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# Fixed point of weak contraction mappings on suprametric spaces

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**Abstract.** Suprametric spaces are a very recent generalization of metric spaces. In this study, various fixed point results are given in suprametric spaces. We prove the existence and uniqueness of fixed point for $\psi - \varphi$-weak contractive mapping. Our results generalize those corresponding in the literature.

## 1. Introduction and Preliminaries

Fixed point theory is considered to be a crucial study area for many researchers in pure and applied mathematics. A large number of authors have written extensively about fixed point theory, especially in the recent century. The famous Banach theorem [9] is one of the most important results of fixed point theory, which has been proved very useful in a variety of problems.

Metric spaces provide an ideal setting for investigating the existence of fixed points for single and multivalued mappings. The axioms of metric spaces are modified in numerous attempts to generalize the metric setting due to the nature of mathematics science. As a result, a number of new space types are established, and numerous metric findings are expanded and generalized to include additional contexts. For instance, partial metric spaces [28] and $b$-metric spaces [17] are important and interesting metric space generalizations. In recent years, there has been substantial progress in fixed point theory and its applications; see papers of [1, 4, 5, 13, 14, 19, 30–32] and references therein.

In 2022, Berzig [11] established the idea of suprametric spaces which is a very interesting generalization of that of the metric spaces. We also refer to [12]. The aim of this paper is to prove some fixed point results in suprametric spaces.

Throughout this study, we denote by $\mathbb{N}$ the set of natural numbers. The symbols $\mathbb{R}$ stands for the set of all real numbers and $\mathbb{R}_0^+$ stands for set of nonnegative real numbers.

We first recall from [11] the definition and some properties of the suprametric spaces.

**Definition 1.1.** *Let $\mathcal{E}$ be a nonempty set and let $d_s : \mathcal{E} \times \mathcal{E} \to \mathbb{R}_0^+$ satisfy*

*($d_s1$)* $d_s(u, v) = 0 \Leftrightarrow u = v$ *(equality);*

*($d_s2$)* $d_s(u, v) = d_s(v, u)$ *(symmetry);*

*($d_s3$)* $d_s(u, v) \leq d_s(u, w) + d_s(w, v) + \rho d_s(u, w) d_s(w, v)$;

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*for some constant $\rho \in \mathbb{R}_0^+$ and for all $u, v, w \in \mathcal{E}$. Then the pair $(\mathcal{E}, d_s)$ is called a suprametric space and $d_s$ is called a* ***suprametric*** *on $\mathcal{E}$.*

**Example 1.2.** *[11] Let $(\mathcal{E}, d)$ is a metric space, and $\alpha, \beta$ be two positive reals. We have following holds;*

*(i)* $d_{s1}^{\alpha}(u,v) = d(u,v)(d(u,v)+\alpha)$, *are suprametrics with constant* $\rho = \frac{2}{\alpha}$.

*(ii)* $d_{s2}^{\beta}(u,v) = \beta\left(e^{d(u,v)} - 1\right)$ *are suprametrics with constant* $\rho = \frac{1}{\beta}$.

*But, $d_1^{\alpha}$ and $d_{s2}^{\beta}$ are not necessarily usual metrics. It is easy to see that, if*

$$d_s = d_{s2}^{1} \text{ or } \left(d_s = d_{s1}^{1}\right)$$

*is defined on $\mathbb{R}$ and $d(u,v) = |u - v|$, we have $d_s(0,1) + d_s(1,2) < d_s(0,2)$.*

Every suprametric with constant $\rho$ is a suprametric with constant $\rho' > \rho$. However, the converse is not always true. We can see the following remark.

**Remark 1.3.** *[11] If $d_s$ is equal $d_{s1}^{1}$ or $d_{s2}^{1}$ with $d(u,v) = |u - v|$, then it is not a suprametric with constant $\rho = \frac{1}{3}$ since we obtain*

$$d_s(0,1) > d_s\left(0, \frac{1}{2}\right) + d_s\left(\frac{1}{2}, 1\right) + \frac{1}{3} d_s(0,1) d_s\left(\frac{1}{2}, 1\right).$$

Now, we present some properties of suprametric spaces.

**Definition 1.4.** *[11] Let $(\mathcal{E}, d_s)$ be a suprametric space. The set*

$$B(u_0, r) := \{u \in \mathcal{E} : d(u_0, u) < r\},$$

*where $r > 0$ and $u_0 \in \mathcal{E}$, is called open ball of radius $r$ and center $u_0$. A subset $Y$ of $\mathcal{E}$ is called open if for any point $v \in Y$, there exists $r > 0$ such that $B(v, r) \subset Y$. The family of all open subsets of $\mathcal{E}$ will be denoted by $\tau$.*

**Proposition 1.5.** *[11] Let $(\mathcal{E}, d_s)$ be a suprametric space. Then each open ball is an open set.*

**Definition 1.6.** *[11] Let $(\mathcal{E}, d_s)$ be a suprametric space. A sequence $\{u_n\}_{n\in\mathbb{N}}$ of elements of $\mathcal{E}$ converges to $u$ if for all $\epsilon > 0$ the ball $B(u, \epsilon)$ contains all but a finite number of terms of the sequence. So, $u$ is a limit point of $\{u_n\}_{n\in\mathbb{N}}$ and we said that $\lim_{n\to\infty} d_s(u_n, u) = 0$.*

**Proposition 1.7.** *[11] Let $(\mathcal{E}, d_s)$ be a suprametric space. If a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{E}$ has a limit, then it is unique.*

**Definition 1.8.** *[11] Let $(\mathcal{E}, d_s)$ be a suprametric space. A mapping $\mathcal{T} : \mathcal{E} \to \mathcal{E}$ is called be continuous at $z$, if for all $\epsilon > 0$ there exists $\zeta > 0$ such that $d_s(\mathcal{T}u, \mathcal{T}z) < \zeta$ whenever $d_s(u, z) < \epsilon$. If $\mathcal{T}$ is continuous at all points of $\mathcal{E}$, then it is said to be continuous.*

It is known that if a function $h : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is an homeomorphism, then $h(0) = 0$ and $h$ is strictly increasing. So, we have following proposition.

