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Existence and Hyers-Ulam stability for boundary value problems of multi-term Caputo fractional differential equations

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Abstract. The present paper is devoted to discussing a class of nonlinear Caputo-type fractional differential equations with two-point type boundary value conditions. We investigate the existence and uniqueness of the solutions by virtue of the classical Schauder alternative principle and the Banach contraction principle. Furthermore, by means of a novel Gronwall-type inequality, we prove the Hyers-Ulam stability of boundary value problems of multi-term Caputo fractional differential equations. Finally, some numerical examples are given to illustrate the results.

1. Introduction

The study of differential equations was mainly in the field of mathematics in the last century. However, in the past few decades, fractional differential equations have been increasingly used to describe the mechanical system, thermal system, control system, rheology, materials, optical, signal processing and other areas. In addition, a variety of fractional derivatives have been developed by scholars to study mathematical models abstracted from the actual situation. For more information and results, we refer to [1–10].

If an equation contains more than one fractional differential term, it can be named a multi-term fractional differential equation. It is universally acknowledged that this type of equation can play a vital role in solving some practical problems. However, the investigations on the application of multi-term fractional differential equations are still relatively scarce (see, e.g., [11–13]). A well-known model is the Bagley-Torvik (B-T) equation, which was formulated by Bagley and Torvik in [14]

$$\lambda_1 y''(t) + \lambda_2 {}_cD^{3/2} y(t) + \lambda_3 y(t) = f(t),$$

where λ_i $(i = 1, 2, 3)$ are given real numbers and f from $[0, 1]$ into $\mathbb{R}$ is a given function with certain constraints. According to further research in [15], the B-T equation has been extensively developed and applied. In [16, 17], the authors discussed the existence of solutions for a class of two-term Caputo type fractional differential equations satisfying a initial value condition or certain boundary value conditions. And this equation can be regarded as a generalized form of the B-T equation.

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The research on Hyers-Ulam stability has developed rapidly in the past thirty years. Initially, this concept of stability was proposed by Hyers [18] and Ulam [19] in the last century. Roughly speaking, if a system is Hyers-Ulam stable, it means that an exact solution of the equation can be found around the approximate solution of the equation. So far, the research and discussion on the Hyers-Ulam stability have existed in the fields of algebra, functions, differentiation, integration, equations, and other areas. In addition, Hyers-Ulam stability is dedicated to studying the entire system of the equation, which is quite different from studying the stability of traditional solutions. We suggest the readers consult [20, 21] and their references. In [22], the authors investigated the existence of solutions for a class of differential equations with mixed Riemann-Liouville type fractional integral and derivative boundary conditions. More importantly, a proof of the Heyers-Ulam stability for differential equations with two-point type boundary value conditions was provided by the authors.

Inspired by the above content, a class of multi-term nonlinear fractional differential equations involving Caputo derivative operators are taken into consideration as follows.

$$\begin{cases} {}_cD^{\gamma_1}\varphi(t) - \xi_c D^{\gamma_2}\varphi(t) + f\big(t,\varphi(t)\big) = 0, \quad 0 < t < 1, \\ \varphi(0) + \varphi(1) = \varphi_0, \end{cases} \tag{1}$$

where ξ and φ_0 are given constants. The operators ${}_cD^{\gamma_1}$ and ${}_cD^{\gamma_2}$ represent the Caputo fractional derivatives. In particular, the coefficients of the operators satisfy $0 < \gamma_2 < \gamma_1 \leqslant 1$, and f from $[0,1]$ into $\mathbb{R}$ is a given function with certain constraints. The differential equation (1) with the fractional derivatives of order γ_1 and γ_2 is a kind of generalization of the B-T equation. As a consequence, some new existence conclusions of the equation (1) can be viewed as a partial extension of the B-T equation in [23, 24]. We discuss the existence and uniqueness of the solutions for the boundary value problems (BVP) of fractional differential equations (1). Additionally, we study the Hyers-Ulam stability of the fractional differential equation (1) with the boundary value condition $\varphi(0) + \varphi(1) = \varphi_0$. The Hyers-Ulam stability of differential equations with initial value conditions is easy to prove. However, the situation is quite different when the restriction conditions are changed from the initial value problems to the boundary value problems. Due to the limitation of boundary value conditions, the research of the Hyers-Ulam stability of equation (1) becomes more complicated. Generally, we make use of the Laplace transform and the classical Gronwall inequality to verify the Hyers-Ulam stability of the equation system. But, it is difficult to give concise proof of the Hyers-Ulam stability at the point $t = 1$ in the present paper. Therefore, the existing Gronwall inequality is not sufficient to solve this kind of boundary value problem, which means that we need to find a suitable method to solve this type of problem. To solve the Hyers-Ulam stability of BVP (1), we construct a novel integral-type Gronwall inequality, which can be considered as a generalization of the results in [25].

The rest of the paper is organized as follows. We are going to present some important definitions and lemmas in section 2. Then, the uniqueness and existence conclusions of BVP (1) will be given in section 3. In section 4, we discuss the Hyers-Ulam stability of the fractional differential equations under the boundary condition $\varphi(0) + \varphi(1) = \varphi_0$. Finally, two numerical examples will be listed to understand the existence and Hyers-Ulam stability of the fractional differential equation (1) in section 5.

2. Preliminaries and lemmas

Some basic lemmas and definitions which are used throughout the article will be presented in this section. Let $C([a,b],\mathbb{R})$ be a Banach space of all continuous functions $\varphi : [a,b] \to \mathbb{R}$ with the norm $\|\varphi\| = \max_{a\leqslant t\leqslant b} |\varphi(t)|$. Additionally, $\Gamma(\cdot)$ denotes the gamma function.

Definition 2.1. *[5] Let $\gamma_1 \in (0,+\infty)$ and $h : [a,b] \to \mathbb{R}$. The integral $J_a^{\gamma_1}$ is defined as*

$$J_a^{\gamma_1} h(t) = \frac{1}{\Gamma(\gamma_1)} \int_a^t (t-s)^{\gamma_1 - 1} h(s) ds, \quad t \in [a,b].$$

Definition 2.2. *[5] Let $\gamma_1 \in (0,+\infty)$ and $h:[a,b]\to\mathbb{R}$. The derivative $D_a^{\gamma_1}$ is defined as*

$$D_a^{\gamma_1}h(t) = D_a^m J_a^{m-\gamma_1}h(t) = \frac{1}{\Gamma(m-\gamma_1)}\frac{d^m}{dt^m}\int_a^t (t-s)^{m-\gamma_1-1}h(s)ds, \quad t\in[a,b],$$

where $m\in\mathbb{N}_+$ and $\gamma_1\in(m-1,m]$.

Definition 2.3. *[5] Let $\gamma_1 \in (0,+\infty)$ and $h:[a,b]\to\mathbb{R}$. The derivative ${}_cD_a^{\gamma_1}$ is defined as*

$${}_cD_a^{\gamma_1}h(t) = \frac{1}{\Gamma(m-\gamma_1)}\int_a^t (t-s)^{m-\gamma_1-1}h^{(m)}(s)ds, \quad t\in[a,b],$$

where $m\in\mathbb{N}_+$ and $\gamma_1\in(m-1,m]$. By the way, ${}_cD_a^{\gamma_1}$ is the Caputo fractional differential operator.

