eta_k|_D}{\alpha_2}\right)\right] \leq' 1\right\}.$$

Let $\alpha = (\alpha_1 + \alpha_2) \in S, \alpha_1 = v'_1 e_1 + v'_2 e_2 \in S_1, \alpha_2 = v''_1 e_1 + v''_2 e_2 \in S_2$ and $\alpha = v_1 e_1 + v_2 e_2$.
Now,

$$\begin{aligned}
g(\xi + \eta) &= \inf\left\{\alpha : \sum_{k=1}^{n}\left[\mathcal{M}_D\left(\frac{|\xi_k + \eta_k|_D}{\alpha}\right)\right] \leq' 1\right\}\\
&= \inf\{v_1 : \alpha \in S\}e_1 + \inf\{v_2 : \alpha \in S\}e_2\\
&= \inf\{v'_1 : \alpha_1 \in S_1\}e_1 + \inf\{v''_1 : \alpha_2 \in S_2\}e_1 + \inf\{v'_2 : \alpha_1 \in S_1\}e_2 + \inf\{v''_2 : \alpha_2 \in S_2\}e_2\\
&= \inf\{v'_1 : \alpha_1 \in S_1\}e_1 + \inf\{v'_2 : \alpha_1 \in S_1\}e_2 + \inf\{v''_1 : \alpha_2 \in S_2\}e_1 + \inf\{v''_2 : \alpha_2 \in S_2\}e_2\\
&= \inf\left\{\alpha_1 : \sum_{k=1}^{n}\left[\mathcal{M}_D\left(\frac{|\xi_k + \eta_k|_D}{\alpha_1}\right)\right] \leq' 1\right\} + \inf\left\{\alpha_2 : \sum_{k=1}^{n}\left[\mathcal{M}_D\left(\frac{|\xi_k + \eta_k|_D}{\alpha_2}\right)\right] \leq' 1\right\}\\
&= g(\xi) + g(\eta).
\end{aligned}$$

Hence, the theorem.

Conclusion. In this article, we have introduced the notion of Cesàro convergence of sequences of bi-complex numbers defined by BC-Orlicz function. We have investigated its different algebraic and topological properties. There are very few articles on sequences of bi-complex numbers.

Acknowledgement The authors thank the reviwer for the comments and suggestions on the article, those improved the presentation of the article.

Declarations

Funding. Not Applicable.

Author contribution. Both the authors have equal contribution in the preparation of this article.

Availability of data and material. Not Applicable.

Code availability. Not Applicable.

Conflicts of interest/Competing interests. We declare that the article is free from Conflicts of interest and Competing interests.

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