**Proposition 1.9.** *[11] Let $(\mathcal{E}, d)$ be a metric space and $(\mathcal{E}, d_s)$ be a suprametric space. Let $h : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be an homeomorphism and that $d_s = h \circ d$, then;*

*(i) A continuous mapping with respect to $d$ is continuous with respect to $d_s$.*

*(ii) A convergent sequence with respect to $d$ converges with respect to $d_s$ to the same point.*

**Definition 1.10.** *[11] Let $(\mathcal{E}, d_s)$ be a suprametric space. A sequence $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{E}$ is a Cauchy sequence if, for all $\epsilon > 0$, there exists some $k \in \mathbb{N}$ such that $d_s(u_n, u_m) < \epsilon$ for all $n, m \geq k$.*

A suprametric space is complete if every Cauchy sequence is convergent.

**Proposition 1.11.** *[11] If $(\mathcal{E}, d)$ be complete metric spaces and $d_s$ is the suprametrics in Example 1.2 then $(\mathcal{E}, d_s)$ is a complete suprametric space.*

**Remark 1.12.** *[11] Let a sequence $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in a complete suprametric $(\mathcal{E}, d_s)$, then there exists $z_* \in \mathcal{E}$ such that $\lim_{n\to\infty} d_s(u_n, z_*) = 0$ and from condition $(d_s3)$ follows that every subsequence $\left\{u_{n(k)}\right\}_{k\in\mathbb{N}}$ converges to $z_*$.*

**Lemma 1.13.** *[11] Every suprametric is continuous.*

Banach's contraction principle in suprametric spaces [11] can be formulated as follows.

**Theorem 1.14.** *[11] Let $\mathcal{T}$ be a mapping on a suprametric space $(\mathcal{E}, d_s)$ into itself and $c \in [0, 1)$ such that:*

$$d_s(\mathcal{T}u, \mathcal{T}v) \leq c\, d_s(u, v)$$

*where $u_n = \mathcal{T}u_{n-1}, n \in \mathbb{N}$, satisfying for every $u_0 \in \mathcal{E}$ and for all $u, v \in \mathcal{E}$. Then $\mathcal{T}$ has a unique fixed point.*

Here, we systematically recall contraction mappings and their notations, which we use in the study. We denote by:

- $\Psi$ the set of continuous and nondecreasing mapping $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that $\psi(t) = 0 \Leftrightarrow t = 0$ for $t \in \mathbb{R}_0^+$.
- $\Phi$ the set of continuous and nondecreasing mapping $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that $\varphi(t) = 0 \Leftrightarrow t = 0$ for $t \in \mathbb{R}_0^+$.

We next recall the Banach's contraction principle [9].

**Theorem 1.15.** *(Banach contraction principle) Let $(\mathcal{E}, d)$ be a metric space and self map $\mathcal{T}$ be a contraction on $\mathcal{E}$, i.e., there exists $\alpha \in [0, 1)$ such that*

$$d(\mathcal{T}u, \mathcal{T}v) \leq \alpha d(u, v), \quad \text{for all } u, v \in \mathcal{E}.$$

*Then $\mathcal{T}$ has a unique fixed point.*

Later, Khan *et al.* [27] generalized the Banach results using altering distance functions, as follows;

**Theorem 1.16.** *($\psi$-contraction) Let $(\mathcal{E}, d)$ be a metric space and self map $\mathcal{T}$ be a $\psi$-contraction on $\mathcal{E}$, i.e., there exists $\psi \in \Psi$ and $\alpha \in [0, 1)$ such that*

$$\psi(d(\mathcal{T}u, \mathcal{T}v)) \leq \alpha\psi(d(u, v)), \quad \text{for all } u, v \in \mathcal{E}.$$

*Then $\mathcal{T}$ has a unique fixed point.*

**Remark 1.17.** *It is clear that if $\psi(t) = t$, for $t \in \mathbb{R}_+$ in Theorem 1.16, Theorem 1.15 will be obtained.*

Alber and Guerre-Delabriere [3] first introduced the concept of $\varphi$-weakly contractive (or weakly contractive) mappings in Hilbert spaces, which is a generalization of Banach's contraction. The $\varphi$-weakly contractive mappings are defined later by Rhoades [34], as follows;

**Theorem 1.18.** *($\varphi$-weak contraction) Let $(\mathcal{E}, d)$ be a complete metric space and $\mathcal{T}$ be a self mapping on $\mathcal{E}$. Suppose that $\mathcal{T}$ is $\varphi$-weak contraction on $\mathcal{E}$, that is, there exists $\varphi \in \Phi$ such that*

$$d(\mathcal{T}u, \mathcal{T}v) \leq d(u, v) - \varphi(d(u, v)), \quad \text{for all } u, v \in \mathcal{E}.$$

*Then $\mathcal{T}$ has a unique fixed point.*

**Remark 1.19.** *It is clear that if $\varphi(t) = (1-k)t$, for $k \in (0,1)$ in Theorem 1.18, Theorem 1.15 will be obtained.*

Afterwards, Dutta and Choudhurry [18] defined the concept of $\psi - \varphi$-weak contraction by generalizing the contractions given above.

**Theorem 1.20.** *($\psi - \varphi$-weak contraction) Let $(\mathcal{E}, d)$ be a complete metric space and $\mathcal{T}$ be a self mapping on $\mathcal{E}$. Suppose that $\mathcal{T}$ is $\psi$-weak contraction on $\mathcal{E}$, that is, there exists $\psi \in \Psi$ and $\varphi \in \Phi$ such that*

$$\psi(d(\mathcal{T}u, \mathcal{T}v)) \leq \psi(d(u,v)) - \varphi(d(u,v)), \quad \text{for all } u, v \in \mathcal{E}.$$

*Then $\mathcal{T}$ has a unique fixed point.*

Recently, many researchers introduced interesting fixed point results for weakly contractive mappings. We refer the reader to [2, 7, 8, 10, 15, 16, 20, 23–26, 29, 36–38].