Lemma 2.4. *[5] Let $\gamma_1>0$ be a fixed constant. Then the equation ${}_cD^{\gamma_1}\varphi(t)=0$ has a general solution φ, and it can be expressed as*

$$\varphi(t)=\sum_{i=0}^{m-1}c_it^i,\ c_i\in\mathbb{R},(i=1,2,...,m-1),\ m=[\gamma_1]+1.$$

Additionally, if further assume that $\varphi\in C^m([0,b];\mathbb{R})$, then we can get

$$J^{\gamma_1}{}_cD^{\gamma_1}\varphi(t)=\varphi(t)+\sum_{i=0}^{m-1}c_it^i,\ c_i\in\mathbb{R},(i=1,2,...,m-1),\ m=[\gamma_1]+1.$$

Lemma 2.5. *Given a function $g\in C([0,1],\mathbb{R})$ and a constant $\xi\neq 2\Gamma(\gamma_1-\gamma_2+1)$, the solution φ of the linear differential equation*

$$\begin{cases} {}_cD^{\gamma_1}\varphi(t)-\xi_cD^{\gamma_2}\varphi(t)+g(t)=0, \\ \varphi(0)+\varphi(1)=\varphi_0, \end{cases} \tag{2}$$

satisfies

$$\varphi(t)=\theta(t)+\int_0^1 H_1(t,s)\varphi(s)ds-\int_0^1 H_2(t,s)g(s)ds,$$

where

$$\theta(t)=\left(\frac{\xi t^{\gamma_1-\gamma_2}-\Gamma(\gamma_1-\gamma_2+1)}{\xi-2\Gamma(\gamma_1-\gamma_2+1)}\right)\varphi_0,$$

$$H_1(t,s)=\frac{\xi}{\Gamma(\gamma_1-\gamma_2)}\begin{cases} \frac{\xi t^{\gamma_1-\gamma_2}-\Gamma(\gamma_1-\gamma_2+1)}{2\Gamma(\gamma_1-\gamma_2+1)-\xi}(1-s)^{\gamma_1-\gamma_2-1}+(t-s)^{\gamma_1-\gamma_2-1}, & 0\leqslant s\leqslant t\leqslant 1, \\ \frac{\xi t^{\gamma_1-\gamma_2}-\Gamma(\gamma_1-\gamma_2+1)}{2\Gamma(\gamma_1-\gamma_2+1)-\xi}(1-s)^{\gamma_1-\gamma_2-1}, & 0\leqslant t\leqslant s\leqslant 1, \end{cases}$$

and

$$H_2(t,s)=\frac{1}{\Gamma(\gamma_1)}\begin{cases} \frac{\xi t^{\gamma_1-\gamma_2}-\Gamma(\gamma_1-\gamma_2+1)}{2\Gamma(\gamma_1-\gamma_2+1)-\xi}(1-s)^{\gamma_1-1}+(t-s)^{\gamma_1-1}, & 0\leqslant s\leqslant t\leqslant 1, \\ \frac{\xi t^{\gamma_1-\gamma_2}-\Gamma(\gamma_1-\gamma_2+1)}{2\Gamma(\gamma_1-\gamma_2+1)-\xi}(1-s)^{\gamma_1-1}, & 0\leqslant t\leqslant s\leqslant 1. \end{cases}$$

Proof. Since $0<\gamma_2<\gamma_1\leqslant 1$, according to Lemma 2.4, we have

$$J^{\gamma_1}{}_cD^{\gamma_1}\varphi(t)=\varphi(t)+c_0,$$

where $c_0\in\mathbb{R}$ is a constant. Integrating the left and right sides of the differential equation (1) by the operator J^{γ_1}, we can get

$$J^{\gamma_1}{}_cD^{\gamma_1}\varphi(t)=\xi J^{\gamma_1}{}_cD^{\gamma_2}\varphi(t)-J^{\gamma_1}g(t), \quad t\in[0,1]. \tag{3}$$

Since

$$J^{\gamma_1}{}_cD^{\gamma_2}\varphi(t) = J^{\gamma_1-\gamma_2}\Big(J^{\gamma_2}{}_cD^{\gamma_2}\varphi(t)\Big) = J^{\gamma_1-\gamma_2}\Big(\varphi(t)+c_0\Big) = J^{\gamma_1-\gamma_2}\varphi(t) + \frac{t^{\gamma_1-\gamma_2}}{\Gamma(\gamma_1-\gamma_2+1)}c_0,$$

we can obtain

$$\varphi(t) = -c_0 + \frac{\xi t^{\gamma_1-\gamma_2}}{\Gamma(\gamma_1-\gamma_2+1)}c_0 + \xi J^{\gamma_1-\gamma_2}\varphi(t) - J^{\gamma_1}g(t). \tag{4}$$

So we have

$$\begin{cases} \varphi(0) = -c_0, \\ \varphi(1) = -c_0 + \frac{\xi}{\Gamma(\gamma_1-\gamma_2+1)}c_0 + \xi J^{\gamma_1-\gamma_2}\varphi(1) - J^{\gamma_1}g(1). \end{cases}$$

Combined with the boundary conditions, we get

$$c_0 = \frac{\Gamma(\gamma_1-\gamma_2+1)\Big(\varphi_0 - \xi J^{\gamma_1-\gamma_2}\varphi(1) + J^{\gamma_1}g(1)\Big)}{\xi - 2\Gamma(\gamma_1-\gamma_2+1)}.$$

Substituting the value of c_0 into the equation (2), we have

$$\begin{aligned}
\varphi(t) &= \frac{\Big(\xi t^{\gamma_1-\gamma_2} - \Gamma(\gamma_1-\gamma_2+1)\Big)\Big(\varphi_0 - \xi J^{\gamma_1-\gamma_2}\varphi(1) + J^{\gamma_1}\varphi(1)\Big)}{\xi - 2\Gamma(\gamma_1-\gamma_2+1)} + \xi J^{\gamma_1-\gamma_2}\varphi(t) - J^{\gamma_1}g(t) \\
&= \left(\frac{\xi t^{\gamma_1-\gamma_2} - \Gamma(\gamma_1-\gamma_2+1)}{\xi - 2\Gamma(\gamma_1-\gamma_2+1)}\right)\varphi_0 - \frac{\Big(\xi t^{\gamma_1-\gamma_2} - \Gamma(\gamma_1-\gamma_2+1)\Big)\xi}{(\xi - 2\Gamma(\gamma_1-\gamma_2+1))\,\Gamma(\gamma_1-\gamma_2)}\int_0^1 (1-s)^{\gamma_1-\gamma_2-1}\varphi(s)ds \\
&\quad + \frac{\Big(\xi t^{\gamma_1-\gamma_2} - \Gamma(\gamma_1-\gamma_2+1)\Big)}{\xi - 2\Gamma(\gamma_1-\gamma_2+1)}\frac{1}{\Gamma(\gamma_1)}\int_0^1 (1-s)^{\gamma_1-1}g(s)ds \\
&\quad + \frac{\xi}{\Gamma(\gamma_1-\gamma_2)}\int_0^t (t-s)^{\gamma_1-\gamma_2-1}\varphi(s)ds - \frac{1}{\Gamma(\gamma_1)}\int_0^t (t-s)^{\gamma_1-1}g(s)ds \\
&= \theta(t) + \int_0^1 H_1(t,s)\varphi(s)ds - \int_0^1 H_2(t,s)g(s)ds.
\end{aligned}$$

The lemma is thus proved. □

Lemma 2.6. *[25] Given continuous functions $\sigma_1(t)$, $\sigma_2(t)$ and a differentiable function $\sigma_3(t)$, further suppose*

$$\begin{cases} \sigma_3'(t) \leqslant \sigma_1(t)\sigma_3(t) + \sigma_2(t), \quad t \geqslant a, \\ \sigma_3(a) \leqslant \sigma_0. \end{cases} \tag{5}$$

Then the assertion given below

$$\sigma_3(t) \leqslant \sigma_0 \exp\left(\int_a^t \sigma_1(s)ds\right) + \int_a^t \sigma_2(s)\exp\left(\int_a^t \sigma_1(\tau)d\tau\right)ds \tag{6}$$

holds, where σ_0 is a constant and $s \in [a, +\infty)$.