## 2. Main Fixed Point Results

Now, we are ready to state and prove our main results.

**Theorem 2.1.** *Let $(\mathcal{E}, d_s)$ be a complete suprametric space and $\mathcal{T}$ be a self mapping on $\mathcal{E}$. Suppose that there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that*

$$\psi(d_s(\mathcal{T}u, \mathcal{T}v)) \leq \psi(d_s(u,v)) - \varphi(d_s(u,v)), \quad \text{for all } u, v \in \mathcal{E}. \tag{1}$$

*Then, $\mathcal{T}$ has a unique fixed point and for all $u_0 \in \mathcal{E}$ the iterative sequence such that $u_p = \mathcal{T}u_{p-1}$, $p \in \mathbb{N}$ converges to fixed point $z$ of $\mathcal{T}$.*

*Proof.* Let $u_0 \in \mathcal{E}$ be arbitrary and define the sequence $u_p \in \mathcal{E}$ as $u_p = \mathcal{T}u_{p-1}$ for all $p \in \mathbb{N}$. Given that $u_p \neq u_{p+1}$ for every $p \in \mathbb{N}$. If we take $u = u_{p-1}$ and $v = u_p$ in condition (1), then we obtain

$$\psi(d_s(u_p, u_{p+1})) = \psi(d_s(\mathcal{T}u_{p-1}, \mathcal{T}u_p)) \leq \psi(d_s(u_{p-1}, u_p)) - \varphi(d_s(u_{p-1}, u_p)), \tag{2}$$

which implies

$$\psi(d_s(u_p, u_{p+1})) \leq \psi(d_s(u_{p-1}, u_p)).$$

Since the monotone property of the $\psi$ function, we get

$$d_s(u_p, u_{p+1}) \leq d_s(u_{p-1}, u_p),$$

for every $p \in \mathbb{N}$. Hence $\{d_s(u_p, u_{p+1})\}$ is an non increasing sequence and bounded. Then, there exists $\mu \geq 0$ such that $\lim_{p\to\infty} d_s(u_p, u_{p+1}) = \mu$. Now we show that $\mu = 0$, then we assume that $\mu > 0$. We have by taking $p \to \infty$ in (2) that,

$$\psi(\mu) \leq \psi(\mu) - \varphi(\mu),$$

which is a contradiction and $\mu = 0$, that is

$$\lim_{p\to\infty} d_s(u_p, u_{p+1}) = 0. \tag{3}$$

Next, we want to prove that $\{u_p\}$ is a Cauchy sequence. Otherwise there exists $\epsilon > 0$ and subsequence $\{u_{q_r}\}$ and $\{u_{p_r}\}$ of $\{u_p\}$ such that for every positive integer $r$ with $p_r > q_r > r$,

$$d_s(u_{p_r}, u_{q_r}) \geq \epsilon, \tag{4}$$

and we have

$$d_s(u_{p_r}, u_{q_r-1}) < \epsilon.$$

Therefore, using (4) and inequality ($d_s$3) for all $r \in \mathbb{N}$, we have

$$\begin{aligned} d_s(u_{p_r}, u_{q_r}) \leq & d_s(u_{p_r}, u_{q_r-1}) + d_s(u_{q_r-1}, u_{q_r}) + \rho d_s(u_{p_r}, u_{q_r-1}) d_s(u_{q_r-1}, u_{q_r}) \\ \leq & \epsilon + d_s(u_{q_r-1}, u_{q_r}) + \rho\epsilon d_s(u_{q_r-1}, u_{q_r}), \end{aligned}$$

letting $r \to \infty$ in the above inequality and using (3), we obtain

$$\lim_{r\to\infty} d_s(u_{p_r}, u_{q_r}) = \epsilon. \tag{5}$$

In addition, by using ($d_s$3), we have

$$\begin{aligned} d_s\left(u_{q_r}, u_{p_r}\right) \leq & d_s\left(u_{q_r}, u_{q_r+1}\right) + d_s\left(u_{q_r+1}, u_{p_r}\right) + \rho d_s\left(u_{q_r}, u_{q_r+1}\right) d_s\left(u_{q_r+1}, u_{p_r}\right) \\ & \leq d_s\left(u_{q_r}, u_{q_r+1}\right) + d_s\left(u_{q_r+1}, u_{p_r+1}\right) + d_s\left(u_{p_r+1}, u_{p_r}\right) + \rho d_s\left(u_{q_r+1}, u_{p_r+1}\right) d_s\left(u_{p_r}, u_{p_r+1}\right) \\ & + \rho d_s\left(u_{q_r}, u_{q_r+1}\right) d_s\left(u_{q_r+1}, u_{p_r}\right) \\ & \leq d_s\left(u_{q_r}, u_{q_r+1}\right) + d_s\left(u_{q_r+1}, u_{p_r+1}\right) + d_s\left(u_{p_r+1}, u_{p_r}\right) + \rho d_s\left(u_{q_r+1}, u_{p_r+1}\right) d_s\left(u_{p_r}, u_{p_r+1}\right) \\ & + \rho d_s\left(u_{q_r} u_{q_r+1}\right)\left(d_s\left(u_{q_r}, u_{q_r+1}\right) + d_s\left(u_{q_r}, u_{p_r}\right) + \rho d_s\left(u_{q_r}, u_{q_r+1}\right) d_s\left(u_{q_r}, u_{p_r}\right)\right) \end{aligned}$$

or equivalently,

$$\begin{aligned} & \left(\left(1 - \rho d_s\left(u_{q_r}, u_{q_r+1}\right) - \rho^2 d_s\left(u_{q_r}, u_{q_r+1}\right)^2\right) d_s\left(u_{q_r}, u_{p_r}\right)\right. \\ & \left. - d_s\left(u_{q_r}, u_{q_r+1}\right) - d_s\left(u_{p_r+1}, u_{p_r}\right) - \rho d_s\left(u_{q_r}, u_{q_r+1}\right)^2\right)\left(1 + \rho d_s\left(u_{p_r}, u_{p_r+1}\right)\right)^{-1} \leq d_s\left(u_{q_r+1}, u_{p_r+1}\right). \end{aligned}$$

and

$$\begin{aligned} d_s\left(u_{q_r+1}, u_{p_r+1}\right) \leq & d_s\left(u_{q_r+1}, u_{q_r}\right) + d_s\left(u_{q_r}, u_{p_r+1}\right) + \rho d_s\left(u_{q_r}, u_{q_r+1}\right) d_s\left(u_{q_r}, u_{p_r+1}\right) \\ \leq & d_s\left(u_{q_r+1}, u_{q_r}\right) + d_s\left(u_{q_r}, u_{p_r}\right) + d_s\left(u_{p_r}, u_{p_r+1}\right) + \rho d_s\left(u_{q_r} u_{p_r}\right) d_s\left(u_{p_r}, u_{p_r+1}\right) \\ & + \rho d_s\left(u_{q_r}, u_{q_r+1}\right) d_s\left(u_{q_r}, u_{p_r+1}\right) \\ \leq & d_s\left(u_{q_r+1}, u_{q_r}\right) + d_s\left(u_{q_r}, u_{p_r}\right) + d_s\left(u_{p_r}, u_{p_r+1}\right) + \rho d_s\left(u_{q_r}, u_{p_r}\right) d_s\left(u_{p_r}, u_{p_r+1}\right) \\ & + \rho d_s\left(u_{q_r}, u_{q_r+1}\right)\left(d_s\left(u_{q_r}, u_{q_r+1}\right) + d_s\left(u_{q_r+1}, u_{p_r+1}\right) + \rho d_s\left(u_{q_r}, u_{q_r+1}\right) d_s\left(u_{q_r+1}, u_{p_r+1}\right)\right) \end{aligned}$$

or equivalently,

$$\begin{aligned} & \left(\left(1 - \rho d_s\left(u_{q_r}, u_{q_r+1}\right) - \rho^2 d_s\left(u_{q_r}, u_{q_r+1}\right)^2\right) d_s\left(u_{q_r+1}, u_{p_r+1}\right)\right. \\ & \leq d_s\left(u_{q_r}, u_{q_r+1}\right) + d_s\left(u_{p_r}, u_{p_r+1}\right) + \rho d_s\left(u_{q_r}, u_{q_r+1}\right)^2 + \left(1 + \rho d_s\left(u_{p_r}, u_{p_r+1}\right)\right) d_s\left(u_{q_r}, u_{p_r}\right). \end{aligned}$$

Therefore letting $r \to \infty$ in the above inequalities and from (3) and (5) we show that

$$\lim_{r\to\infty} d_s(u_{p_{r+1}}, u_{q_{r+1}}) = \epsilon.$$

In the condition (1), taking $u = u_{p_r}$ and $v = u_{q_r}$ then we obtain

$$\psi(d_s(\mathcal{T}u_{p_r}, \mathcal{T}u_{q_r})) \leq \psi(d_s(u_{q_r}, u_{p_r})) - \varphi(d_s(u_{q_r}, u_{p_r})),$$

using the continuous of $\psi$ and $\varphi$ and letting $r \to \infty$, we show that

$$\psi(\epsilon) \leq \psi(\epsilon) - \varphi(\epsilon),$$

that is, $\epsilon = 0$, which contradicts the assumption that $\{u_p\}$ is not a Cauchy sequence. Hence, the sequence $\{u_p\}$ is a Cauchy sequence and since $(\mathcal{E}, d_s)$ is a complete suprametric space, so there exists $z \in \mathcal{T}$ such that $u_p \to z$. Now we show that $z$ is a fixed point of $\mathcal{T}$. From the contractive condition and Proposition (1.7) and Remark (1.12), we get

$$\psi(d_s(\mathcal{T}u_{p_r}, \mathcal{T}z)) \leq \psi(d_s(u_{p_r}, z)) - \varphi(d_s(u_{p_r}, z)),$$

then letting $r \to \infty$, we get $\psi(d_s(\mathcal{T}z, z)) = 0$. Then $d_s(\mathcal{T}z, z) = 0$, $\mathcal{T}z = z$.

To prove the uniqueness of fixed point, assume that $z$ and $z^*$ are two different fixed point of $\mathcal{T}$. So, using the contractive condition and the properties of the functions $\psi$ and $\varphi$, we have

$$\psi(d_s(z, z^*)) = \psi(d_s(\mathcal{T}z, \mathcal{T}z^*)) \leq \psi(d_s(z, z^*)) - \varphi(d_s(z, z^*)),$$

which is a contraction with the assumption that $z \neq z^*$. Therefore, the fixed point of $\mathcal{T}$ is unique. □

**Theorem 2.2.** *Let $(\mathcal{E}, d_s)$ be a complete suprametric space and $\mathcal{T}$ be a self mapping on $\mathcal{E}$. Suppose that there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that*

$$\psi(d_s(\mathcal{T}u, \mathcal{T}v)) \leq \psi(\max\{d_s(u, v), d_s(\mathcal{T}u, u), d_s(\mathcal{T}v, v)\}) - \varphi(\max\{d_s(u, v), d_s(\mathcal{T}u, u), d_s(\mathcal{T}v, v)\}), \text{ for all } u, v \in \mathcal{E}. \tag{6}$$

*Then, $\mathcal{T}$ has a unique fixed point and for all $u_0 \in \mathcal{E}$ the iterative sequence such that $u_\vartheta = \mathcal{T}u_{\vartheta-1}$, $\vartheta \in \mathbb{N}$ converges to the fixed point of $\mathcal{T}$.*