Based on the inequality introduced above and the model of differential equations in this paper, we are going to construct and prove a new Gronwall-type inequality that can be used to study a class of integro-differential equations with boundary conditions.

Lemma 2.7. *Let $\nu(t)$ be a nonnegative function on $[a, T]$. Suppose that the inequality given below*

$$\nu(t) \leqslant \zeta_1 + \zeta_2 \int_a^t \nu(s)ds + \zeta_3 \int_a^T \nu(s)ds, \quad \zeta_i \in \mathbb{R} \ (i = 1, 2, 3), \tag{7}$$

holds, and

$$q_0 = \frac{\zeta_3}{\zeta_2}\Big(\exp\Big(\zeta_2(T-a)\Big) - 1\Big) < 1.$$

Then, we have the following assertion

$$\nu(t) \leqslant \frac{\zeta_1}{1-q_0}\exp\Big(\zeta_2(T-a)\Big).$$

Proof. Let $\mu(t) = \zeta_1 + \zeta_2 \int_a^t \nu(s)ds + \zeta_3 \int_a^T \nu(s)ds$. Then we obtain

$$\nu(t) \leqslant \mu(t), \tag{8}$$

and

$$\mu(a) = \zeta_1 + \zeta_3 \int_a^T \nu(s)ds. \tag{9}$$

Hence, we obtain $\mu'(t) = \zeta_2\nu(t) \leqslant \zeta_2\mu(t)$. By Lemma 2.6,

$$\mu(t) \leqslant \mu(a)\exp\left(\int_a^t \zeta_2 ds\right) = \mu(a)\exp\Big(\zeta_2(t-a)\Big). \tag{10}$$

Substituting (10) into (9) yields

$$\mu(a) = \zeta_1 + \zeta_3 \int_a^T \nu(s)ds \leqslant \zeta_1 + \zeta_3 \int_a^T \mu(a)\exp\Big(\zeta_2(s-a)\Big)ds \leqslant \zeta_1 + \mu(a)\frac{\zeta_3}{\zeta_2}\Big(\exp\Big(\zeta_2(T-a)\Big) - 1\Big).$$

Hence,

$$\mu(a)\left(1 - \frac{\zeta_3}{\zeta_2}\Big(\exp\Big(\zeta_2(T-a)\Big) - 1\Big)\right) \leqslant \zeta_1.$$

It follows that

$$\mu(a) \leqslant \frac{\zeta_1}{1 - \frac{\zeta_3}{\zeta_2}\Big(\exp\Big(\zeta_2(T-a)\Big) - 1\Big)} = \frac{\zeta_1}{1-q_0},$$

where $q_0 = \frac{\zeta_3}{\zeta_2}\Big(\exp\Big(\zeta_2(T-a)\Big) - 1\Big)$. Substituting (10) into (9) yields

$$\nu(t) \leqslant \frac{\zeta_1}{1-q_0}\exp\Big(\zeta_2(T-a)\Big).$$

The lemma is thus proved. □

3. Existence results

The uniqueness and existence of the BVP (1) are discussed in this section. To begin with, an important definition is given as follows.

Definition 3.1. *If φ satisfies the equation*

$$\varphi(t) = \theta(t) + \int_0^1 H_1(t,s)\varphi(s)ds - \int_0^1 H_2(t,s)f\Big(s,\varphi(s)\Big)ds, \quad t \in [0,1],$$

then φ is the solution of BVP (1).

For convenience, we first list the hypotheses.

(H1) Function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H2) There exists a fixed number $k > 0$ such that

$$|f(t,\varphi_1) - f(t,\varphi_2)| \leqslant k\left|\varphi_1 - \varphi_2\right|, \quad t \in [0,1],$$

for all $\varphi_1, \varphi_2 \in \mathbb{R}$.

(H3) There exist nonnegative and continuous functions $k(t)$ and $l(t)$ such that

$$|f(t,\varphi)| \leqslant k(t) + l(t)\left|\varphi\right|,$$

for each $(t,\varphi) \in [0,1] \times \mathbb{R}$.

(H4) There exist continuous functions $\psi \in C([0,1],\mathbb{R}^+)$ and nondecreasing functions $\omega : \mathbb{R} \to \mathbb{R}^+$ such that

$$|f(t,\varphi)| \leqslant \psi(t)\omega(|\varphi|),$$

for each $(t,\varphi) \in [0,1] \times \mathbb{R}$.

Theorem 3.2. *Suppose that conditions (H1) and (H2) hold. If further assume that*

$$M_1 + kM_2 < 1, \quad t \in [0,1], \tag{11}$$

then the BVP (1) *has a unique solution.*

Proof. Define an operator $\mathcal{P}$ on $C([0,1],\mathbb{R})$ by

$$\mathcal{P}\varphi(t) = \theta(t) + \int_0^1 H_1(t,s)\varphi(s)ds - \int_0^1 H_2(t,s)f\Big(s,\varphi(s)\Big)ds.$$

for $\varphi \in C([0,1],\mathbb{R})$. Since f and φ are continuous, and H_1 and H_2 are integrable, it is easy to prove that $\mathcal{P}\varphi$ is continuous on $[0,1]$, i.e., $\mathcal{P}$ maps $C([0,1],\mathbb{R})$ into itself. According to the definition of the operator $\mathcal{P}$, if there exists a unique fixed point $\varphi \in \mathcal{P}$, then φ is the solution of BVP (1). Taking any $\varphi_1, \varphi_2 \in C([0,1],\mathbb{R})$, then the condition (H2) implies

$$\begin{aligned} |\mathcal{P}\varphi_1(t) - \mathcal{P}\varphi_2(t)| &\leqslant \int_0^1 |H_1(t,s)|\,|\varphi_1(s) - \varphi_2(s)|ds + \int_0^1 |H_2(t,s)|\,|f\Big(s,\varphi_1(s)\Big) - f\Big(s,\varphi_2(s)\Big)|ds \\ &\leqslant \left\|\varphi_1 - \varphi_2\right\| \int_0^1 |H_1(t,s)|ds + k\left\|\varphi_1 - \varphi_2\right\| \int_0^1 |H_2(t,s)|\,ds \\ &\leqslant (M_1 + kM_2)\left\|\varphi_1 - \varphi_2\right\|. \end{aligned}$$

Hence,

$$\|\mathcal{P}\varphi_1 - \mathcal{P}\varphi_2\| \leqslant (M_1 + kM_2)\left\|\varphi_1 - \varphi_2\right\|.$$

According to condition (11), $\mathcal{P}$ is a mapping of contraction. By the principle of Banach contraction, we can get the conclusions immediately. The theorem is thus proved. □

Theorem 3.3. *Suppose that conditions (H1) and (H3) hold. If further assume that*

$$M_1 + M_2 \|l\| < 1, \quad t \in [0,1], \tag{12}$$

then there exists at least one solution of BVP (1).

Proof. This theorem can be verified by the following three steps.

Step 1 According to the definition of the operator $\mathcal{P}$ and the well-known Lebesgue dominated convergence theorem, $\mathcal{P}$ is a continuous operator. The detailed proof will not be listed here.

Step 2 Next, we need to verify that $\mathcal{P}$ is a compact operator. Taking any bounded subset $\mathcal{H} \subseteq C([0,1],\mathbb{R})$, then there exists a constant $\eta > 0$ satisfying that $\mathcal{H} \subseteq T_\eta = \{\varphi \in C([0,1],\mathbb{R}) : \|\varphi\| \leqslant \eta\}$. Obviously, the subset T_η is convex, bounded and closed in $C([0,1],\mathbb{R})$. So we get

$$\begin{aligned}|\mathcal{P}\varphi(t)| &\leqslant |\theta(t)| + \int_0^1 |H_1(t,s)|\,|\varphi(s)|ds + \int_0^1 |H_2(t,s)|\,|f\big(s,\varphi(s)\big)|ds \\ &\leqslant \eta M_1 + M_2\left(\|k\| + \eta\,\|l\|\right) + M_3,\end{aligned}$$

which implies that $\|\mathcal{P}\varphi\| \leqslant \eta M_1 + M_2\left(\|k\| + \eta\,\|l\|\right) + M_3$. This indicates that $\mathcal{P}T_\eta$ is uniformly bounded. Given any $0 \leqslant t_1 < t_2 \leqslant 1$ and any $u \in T_\eta$, then

$$\begin{aligned}|\mathcal{P}\varphi(t_2) - \mathcal{P}\varphi(t_1)| &\leqslant |\theta(t_2) - \theta(t_1)| + \left|\int_0^1 \big(H_1(t_2,s) - H_1(t_1,s)\big)\varphi(s)ds\right| + \left|\int_0^1 \big(H_2(t_2,s) - H_2(t_1,s)\big)f\big(s,\varphi(s)\big)ds\right| \\ &\leqslant |\theta(t_2) - \theta(t_1)| + \frac{\xi^2\big(t_2^{\gamma_1-\gamma_2} - t_1^{\gamma_1-\gamma_2}\big)}{\Gamma(\gamma_1-\gamma_2)|2\Gamma(\gamma_1-\gamma_2+1) - \xi|}\int_0^1 (1-s)^{\gamma_1-\gamma_2-1}|\varphi(s)|ds \\ &\quad + \frac{|\xi|}{\Gamma(\gamma_1-\gamma_2)}\left|\int_0^{t_2}(t_2-s)^{\gamma_1-\gamma_2-1}\varphi(s)ds - \int_0^{t_1}(t_1-s)^{\gamma_1-\gamma_2-1}\varphi(s)ds\right| \\ &\quad + \frac{|\xi|\big(t_2^{\gamma_1-\gamma_2} - t_1^{\gamma_1-\gamma_2}\big)}{\Gamma(\gamma_1)|2\Gamma(\gamma_1-\gamma_2+1) - \xi|}\int_0^1 (1-s)^{\gamma_1-1}|f\big(s,\varphi(s)\big)|ds \\ &\quad + \frac{1}{\Gamma(\gamma_1)}\left|\int_0^{t_2}(t_2-s)^{\gamma_1-1}f\big(s,\varphi(s)\big)ds - \int_0^{t_1}(t_1-s)^{\gamma_1-1}f\big(s,\varphi(s)\big)ds\right| \\ &:= I_1 + I_2 + I_3 + I_4 + I_5,\end{aligned}$$

where

$$\begin{aligned}I_1 &= |\theta(t_2) - \theta(t_1)|, \\ I_2 &= \frac{\xi^2\big(t_2^{\gamma_1-\gamma_2} - t_1^{\gamma_1-\gamma_2}\big)}{\Gamma(\gamma_1-\gamma_2)|2\Gamma(\gamma_1-\gamma_2+1) - \xi|}\int_0^1 (1-s)^{\gamma_1-\gamma_2-1}|\varphi(s)|ds, \\ I_3 &= \frac{|\xi|}{\Gamma(\gamma_1-\gamma_2)}\left|\int_0^{t_2}(t_2-s)^{\gamma_1-\gamma_2-1}\varphi(s)ds - \int_0^{t_1}(t_1-s)^{\gamma_1-\gamma_2-1}\varphi(s)ds\right|, \\ I_4 &= \frac{|\xi|\big(t_2^{\gamma_1-\gamma_2} - t_1^{\gamma_1-\gamma_2}\big)}{\Gamma(\gamma_1)|2\Gamma(\gamma_1-\gamma_2+1) - \xi|}\int_0^1 (1-s)^{\gamma_1-1}|f\big(s,\varphi(s)\big)|ds, \\ I_5 &= \frac{1}{\Gamma(\gamma_1)}\left|\int_0^{t_2}(t_2-s)^{\gamma_1-1}f\big(s,\varphi(s)\big)ds - \int_0^{t_1}(t_1-s)^{\gamma_1-1}f\big(s,\varphi(s)\big)ds\right|.\end{aligned}$$

On the one hand, thanks to θ is a polynomial function, it can be verified that

$$\lim_{t_2-t_1\to 0} I_1 = \lim_{t_2-t_1\to 0}|\theta(t_2) - \theta(t_1)| = 0.$$

On the other hand, according to the conditions (H1) and (H3), we have

$$I_2 \leqslant \frac{\xi^2(t_2^{\gamma_1-\gamma_2}-t_1^{\gamma_1-\gamma_2})}{\Gamma(\gamma_1-\gamma_2)|2\Gamma(\gamma_1-\gamma_2+1)-\xi|}\left\|\varphi\right\|\int_0^1(1-s)^{\gamma_1-\gamma_2-1}ds$$
$$\leqslant \frac{\eta\xi^2}{\Gamma(\gamma_1-\gamma_2+1)|2\Gamma(\gamma_1-\gamma_2+1)-\xi|}(t_2^{\gamma_1-\gamma_2}-t_1^{\gamma_1-\gamma_2}).$$

With the same conditions, we also have

$$I_3 \leqslant \frac{|\xi|}{\Gamma(\gamma_1-\gamma_2)}\int_0^{t_1}\left|(t_2-s)^{\gamma_1-\gamma_2-1}-(t_1-s)^{\gamma_1-\gamma_2-1}\right||\varphi(s)|ds+\frac{|\xi|}{\Gamma(\gamma_1-\gamma_2)}\int_{t_1}^{t_2}\left|(t_2-s)^{\gamma_1-\gamma_2-1}\right||\varphi(s)|ds$$
$$\leqslant \frac{|\xi|\left\|\varphi\right\|}{\Gamma(\gamma_1-\gamma_2)}\int_0^{t_1}\left|(t_2-s)^{\gamma_1-\gamma_2-1}-(t_1-s)^{\gamma_1-\gamma_2-1}\right|ds+\frac{|\xi|\left\|\varphi\right\|}{\Gamma(\gamma_1-\gamma_2)}\int_{t_1}^{t_2}\left|(t_2-s)^{\gamma_1-\gamma_2-1}\right|ds$$
$$\leqslant \frac{\eta|\xi|}{\Gamma(\gamma_1-\gamma_2+1)}\left(t_2^{\gamma_1-\gamma_2}-t_1^{\gamma_1-\gamma_2}+2(t_2-t_1)^{\gamma_1-\gamma_2}\right).$$

Similarly, we can draw the bounds of the terms I_4 and I_5.

$$I_4 \leqslant \frac{|\xi|(t_2^{\gamma_1-\gamma_2}-t_1^{\gamma_1-\gamma_2})}{\Gamma(\gamma_1)|2\Gamma(\gamma_1-\gamma_2+1)-\xi|}\int_0^1(1-s)^{\gamma_1-1}\left(|k(s)|+|l(s)|\left\|\varphi\right\|\right)ds$$
$$\leqslant \frac{(\|k\|+\eta\|l\|)|\xi|}{\Gamma(\gamma_1+1)|2\Gamma(\gamma_1-\gamma_2+1)-\xi|}(t_2^{\gamma_1-\gamma_2}-t_1^{\gamma_1-\gamma_2}),$$
$$I_5 \leqslant \frac{(\|k\|+\eta\|l\|)}{\Gamma(\gamma_1)}\int_0^{t_1}\left|(t_2-s)^{\gamma_1-1}-(t_1-s)^{\gamma_1-1}\right|ds+\frac{(\|k\|+\eta\|l\|)}{\Gamma(\gamma_1)}\int_{t_1}^{t_2}\left|(t_2-s)^{\gamma_1-1}\right|ds$$
$$\leqslant \frac{(\|k\|+\eta\|l\|)}{\Gamma(\gamma_1+1)}\left(t_2^{\gamma_1}-t_1^{\gamma_1}+2(t_2-t_1)^{\gamma_1}\right).$$

Obviously, I_i $(i=2,3,4,5)$ tend to 0 as $t_2-t_1\to 0$, and

$$\lim_{t_2-t_1\to 0}|\mathcal{P}\varphi(t_2)-\mathcal{P}\varphi(t_1)|=0,$$

as a consequence, there is no doubt that $\mathcal{P}T_\eta$ is equicontinuous. Combined with the results above, we can obtain that $\mathcal{P}T_\eta$ is compact by applying the Arzela-Ascoli theorem.