*Proof.* Let $u_0 \in \mathcal{E}$ be arbitrary and define the sequence $u_\vartheta \in \mathcal{E}$ as $u_\vartheta = \mathcal{T}u_{\vartheta-1}$ for every $\vartheta \in \mathbb{N}$. Assume that $u_\vartheta \neq u_{\vartheta+1}$ for every $\vartheta \in \mathbb{N}$. If we give $u = u_{\vartheta-1}$ and $v = u_\vartheta$ in (6), then we deduce

$$\begin{aligned}\psi(d_s(u_\vartheta, u_{\vartheta+1})) = \psi(d_s(\mathcal{T}u_{\vartheta-1}, \mathcal{T}u_\vartheta)) \leq &\psi(\max\{d_s(u_{\vartheta-1}, u_\vartheta), d_s(\mathcal{T}u_{\vartheta-1}, u_{\vartheta-1}), d_s(\mathcal{T}u_\vartheta, u_\vartheta)\}) \\ &- \varphi(\max\{d_s(u_{\vartheta-1}, u_\vartheta), d_s(\mathcal{T}u_{\vartheta-1}, u_{\vartheta-1}), d_s(\mathcal{T}u_\vartheta, u_\vartheta)\}).\end{aligned} \tag{7}$$

If $\max\{d_s(u_{\vartheta-1}, u_\vartheta), d_s(\mathcal{T}u_{\vartheta-1}, u_{\vartheta-1}), d_s(\mathcal{T}u_\vartheta, u_\vartheta)\} = d_s(u_\vartheta, u_{\vartheta+1})$, from (7) we deduce

$$\psi(d_s(u_\vartheta, u_{\vartheta+1})) \leq \psi(d_s(u_\vartheta, u_{\vartheta+1})) - \varphi(d_s(u_\vartheta, u_{\vartheta+1})),$$

this is imposible so, $\max\{d_s(u_{\vartheta-1}, u_\vartheta), d_s(\mathcal{T}u_{\vartheta-1}, u_{\vartheta-1}), d_s(\mathcal{T}u_\vartheta, u_\vartheta)\} = d_s(u_{\vartheta-1}, u_\vartheta)$. Then, we can write,

$$\psi(d_s(u_\vartheta, u_{\vartheta+1})) \leq \psi(d_s(u_\vartheta, u_{\vartheta-1})) - \varphi(d_s(u_\vartheta, u_{\vartheta-1})),$$

which implies

$$\psi(d_s(u_\vartheta, u_{\vartheta+1})) \leq \psi(d_s(u_{\vartheta-1}, u_\vartheta)).$$

From the monotone property of the $\psi$ function, we deduce

$$d_s(u_\vartheta, u_{\vartheta+1}) \leq d_s(u_{\vartheta-1}, u_\vartheta),$$

for every $\vartheta \in \mathbb{N}$. Hence $\{d_s(u_\vartheta, u_{\vartheta+1})\}$ is an non increasing sequence and bounded. Then, there exists $\wp \geq 0$ such that $\lim_{\vartheta\to\infty} d_s(u_\vartheta, u_{\vartheta+1}) = \wp$. We obtain that $\wp = 0$. We given that $\wp > 0$. We have by putting $\vartheta \to \infty$ in (7).

$$\psi(\wp) \leq \psi(\wp) - \varphi(\wp),$$

which is a contradiction and $\wp = 0$, then,

$$\lim_{\vartheta\to\infty} d_s(u_\vartheta, u_{\vartheta+1}) = 0. \tag{8}$$

Similarly, we show that $\{u_\vartheta\}$ is a Cauchy sequence. Let $\{u_\vartheta\}$ not a Cauchy sequence, there exists $\epsilon > 0$ and subsequence $\{u_{w_j}\}$ and $\{u_{\vartheta_j}\}$ of $\{u_\vartheta\}$ such that for every positive integer $j$ with $\vartheta_j > w_j > j$,

$$d_s(u_{\vartheta_j}, u_{w_j}) \geq \epsilon, \tag{9}$$

and this show that

$$d_s(u_{\vartheta_j}, u_{w_j-1}) < \epsilon.$$

Consequently, by using (9) and inequality ($d_s$3) for all $j \in \mathbb{N}$, we get

$$\begin{aligned} d_s(u_{\vartheta_j}, u_{w_j}) &\leq d_s(u_{\vartheta_j}, u_{w_{j-1}}) + d_s(u_{w_{j-1}}, u_{w_j}) + \rho d_s(u_{\vartheta_j}, u_{w_{j-1}}) d_s(u_{w_{j-1}}, u_{w_j}) \\ &\leq \epsilon + d_s(u_{w_{j-1}}, u_{w_j}) + \rho\epsilon d_s(u_{w_{j-1}}, u_{w_j}), \end{aligned}$$

letting $j \to \infty$ in the above inequality and using (8), we obtain

$$\lim_{j\to\infty} d_s(u_{\vartheta_j}, u_{w_j}) = \epsilon. \tag{10}$$

Also, by from ($d_s$3), we deduce

$$\begin{aligned} d_s\left(u_{w_j}, u_{\vartheta_j}\right) &\leq d_s\left(u_{w_j}, u_{w_j+1}\right) + d_s\left(u_{w_j+1}, u_{\vartheta_j}\right) + \rho d_s\left(u_{w_j}, u_{w_j+1}\right) d_s\left(u_{w_j+1}, u_{\vartheta_j}\right) \\ &\leq d_s\left(u_{w_j}, u_{w_j+1}\right) + d_s\left(u_{w_j+1}, u_{\vartheta_j+1}\right) + d_s\left(u_{\vartheta_j+1}, u_{\vartheta_j}\right) + \rho d_s\left(u_{w_j+1}, u_{\vartheta_j+1}\right) d_s\left(u_{\vartheta_j}, u_{\vartheta_j+1}\right) \\ &\quad + \rho d_s\left(u_{w_j}, u_{w_j+1}\right) d_s\left(u_{w_j+1}, u_{\vartheta_j}\right) \\ &\leq d_s\left(u_{w_j}, u_{w_j+1}\right) + d_s\left(u_{w_j+1}, u_{\vartheta_j+1}\right) + d_s\left(u_{\vartheta_j+1}, u_{\vartheta_j}\right) + \rho d_s\left(u_{w_j+1}, u_{\vartheta_j+1}\right) d_s\left(u_{\vartheta_j}, u_{\vartheta_j+1}\right) \\ &\quad + \rho d_s\left(u_{w_j} u_{w_j+1}\right)\left(d_s\left(u_{w_j}, u_{w_j+1}\right) + d_s\left(u_{w_j}, u_{\vartheta_j}\right) + \rho d_s\left(u_{w_j}, u_{w_j+1}\right) d_s\left(u_{w_j}, u_{\vartheta_j}\right)\right) \end{aligned}$$

or equivalently,

$$\begin{aligned} &\left(\left(1 - \rho d_s\left(u_{w_j}, u_{w_j+1}\right) - \rho^2 d_s\left(u_{w_j}, u_{w_j+1}\right)^2\right) d_s\left(u_{w_j}, u_{\vartheta_j}\right)\right. \\ &\left. - d_s\left(u_{w_j}, u_{w_j+1}\right) - d_s\left(u_{\vartheta_j+1}, u_{\vartheta_j}\right) - \rho d_s\left(u_{w_j}, u_{w_j+1}\right)^2\right)\left(1 + \rho d_s\left(u_{\vartheta_j}, u_{\vartheta_j+1}\right)\right)^{-1} \leq d_s\left(u_{w_j+1}, u_{\vartheta_j+1}\right). \end{aligned}$$

and

$$\begin{aligned} d_s\left(u_{w_j+1}, u_{\vartheta_j+1}\right) &\leq d_s\left(u_{w_j+1}, u_{w_j}\right) + d_s\left(u_{w_j}, u_{\vartheta_j+1}\right) + \rho d_s\left(u_{w_j}, u_{w_j+1}\right) d_s\left(u_{w_j}, u_{\vartheta_j+1}\right) \\ &\leq d_s\left(u_{w_j+1}, u_{w_j}\right) + d_s\left(u_{w_j}, u_{\vartheta_j}\right) + d_s\left(u_{\vartheta_j}, u_{\vartheta_j+1}\right) + \rho d_s\left(u_{w_j} u_{\vartheta_j}\right) d_s\left(u_{\vartheta_j}, u_{\vartheta_j+1}\right) \\ &\quad + \rho d_s\left(u_{w_j}, u_{w_j+1}\right) d_s\left(u_{w_j}, u_{\vartheta_j+1}\right) \\ &\leq d_s\left(u_{w_j+1}, u_{w_j}\right) + d_s\left(u_{w_j}, u_{\vartheta_j}\right) + d_s\left(u_{\vartheta_j}, u_{\vartheta_j+1}\right) + \rho d_s\left(u_{w_j}, u_{\vartheta_j}\right) d_s\left(u_{\vartheta_j}, u_{\vartheta_j+1}\right) \\ &\quad + \rho d_s\left(u_{w_j}, u_{w_j+1}\right)\left(d_s\left(u_{w_j}, u_{w_j+1}\right) + d_s\left(u_{w_j+1}, u_{\vartheta_j+1}\right) + \rho d_s\left(u_{w_j}, u_{w_j+1}\right) d_s\left(u_{w_j+1}, u_{\vartheta_j+1}\right)\right) \end{aligned}$$

or equivalently,

$$\begin{aligned} &\left(\left(1 - \rho d_s\left(u_{w_j}, u_{w_j+1}\right) - \rho^2 d_s\left(u_{w_j}, u_{w_j+1}\right)^2\right) d_s\left(u_{w_j+1}, u_{\vartheta_j+1}\right)\right. \\ &\leq d_s\left(u_{w_j}, u_{w_j+1}\right) + d_s\left(u_{\vartheta_j}, u_{\vartheta_j+1}\right) + \rho d_s\left(u_{w_j}, u_{w_j+1}\right)^2 + \left(1 + \rho d_s\left(u_{\vartheta_j}, u_{\vartheta_j+1}\right)\right) d_s\left(u_{w_j}, u_{\vartheta_j}\right). \end{aligned}$$

Therefore taking $j \to \infty$ in the above inequalities and from (8) and (10) we obtain

$$\lim_{j\to\infty} d_s(u_{\vartheta_{j+1}}, u_{w_{j+1}}) = \epsilon.$$

In the contraction (6), putting $u = u_{\vartheta_j}$ and $v = u_{w_j}$ so we deduce

$$\begin{aligned}\psi(d_s(\mathcal{T}u_{\vartheta_j}, \mathcal{T}u_{w_j})) \leq &\psi(\max\{d_s(u_{w_j}, u_{\vartheta_j}), d_s(\mathcal{T}u_{w_j}, u_{w_j}), d_s(\mathcal{T}u_{\vartheta_j}, u_{\vartheta_j})\}) \\ &- \varphi(\max\{d_s(u_{w_j}, u_{\vartheta_j}), d_s(\mathcal{T}u_{w_j}, u_{w_j}), d_s(\mathcal{T}u_{\vartheta_j}, u_{\vartheta_j})\}),\end{aligned}$$

using the continuity of $\psi$ and $\varphi$ and taking the limit $j \to \infty$, we have

$$\psi(\epsilon) \leq \psi(\epsilon) - \varphi(\epsilon),$$

that is $\epsilon = 0$, which contradicts the suppose that $\{u_\vartheta\}$ is not a Cauchy sequence. Then, the sequence $\{u_\vartheta\}$ is a Cauchy sequence and as $(\mathcal{E}, d_s)$ is a complete suprametric space, so there exists $p \in \mathcal{T}$ such that $u_\vartheta \to p$. We show that $p$ is a fixed point of $\mathcal{T}$. By using the contractive condition and Proposition (1.7) and Remark (1.12), we get

$$\psi(d_s(\mathcal{T}u_{\vartheta_j}, \mathcal{T}p)) \leq \psi(\max\{d_s(u_{\vartheta_j}, p), d_s(\mathcal{T}u_{\vartheta_j}, u_{\vartheta_j}), d_s(\mathcal{T}p, p)\}) - \varphi(\max\{d_s(u_{\vartheta_j}, p), d_s(\mathcal{T}u_{\vartheta_j}, u_{\vartheta_j}), d_s(\mathcal{T}p, p)\}),$$