Step 3 It has been verified that $\mathcal{P}$ satisfies the complete continuity, then we will complete the proof of the whole theorem by reduction to absurdity. According to the condition $M_1+M_2\|l\|<1$, we can deduce that there exists a fixed constant $N>0$ such that

$$M_1N+M_2\|k\|+M_2\|l\|N+M_3<N.$$

Define the set $\mathcal{E}=\{\varphi\in C([0,1],\mathbb{R}):\left\|\varphi\right\|<N\}$. So, the operator $\mathcal{P}:\overline{\mathcal{E}}\to C([0,1],\mathbb{R})$ satisfies the complete continuity. Assume the equation $\varphi=\lambda\mathcal{P}\varphi$ holds for some $\varphi\in\overline{\mathcal{E}}$ and $\lambda\in(0,1)$. Then we obtain

$$|\varphi(t)|=|\lambda\mathcal{P}\varphi(t)|\leqslant|\mathcal{P}\varphi(t)|$$
$$\leqslant\left|\int_0^1H_1(t,s)\varphi(s)ds\right|+\left|\int_0^1H_2(t,s)f\big(s,\varphi(s)\big)ds\right|+|\theta(t)|$$
$$\leqslant M_1\left\|\varphi\right\|+M_2\left(\|k\|+\|l\|\left\|\varphi\right\|\right)+M_3.$$

Hence,

$$N=\|\varphi\|\leqslant M_1\left\|\varphi\right\|+M_2\|k\|+M_2\|l\|\|\varphi\|+M_3<N,$$

which implies a contradiction. So we get $\varphi\neq\lambda\mathcal{P}\varphi$, for any $u\in\overline{\mathcal{E}}$ and $\lambda\in(0,1)$. By the Leray-Schauder alternative, we infer that there exists at least a fixed point φ in $\mathcal{P}$. On account of that the fixed point φ can also represent the solution of the BVP (1), the proof of the theorem is finished. □

Theorem 3.4. *Suppose that conditions (H1) and (H4) hold. If further assume that*

$$M_1 + M_2 \left\|\psi\right\| \lim_{r\to\infty} sup \frac{\omega(r)}{r} < 1, \quad t \in [0,1], \tag{13}$$

then there exists at least one solution of BVP (1).

Proof. Similar to the proof of Theorem 3.3, it is not difficult to verify that the operator $\mathcal{P}$ is continuous. Additionally, subset T_η is convex, bounded and closed in $C([0,1],\mathbb{R})$. Then we get

$$\begin{aligned}|\mathcal{P}\varphi(t)| &\leqslant |\theta(t)| + \int_0^1 |H_1(t,s)|\,|\varphi(s)|ds + \int_0^1 |H_2(t,s)|\,\big|f\big(s,\varphi(s)\big)\big|ds \\ &\leqslant \eta M_1 + M_2 \left\|\psi\right\| \omega(\eta) + M_3,\end{aligned}$$

which implies $\|\mathcal{P}u\| \leqslant \eta M_1 + M_2 \left\|\varphi\right\| \omega(\eta) + M_3$. This indicates that $\mathcal{P}T_\eta$ is uniformly bounded. Given any $0 \leqslant t_1 < t_2 \leqslant 1$ and $u \in T_\eta$, then

$$\begin{aligned}|\mathcal{P}\varphi(t_2) - \mathcal{P}\varphi(t_1)| &\leqslant |\theta(t_2) - \theta(t_1)| + \left|\int_0^1 \Big(H_1(t_2,s) - H_1(t_1,s)\Big)\varphi(s)ds\right| + \left|\int_0^1 \Big(H_2(t_2,s) - H_2(t_1,s)\Big) f\big(s,\varphi(s)\big)ds\right| \\ &\leqslant |\theta(t_2) - \theta(t_1)| + \frac{\xi^2(t_2^{\gamma_1-\gamma_2} - t_1^{\gamma_1-\gamma_2})}{\Gamma(\gamma_1-\gamma_2)|2\Gamma(\gamma_1-\gamma_2+1) - \xi|} \int_0^1 (1-s)^{\gamma_1-\gamma_2-1}|\varphi(s)|ds \\ &\quad + \frac{|\xi|}{\Gamma(\gamma_1-\gamma_2)} \left|\int_0^{t_2} (t_2-s)^{\gamma_1-\gamma_2-1}\varphi(s)ds - \int_0^{t_1} (t_1-s)^{\gamma_1-\gamma_2-1}\varphi(s)ds\right| \\ &\quad + \frac{|\xi|(t_2^{\gamma_1-\gamma_2} - t_1^{\gamma_1-\gamma_2})}{\Gamma(\gamma_1)|2\Gamma(\gamma_1-\gamma_2+1) - \xi|} \int_0^1 (1-s)^{\gamma_1-1}\big|f\big(s,\varphi(s)\big)\big|ds \\ &\quad + \frac{1}{\Gamma(\gamma_1)} \left|\int_0^{t_2} (t_2-s)^{\gamma_1-1} f\big(s,\varphi(s)\big)ds - \int_0^{t_1} (t_1-s)^{\gamma_1-1} f\big(s,\varphi(s)\big)ds\right| \\ &:= J_1 + J_2 + J_3 + J_4 + J_5.\end{aligned}$$

Owing to that θ is a polynomial function, it can be verified that

$$\lim_{t_2-t_1\to 0} J_1 = \lim_{t_2-t_1\to 0} |\theta(t_2) - \theta(t_1)| = 0.$$

It follows from the conditions (H1)) and (H4) that

$$\begin{aligned}J_2 &\leqslant \frac{\xi^2(t_2^{\gamma_1-\gamma_2} - t_1^{\gamma_1-\gamma_2})}{\Gamma(\gamma_1-\gamma_2)|2\Gamma(\gamma_1-\gamma_2+1) - \xi|} \left\|\varphi\right\| \int_0^1 (1-s)^{\gamma_1-\gamma_2-1}ds \\ &\leqslant \frac{\eta\xi^2}{\Gamma(\gamma_1-\gamma_2+1)|2\Gamma(\gamma_1-\gamma_2+1) - \xi|}(t_2^{\gamma_1-\gamma_2} - t_1^{\gamma_1-\gamma_2}).\end{aligned}$$