so, $j \to \infty$, we get

$$\psi(d_s(\mathcal{T}u_{\vartheta_j}, \mathcal{T}p)) \leq \psi(d_s(\mathcal{T}p, p)) - \varphi(d_s(\mathcal{T}p, p)),$$

hereby, $d_s(\mathcal{T}p, p) = 0$, i.e, $\mathcal{T}p = p$. To show the uniqueness of fixed point, given that $p$ and $p^*$ are two different fixed point of $\mathcal{T}$. So, from contractive condition and the properties of the functions $\psi$ and $\varphi$, we get

$$\begin{aligned}\psi(d_s(p, p^*)) = \psi(d_s(\mathcal{T}p, \mathcal{T}p^*)) \leq &\psi(\max\{d_s(\mathcal{T}p, \mathcal{T}p^*), d_s(\mathcal{T}p, p), d_s(\mathcal{T}p^*, p^*)\}) \\ &- \varphi(\max\{d_s(\mathcal{T}p, \mathcal{T}p^*), d_s(\mathcal{T}p, p), d_s(\mathcal{T}p^*, p^*)\}),\end{aligned}$$

which is a contraction then, the fixed point of $\mathcal{T}$ is unique. □

Now, in the Theorem 2.2, by taking $\psi(w) = w$, we obtain the following the result.

**Corollary 2.3.** *Let $(\mathcal{E}, d_s)$ be a complete suprametric space and $\mathcal{T}$ be a self mapping on $\mathcal{E}$. Suppose that there exists $\varphi \in \Phi$ such that*

$$d_s(\mathcal{T}u, \mathcal{T}v) \leq \max\{d_s(u, v), d_s(\mathcal{T}u, u), d_s(\mathcal{T}v, v)\} - \varphi(\max\{d_s(u, v), d_s(\mathcal{T}u, u), d_s(\mathcal{T}v, v)\}), \quad \text{for all } u, v \in \mathcal{E}.$$

*Then, $\mathcal{T}$ has a unique fixed point and for all $u_0 \in \mathcal{E}$ the iterative sequence such that $u_p = \mathcal{T}u_{p-1}$, $p \in \mathbb{N}$ converges to the fixed point of $\mathcal{T}$.*

Now, in the Theorem 2.1, by taking $\psi(w) = w$, we obtain the following Alber and Guerre-Delabriere type result in suprametric space.

**Corollary 2.4.** *Let $(\mathcal{E}, d_s)$ be a suprametric space and $\mathcal{T}$ be a self mapping on $\mathcal{E}$. Suppose that there exists $\varphi \in \Phi$ such that*

$$d_s(\mathcal{T}u, \mathcal{T}v) \leq d_s(u, v) - \varphi(d_s(u, v)), \quad \text{for all } u, v \in \mathcal{E}.$$

*Then, $\mathcal{T}$ has a unique fixed point and for all $u_0 \in \mathcal{E}$ the iterative sequence such that $u_p = \mathcal{T}u_{p-1}$, $p \in \mathbb{N}$ converges to fixed point $z$ of $\mathcal{T}$.*

In the Corollary 2.4, putting $\varphi(s) = (1 - k)\varphi(s)$, for $s \in [0, \infty)$, where $k \in [0, 1)$. Then, we obtain the following Khan type result in suprametric space.

**Corollary 2.5.** *Let $(\mathcal{E}, d_s)$ be a suprametric space and $\mathcal{T}$ be a self mapping on $\mathcal{E}$. Suppose that there exists $\psi \in \Psi$ such that*

$$\psi(d_s(\mathcal{T}u, \mathcal{T}v)) \leq k\psi(d_s(u, v)), \quad \text{for all } u, v \in \mathcal{E}.$$

*Then, $\mathcal{T}$ has a unique fixed point and for all $u_0 \in \mathcal{E}$ the iterative sequence such that $u_p = \mathcal{T}u_{p-1}$, $p \in \mathbb{N}$ converges to fixed point $z$ of $\mathcal{T}$.*

**Remark 2.6.** *In the Corollary 2.5, by putting $\psi(w) = w$, we obtain the Banach Contraction Principle in suprametric space of Theorem 1.14.*

## 3. Conclusion

We established some fixed point results on suprametric spaces, which is a very new metric generalization. We prove the existence of a fixed point for the self-mappings satisfying a $\psi - \varphi$-weak contractive condition. Our results extend previous results given by Rhoades [34] and Dutta-Choudhury [18]. In future studies, some fixed point results can be taken up for other well-known contraction mappings on suprametric spaces.

## References


[1] M. Abbas, M. Berzig, T. Nazir, E. Karapınar, Iterative approximation of fixed points for Presic type F-contraction operators, *U.P.B. Sci. Bull., Series A*, **78**(2), (2016), 147-160

[2] T. Abdeljawad, E. Karapınar, K. Taş, Existence and uniqueness of a common fixed point on partial metric spaces. *App. Math. Letters*, 24(11), (2011), 1900-1904.

[3] Y.I. Alber, S. Guerre-Delabriere, *Principle of weakly contractive maps in Hilbert spaces*. In New Results in Operator Theory and Its Applications: The Israel M. Glazman Memorial Volume (pp. 7-22). Birkhäuser Basel, 1997.

[4] M. Asadi, A. Khalesi, Lower semi-continuity in a generalized metric space. *Advances in the Theory of Nonlinear Anal. its App.*, **6**(1), (2022), 143-147.

[5] M. Asadi, P. Salimi, Some fixed point and common fixed point theorems on G-metric spaces. *Nonlinear Funct. Anal. Appl*, **21**(3), (2016), 523-530.

[6] H. Aydi, M. Jellali, E. Karapınar, Common fixed points for generalized $\alpha$-implicit contractions in partial metric spaces: consequences and application. *RACSAM*, **109**(2), (2015) 367-384.

[7] H. Aydi, M.F. Bota, E. Karapınar, S. Moradi, A common fixed point for weak $\varphi$-contractions on b-metric spaces. *Fixed Point Theory*, **13**(2), (2012), 337-346.

[8] H. Aydi, E. Karapınar, W. Shatanawi, Coupled fixed point results for $(\psi - \varphi)$-weakly contractive condition in ordered partial metric spaces. *Computers & Math. with App.*, **62**(12), (2011), 4449-4460.

[9] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundamenta Mathematicae*, **3** (1922), 133–181.

[10] S. Bedre, Remarks on F-weak contractions and discontinuity at the fixed point. *Advances in the Theory of Nonlinear Analysis and its App.*, **4**(4), (2020), 260-265.

[11] M. Berzig, First Results in Suprametric Spaces with Applications. *Mediterranean J. of Math.*, **19**(5), (2022) 226.

[12] M. Berzig, Nonlinear contraction in *b*-suprametric spaces. *arXiv preprint* arXiv:2304.08507, (2023).

[13] M. Berzig, E. Karapınar, A.F. Roldán-López-de-Hierro, Some fixed point theorems in Branciari metric spaces. *Math. Slovaca*, **67**(5), (2017), 1189-1202.

[14] M. Berzig, E. Karapınar, Note on" Modified $\alpha-\psi$-Contractive Mappings with Applications". *Thai J. of Math.*, **13**(1), (2014), 147-152.

[15] M. Berzig, E. Karapınar, Fixed point results for $(\alpha\psi, \beta\varphi)$-contractive mappings for a generalized altering distance. *Fixed Point Theory and App.*, **2013**, (2013), 1-18.

[16] M. Berzig, E. Karapınar, A.F. Roldán-López-de-Hierro, Discussion on generalized-$(\alpha, \psi, \beta)$-contractive mappings via generalized altering distance function and related fixed point theorems. *Abstract and Applied Analysis*, **2014**, (2014).

[17] S. Czerwik, Contraction mappings in *b*-metric spaces. *Acta Math. Inf. univ. Ostraviensis*, **1**(1), (1993) 5-11.

[18] P.N. Dutta, B.S. Choudhury, A generalisation of contraction principle in metric spaces. *Fixed Point Theory and Algorithms for Sci. and Eng.*, (2008)

[19] M. Eshraghisamani, S.M. Vaezpour, M. Asadi, New fixed point results on Branciari metric spaces. *J. Math. Anal*, **8**(6), (2017), 132-141.

[20] K. Hammache, E. Karapınar, A. Ould-Hammouda, On admissible weak contractions in *b*-metric-like space. *J. Math. Anal*, 8(3), (2017), 167-180.

[21] E. Karapınar, P. Kumam, P. Salimi, On $\alpha - \psi$-Meir-Keeler contractive mappings. *Fixed Point Theory and App.*, **2013**(1), (2013), 1-12.

[22] E. Karapınar, $\alpha - \psi$-Geraghty contraction type mappings and some related fixed point results. *Filomat*, **28**(1), (2014), 37-48.

[23] E. Karapınar, Fixed point theory for cyclic weak $\varphi$-contraction. *App. Math. Letters*, **24**(6), (2011), 822-825.

[24] E. Karapınar, Quadruple fixed point theorems for weak $\varphi$-contractions. *Int. Scholarly Res. Not.*, **2011**, (2011).

[25] E. Karapınar, Weak $\varphi$-contraction on partial metric spaces, *J. of Comp. Analysis App.*, **14**(1), (2012).

[26] E. Karapınar, R.P. Agarwal, *Fixed Point Theory in Generalized Metric Spaces.* (2022) Springer Nature.

[27] M.S. Khan, M. Swaleh, S. Sessa, Fixed point theorems by altering distances between the points. *Bul. of the Australian Math. Soc.*, **30**(1), (1984), 1-9.

[28] S.G. Matthews, *Partial metric spaces*, Research Report 212, Dept. of Computer Science, University of Warwick, (1992).

[29] J.R. Morales, E.M. Rojas, Common fixed points for $(\psi - \varphi)$-weak contractions type in $b$-metric spaces. *Arabian J. of Math.*, **10**(3), (2021), 639-658.

[30] H. Monfared, M. Asadi, A. Farajzadeh, New Generalization of Darbo's Fixed Point Theorem via $\alpha$-admissible Simulation Functions with Application. *Sahand Comm. Math. Anal.*, **17**(2), (2020), 161-171.

[31] H. Monfareda, M. Asadib, M. Azhinia, $F(\psi, \varphi)$-contractions for $\alpha$-admissible mappings on metric spaces and related fixed point results. *Comm. Nonlinear An.*, **2** (2016), 86-94.

[32] F. Nikbakhtsarvestani, S.M. Vaezpour, M. Asadi, $F(\psi, \varphi)$-contraction in terms of measure of noncompactness with application for nonlinear integral equations. *J. Ineq. App.*, **2017**, (2017), 1-17.

[33] O. Popescu, Some new fixed point theorems for $\alpha$-Geraghty contraction type maps in metric spaces. *Fixed Point Theory and App.*, **2014**(1), (2014), 1-12.

[34] B. Rhoades, Some theorems on weakly contractive maps. *Nonlinear Analysis: Theory, Methods App.*, **47**(4), (2001), 2683-2693.

[35] B. Samet, C. Vetro, P. Vetro, Fixed point theorems for $\alpha - \psi$-contractive type mappings. *Nonlinear Analysis: Theory, Methods App.*, **75**(4), (2012) 2154-2165.

[36] C. Vetro, S. Chauhan, E. Karapınar, W. Shatanawi, Fixed points of weakly compatible mappings satisfying generalized $\varphi$-weak contractions. *Bul. of the Malaysian Math. Sci. Soc.*, **38**(3), (2015), 1085-1105.

[37] L. Wangwe, S. Kumar, Common Fixed Point Theorem for Hybrid Pair of Mappings in a Generalized $(F, \xi, \eta)$-contraction in weak Partial $b$-Metric Spaces with an Application. *Advances in the Theory of Nonlinear Analysis and its App.*, 5(4), (2021), 531-550.

[38] Q. Zhang, Y. Song, Fixed point theory for generalized $\varphi$-weak contractions. *App. Math. Letters*, **22**(1), (2009), 75-78.