As for the term J_3, the conditions indicate that

$$\begin{aligned}J_3 &\leqslant \frac{|\xi|}{\Gamma(\gamma_1-\gamma_2)} \int_0^{t_1} \left|(t_2-s)^{\gamma_1-\gamma_2-1} - (t_1-s)^{\gamma_1-\gamma_2-1}\right| |\varphi(s)|ds + \frac{|\xi|}{\Gamma(\gamma_1-\gamma_2)} \int_{t_1}^{t_2} \left|(t_2-s)^{\gamma_1-\gamma_2-1}\right| |\varphi(s)|ds \\ &\leqslant \frac{|\xi| \left\|\varphi\right\|}{\Gamma(\gamma_1-\gamma_2)} \int_0^{t_1} \left|(t_2-s)^{\gamma_1-\gamma_2-1} - (t_1-s)^{\gamma_1-\gamma_2-1}\right| ds + \frac{|\xi| \left\|\varphi\right\|}{\Gamma(\gamma_1-\gamma_2)} \int_{t_1}^{t_2} \left|(t_2-s)^{\gamma_1-\gamma_2-1}\right| ds \\ &\leqslant \frac{\eta\,|\xi|}{\Gamma(\gamma_1-\gamma_2+1)}\Big(t_2^{\gamma_1-\gamma_2} - t_1^{\gamma_1-\gamma_2} + 2(t_2-t_1)^{\gamma_1-\gamma_2}\Big).\end{aligned}$$

Similar calculations with respect to J_4 and J_5 can be shown as follows.

$$J_4 \leqslant \frac{|\xi|(t_2^{\gamma_1-\gamma_2} - t_1^{\gamma_1-\gamma_2})}{\Gamma(\gamma_1)|2\Gamma(\gamma_1-\gamma_2+1)-\xi|}\int_0^1 (1-s)^{\gamma_1-1}\left\|\psi\right\|\omega(\left\|\varphi\right\|)ds \leqslant \frac{\omega(\eta)\left\|\psi\right\||\xi|}{\Gamma(\gamma_1+1)|2\Gamma(\gamma_1-\gamma_2+1)-\xi|}(t_2^{\gamma_1-\gamma_2} - t_1^{\gamma_1-\gamma_2}),$$

$$\begin{aligned} J_5 &\leqslant \frac{\left\|\psi\right\|\omega(\eta)}{\Gamma(\gamma_1)}\int_0^{t_1}\left|(t_2-s)^{\gamma_1-1} - (t_1-s)^{\gamma_1-1}\right|ds + \frac{\left\|\psi\right\|\omega(\eta)}{\Gamma(\gamma_1)}\int_{t_1}^{t_2}\left|(t_2-s)^{\gamma_1-1}\right|ds \\ &\leqslant \frac{\left\|\psi\right\|\omega(\eta)}{\Gamma(\gamma_1+1)}\Big(t_2^{\gamma_1} - t_1^{\gamma_1} + 2(t_2-t_1)^{\gamma_1}\Big). \end{aligned}$$

Obviously, J_i $(i = 2, 3, 4, 5)$ tend to 0 as $t_2 - t_1 \to 0$ and

$$\lim_{t_2-t_1\to 0} |\mathcal{P}u(t_2) - \mathcal{P}u(t_1)| = 0,$$

as a consequence, there is no doubt that $\mathcal{T}B_\eta$ is equicontinuous. Combined with the results above, we can obtain that $\mathcal{P}T_\eta$ is compact by applying the Arzela-Ascoli theorem.

It has been verified that $\mathcal{P}$ satisfies the complete continuity, then we will complete the proof of the whole theorem by reduction to absurdity. According to the condition

$$M_1 + M_2\left\|\psi\right\|\lim_{r\to\infty} sup\frac{\omega(r)}{r} < 1,$$

we can deduce that there is a fixed number $N > 0$ such that $M_1N + M_2\left\|\psi\right\|\omega(N) + M_3 < N$. Define the set $\mathcal{E} = \{\varphi \in C([0,1],\mathbb{R}) : \left\|\varphi\right\| < N\}$. So the operator $\mathcal{P} : \overline{\mathcal{E}} \to C([0,1],\mathbb{R})$ satisfies the complete continuity. Assume the equation $\varphi = \lambda\mathcal{P}\varphi$ holds for some $\varphi \in \overline{\mathcal{E}}$ and $\lambda \in (0,1)$. Then we obtain

$$\begin{aligned} |\varphi(t)| = |\lambda\mathcal{P}\varphi(t)| &\leqslant |\mathcal{P}\varphi(t)| \\ &\leqslant |\theta(t)| + \left|\int_0^1 H_1(t,s)\varphi(s)ds\right| + \left|\int_0^1 H_2(t,s)f\Big(s,\varphi(s)\Big)ds\right| \\ &\leqslant M_1\left\|\varphi\right\| + M_2\left\|\psi\right\|\omega(\left\|\varphi\right\|) + M_3. \end{aligned}$$

Hence $N = \left\|\varphi\right\| \leqslant M_1\left\|\varphi\right\| + M_2\left\|\psi\right\|\omega(\left\|\varphi\right\|) + M_3 < N$, which implies a contradiction. So, we get $\varphi \neq \lambda\mathcal{P}\varphi$ for any $u \in \overline{\mathcal{E}}$ and $\lambda \in (0,1)$. By the Leray-Schauder alternative, we infer that there exists at least a fixed point φ in $\mathcal{P}$. On account of that the fixed point φ can also represent the solution of the BVP (1), proof of the theorem is finished. □

4. Stability analysis

The Hyers-Ulam stability of the differential equation given below is discussed and proved in this section:

$$\begin{cases} {}_cD^{\gamma_1}\varphi(t) - \xi_c D^{\gamma_2}\varphi(t) + f\Big(t,\varphi(t)\Big) = 0, & t \in [0,1], \\ \varphi(0) + \varphi(1) = \varphi_0. \end{cases} \tag{14}$$

Definition 4.1. *The BVP* (14) *is Hyers-Ulam stable if there exists a fixed positive constant c, such that for any* $\varepsilon > 0$, *and for each function* φ_2 *that satisfies the inequality*

$$\left|{}_cD^{\gamma_1}\varphi_2(t) - \xi_c D^{\gamma_2}\varphi_2(t) + f\Big(t,\varphi_2(t)\Big)\right| \leqslant \varepsilon, \quad t \in [0,1],$$

there exists a solution φ_1 *of the BVP* (14) *with* $|\varphi_2(t) - \varphi_1(t)| \leqslant c\varepsilon$, $t \in [0,1]$, *where* φ_1, φ_2 *are continuous functions.*

Theorem 4.2. *Further assume that $M_1 < 1$, $\xi \neq 2\Gamma(\gamma_1 - \gamma_2 + 1)$ and conditions (H1), (H2) hold. Then the BVP* (14) *is Hyers-Ulam stable.*

Proof. For any $\varepsilon > 0$, and each φ_2 that satisfies the following inequality

$$\left|{}_cD^{\gamma_1}\varphi_2(t) - \xi_cD^{\gamma_2}\varphi_2(t) + f\Big(t, \varphi_2(t)\Big)\right| \leqslant \varepsilon, \quad t \in [0, 1],$$

a function $g(t) ={}_c D^{\gamma_1}\varphi_2(t) - \xi_cD^{\gamma_2}\varphi_2(t) + f\Big(t, \varphi_2(t)\Big)$ can be found, then we get $|g(t)| \leqslant \varepsilon$. It implies

$$\begin{aligned}\varphi_2(t) &= \xi J^{\gamma_1-\gamma_2}\varphi_2(t) - J^{\gamma_1} f\Big(t, \varphi_2(t)\Big) + J^{\gamma_1} g(t)\\ &\quad + \frac{\xi t^{\gamma_1-\gamma_2} - \Gamma(\gamma_1 - \gamma_2 + 1)}{\xi - 2\Gamma(\gamma_1 - \gamma_2 + 1)}\Big(\varphi_0 - \xi J^{\gamma_1-\gamma_2}\varphi_2(1) + J^{\gamma_1} f\Big(1, \varphi_2(1)\Big) - J^{\gamma_1} g(1)\Big).\end{aligned}$$

According to the theorem 3.2, it has been verified that there is a unique solution φ_1 of BVP (14), which satisfies the boundary value condition $\varphi(0) + \varphi(1) = \varphi_0$, then φ_1 can be given by the following integral equation

$$\varphi_1(t) = \xi J^{\gamma_1-\gamma_2}\varphi_1(t) - J^{\gamma_1} f\Big(t, \varphi_1(t)\Big) + \frac{\xi t^{\gamma_1-\gamma_2} - \Gamma(\gamma_1 - \gamma_2 + 1)}{\xi - 2\Gamma(\gamma_1 - \gamma_2 + 1)}\Big(\varphi_0 - \xi J^{\gamma_1-\gamma_2}\varphi_1(1) + J^{\gamma_1} f\Big(1, \varphi_1(1)\Big)\Big).$$

Then

$$\begin{aligned}\Big|\varphi_2(t) - \varphi_1(t)\Big| &\leqslant |\xi|\, J^{\gamma_1-\gamma_2}\Big|\varphi_2(t) - \varphi_1(t)\Big| + J^{\gamma_1}\Big|f\left(t, \varphi_2(t)\right) - f\left(t, \varphi_1(t)\right)\Big|\\ &\quad + \frac{|\xi|\, t^{\gamma_1-\gamma_2} + \Gamma(\gamma_1 - \gamma_2 + 1)}{\Big|\xi - 2\Gamma(\gamma_1 - \gamma_2 + 1)\Big|}\left(|\xi| J^{\gamma_1-\gamma_2}\Big|\varphi_2(1) - \varphi_1(1)\Big|\right)\\ &\quad + \frac{|\xi|\, t^{\gamma_1-\gamma_2} + \Gamma(\gamma_1 - \gamma_2 + 1)}{\Big|\xi - 2\Gamma(\gamma_1 - \gamma_2 + 1)\Big|}\left(J^{\gamma_1}\Big|f\Big(1, \varphi_2(1)\Big) - f\Big(1, \varphi_1(1)\Big)\Big|\right)\\ &\quad + \frac{|\xi|\, t^{\gamma_1-\gamma_2} + \Gamma(\gamma_1 - \gamma_2 + 1)}{\Big|\xi - 2\Gamma(\gamma_1 - \gamma_2 + 1)\Big|} J^{\gamma_1}\Big|g(1)\Big| + J^{\gamma_1}\Big|g(t)\Big|\\ &\leqslant |\xi| \int_0^t \left(\frac{(t-s)^{\gamma_1-\gamma_2-1}}{\Gamma(\gamma_1 - \gamma_2)} + \frac{k}{|\xi|}\frac{(t-s)^{\gamma_1-1}}{\Gamma(\gamma_1)}\right)\Big|\varphi_2(s) - \varphi_1(s)\Big|\, ds\\ &\quad + |\xi| \frac{|\xi| t^{\gamma_1-\gamma_2} + \Gamma(\gamma_1 - \gamma_2 + 1)}{\Big|\xi - 2\Gamma(\gamma_1 - \gamma_2 + 1)\Big|}\int_0^1 \left(\frac{(1-s)^{\gamma_1-\gamma_2-1}}{\Gamma(\gamma_1 - \gamma_2)}\right)\Big|\varphi_2(s) - \varphi_1(s)\Big|\, ds\\ &\quad + \frac{|\xi| t^{\gamma_1-\gamma_2} + \Gamma(\gamma_1 - \gamma_2 + 1)}{\Big|\xi - 2\Gamma(\gamma_1 - \gamma_2 + 1)\Big|}\int_0^1 \left(k\frac{(1-s)^{\gamma_1-1}}{\Gamma(\gamma_1)}\right)\Big|\varphi_2(s) - \varphi_1(s)\Big|\, ds\\ &\quad + |\xi| \frac{|\xi| t^{\gamma_1-\gamma_2} + \Gamma(\gamma_1 - \gamma_2 + 1)}{\Big|\xi - 2\Gamma(\gamma_1 - \gamma_2 + 1)\Big|}\int_0^1 \frac{(1-s)^{\gamma_1-1}}{\Gamma(\gamma_1)}\Big|g(s)\Big|\, ds + \int_0^t \frac{(t-s)^{\gamma_1-1}}{\Gamma(\gamma_1)}\Big|g(s)\Big|\, ds\\ &\leqslant |\xi| \int_0^t \left(\frac{(t-s)^{\gamma_1-\gamma_2-1}}{\Gamma(\gamma_1 - \gamma_2)} + \frac{k}{|\xi|}\frac{(t-s)^{\gamma_1-1}}{\Gamma(\gamma_1)}\right)\Big|\varphi_2(s) - \varphi_1(s)\Big|\, ds\\ &\quad + |\xi| \frac{|\xi| + \Gamma(\gamma_1 - \gamma_2 + 1)}{|\xi - 2\Gamma(\gamma_1 - \gamma_2 + 1)|}\int_0^1 \left(\frac{(1-s)^{\gamma_1-\gamma_2-1}}{\Gamma(\gamma_1 - \gamma_2)}\right)\Big|\varphi_2(s) - \varphi_1(s)\Big|\, ds\\ &\quad + \frac{|\xi| + \Gamma(\gamma_1 - \gamma_2 + 1)}{|\xi - 2\Gamma(\gamma_1 - \gamma_2 + 1)|}\int_0^1 \left(k\frac{(1-s)^{\gamma_1-1}}{\Gamma(\gamma_1)}\right)\Big|\varphi_2(s) - \varphi_1(s)\Big|\, ds\\ &\quad + \left(|\xi| \frac{|\xi| + \Gamma(\gamma_1 - \gamma_2 + 1)}{\Big|\xi - 2\Gamma(\gamma_1 - \gamma_2 + 1)\Big|} + 1\right)\frac{1}{\Gamma(\gamma_1 + 1)}\varepsilon\end{aligned}$$

$$
\begin{aligned}
&\leqslant |\xi| \int_0^t \left(\frac{(t-s)^{\gamma_1-\gamma_2-1}}{\Gamma(\gamma_1-\gamma_2)} + \frac{k}{|\xi|} \frac{(t-s)^{\gamma_1-1}}{\Gamma(\gamma_1)} \right) \Big| \varphi_2(s) - \varphi_1(s) \Big| ds \\
&\quad + |\xi| L_0 \int_0^1 \left(\frac{(1-s)^{\gamma_1-\gamma_2-1}}{\Gamma(\gamma_1-\gamma_2)} + \frac{k}{|\xi|} \frac{(1-s)^{\gamma_1-1}}{\Gamma(\gamma_1)} \right) \Big| \varphi_2(s) - \varphi_1(s) \Big| ds + M\varepsilon,
\end{aligned}
$$

where

$$
L_0 := \frac{|\xi| + \Gamma(\gamma_1-\gamma_2+1)}{|\xi - 2\Gamma(\gamma_1-\gamma_2+1)|}, \qquad M := \frac{1}{\Gamma(\gamma_1+1)} \left(1 + |\xi| \frac{|\xi| + \Gamma(\gamma_1-\gamma_2+1)}{|\xi - 2\Gamma(\gamma_1-\gamma_2+1)|} \right).
$$

First and foremost, let $x(t) := |\varphi_2(t) - \varphi_1(t)|$, choosing fixed constants $p, q \in (1, +\infty)$ such that $\gamma_1 - \gamma_2 + \frac{1}{q} > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. By Hölder inequality, we can get

$$
\begin{aligned}
x(t) &\leqslant M\varepsilon + \frac{|\xi|}{\Gamma(\gamma_1-\gamma_2)} \left(\int_0^t (t-s)^{(\gamma_1-\gamma_2-1)q} ds \right)^{\frac{1}{q}} \left(\int_0^t x^p(s) ds \right)^{\frac{1}{p}} + \frac{k}{\Gamma(\gamma_1)} \left(\int_0^t (t-s)^{(\gamma_1-1)q} ds \right)^{\frac{1}{q}} \left(\int_0^t x^p(s) ds \right)^{\frac{1}{p}} \\
&\quad + \frac{L_0|\xi|}{\Gamma(\gamma_1-\gamma_2)} \left(\int_0^1 (1-s)^{(\gamma_1-\gamma_2-1)q} ds \right)^{\frac{1}{q}} \left(\int_0^t x^p(s) ds \right)^{\frac{1}{p}} + \frac{L_0 k}{\Gamma(\gamma_1)} \left(\int_0^1 (1-s)^{(\gamma_1-1)q} ds \right)^{\frac{1}{q}} \left(\int_0^t x^p(s) ds \right)^{\frac{1}{p}} \\
&\leqslant M\varepsilon + Q_1 \left(\int_0^t x^p(s) ds \right)^{\frac{1}{p}} + Q_2 \left(\int_0^1 x^p(s) ds \right)^{\frac{1}{p}},
\end{aligned}
$$

where

$$
Q_1 = \frac{|\xi|}{\Gamma(\gamma_1-\gamma_2)\Big(1 + (\gamma_1-\gamma_2-1)q\Big)^{\frac{1}{q}}} + \frac{k}{\Gamma(\gamma_1)\Big(1 + (\gamma_1-1)q\Big)^{\frac{1}{q}}},
$$

and

$$
Q_2 = \frac{L_0|\xi|}{\Gamma(\gamma_1-\gamma_2)\Big(1 + (\gamma_1-\gamma_2-1)q\Big)^{\frac{1}{q}}} + \frac{L_0 k}{\Gamma(\gamma_1)\Big(1 + (\gamma_1-1)q\Big)^{\frac{1}{q}}}.
$$

On the basis of the inequality above, by Jensen's inequality, we conclude that

$$
\begin{aligned}
x^p(t) &\leqslant \left(M\varepsilon + Q_1 \left(\int_0^t x^p(s) ds \right)^{\frac{1}{p}} + Q_2 \left(\int_0^1 x^p(s) ds \right)^{\frac{1}{p}} \right)^p \\
&= 3^p \left(\frac{1}{3} M\varepsilon + \frac{1}{3} Q_1 \left(\int_0^t x^p(s) ds \right)^{\frac{1}{p}} + \frac{1}{3} Q_2 \left(\int_0^1 x^p(s) ds \right)^{\frac{1}{p}} \right)^p \\
&\leqslant 3^{p-1} (M\varepsilon)^p + 3^{p-1} Q_1^p \int_0^t x^p(s) ds + 3^{p-1} Q_2^p \int_0^1 x^p(s) ds.
\end{aligned}
$$

Finally, applying the integral inequality in Lemma 2.7, we can obtain that

$$
x^p(t) \leqslant \frac{3^{p-1}(M\varepsilon)^p \exp(3^{p-1} Q_1^p)}{1 - L_0^p\Big(\exp(3^{p-1} Q_1^p) - 1\Big)}.
$$

It follows that

$$
x(t) \leqslant \frac{M\Big(3^{p-1} \exp(3^{p-1} Q_1^p)\Big)^{\frac{1}{p}}}{\Big(1 - L_0^p\Big(\exp(3^{p-1} Q_1^p) - 1\Big)\Big)^{\frac{1}{p}}} \varepsilon.
$$

Set

$$c := \frac{M\Big(3^{p-1}\exp(3^{p-1}Q_1^p)\Big)^{\frac{1}{p}}}{\Big(1 - L_0^p\Big(\exp(3^{p-1}Q_1^p) - 1\Big)\Big)^{\frac{1}{p}}}.$$

The inequality $|\varphi_2(t) - \varphi_1(t)| \leqslant c\varepsilon$ is confirmed, which means the theorem is proved. □

5. Examples

Example 5.1. *The following two-term Caputo fractional differential equation is taken into the consideration*

$$\begin{cases} {}_cD^{\frac{3}{4}}\varphi(t) - \frac{1}{10}{}_cD^{\frac{1}{5}}\varphi(t) + \frac{1}{3}t^2\sin\varphi(t) = 0, \quad 0 < t < 1, \\ \varphi(0) + \varphi(1) = 1, \end{cases} \tag{15}$$

where $\gamma_1 = 3/4$*,* $\gamma_2 = 1/5$*,* $\xi = 1/10$*,* $\varphi_0 = 1$ *and* $f(t,\varphi) = (1/3)t^2\sin\varphi(t)$*. We are going to verify the uniqueness and Hyers-Ulam stability of the BVP* (15)*. Firstly, it can be easily proved that*

$$|f(t,\varphi_1) - f(t,\varphi_2)| < \frac{1}{3}|\varphi_1 - \varphi_2| < |\varphi_1 - \varphi_2|,$$

which implies that condition (H2) is satisfied. So we can set $k = 1/10$*. It is not difficult to figure out that* $M_1 < 0.1834$ *and* $M_2 < 1.7734$*. Hence, we can get*

$$M_1 + kM_2 < 0.3607 < 1.$$

According to Theorem 3.2, we can claim that BVP (15) *has a unique solution on the interval* $(0,1)$*. Additionally, set* $p = q = 2$*, we can calculate that*

$$L_0^p\Big(\exp(3^{p-1}Q_1^p) - 1\Big) < 0.2969 < 1.$$

By lemma 2.7 and Theorem 4.2, the BVP (15) *with* $\varphi(0) + \varphi(1) = 1$ *is Hyers-Ulam stable.*

Example 5.2. *Considering the nonlinear fractional differential equation with Caputo derivatives given below*

$$\begin{cases} {}_cD^{\frac{1}{2}}\varphi(t) - \frac{1}{5}{}_cD^{\frac{1}{6}}\varphi(t) + \frac{1}{4}t^2\varphi^2 = 0, \quad 0 < t < 1, \\ \varphi(0) + \varphi(1) = 0, \end{cases} \tag{16}$$

where $\gamma_1 = 1/2$*,* $\gamma_2 = 1/6$*,* $\xi = 1/5$*,* $\varphi_0 = 0$ *and* $f(t,\varphi) = (1/4)t^2\varphi(t)$*. We will check the existence of solutions to the BVP* (16) *in a different way of Example 5.1. Directly, we can get* $M_1 < 0.2492$ *and* $M_2 < 1.9894$ *by calculating. Set* $\psi(t) = (1/4)t^2$ *and* $\omega(t) = \varphi^2(t)$*. It indicates that*

$$M_1 + M_2\left\|\psi\right\| \lim_{r\to\infty} sup\frac{\omega(r)}{r} < 1.$$

Therefore, according to Theorem 3.4, there exists at least one solution of BVP (16) *on the interval (0,1).*

6. Conclusion

In the present paper, we mainly discuss and investigate a class of nonlinear differential equations with multi-term fractional derivatives. The Green's function of the fractional equation is given at the beginning of this paper. Some existence conclusions of the given integral equations are achieved by the Banach contraction principle and Leray-Schauder alternative theorem. According to the characteristics of the system, the Hyers-Ulam stability is studied. As a theoretical tool, we also introduce a Gronwall-type inequality which can be used for studying differential and integral equations.

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