Filomat 37:28 (2023), 9657–9668
https://doi.org/10.2298/FIL2328657G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The algebraic surfaces of the minimal-maximal surfaces

Erhan Güler[a], **Ömer Kişi**[a]

[a] *Bartın University, Faculty of Sciences, Department of Mathematics, 74100 Bartın, Turkey*

Abstract. Considering soft computing, the Weierstrass data $\left(\zeta^{-1/2}, \zeta^{1/2}\right)$ gives two different minimal surface equations and figures. By using hard computing, we give the family of minimal and spacelike maximal surfaces $\mathcal{S}_{(m,n)}$ for natural numbers m and n in Euclidean and Minkowski 3-spaces $\mathbb{E}^3$, $\mathbb{E}^{2,1}$, respectively. We obtain the classes and degrees of surfaces $\mathcal{S}_{(m,n)}$. Considering the integral free form of Weierstrass, we define some algebraic functions for $\mathcal{S}_{(m,n)}$. Indicating several maximal surfaces of value (m,n) are algebraic, we recall Weierstrass-type representations for maximal surfaces in $\mathbb{E}^{2,1}$, and give explicit parametrizations for spacelike maximal surfaces of value (m,n). Finally, we compute the implicit equations, degree, and class of the spacelike maximal surfaces $\mathcal{S}_{(0,1)}$, $\mathcal{S}_{(1,1)}$ and $\mathcal{S}_{(2,1)}$ in terms of their cartesian or inhomogeneous tangential coordinates in $\mathbb{E}^{2,1}$.

1. Introduction

Researchers and scientists, especially mathematicians and geometers, have been interested in explicit (i.e., parametric) surfaces as well as implicit (i.e., algebraic) surfaces for centuries. Although some of them have focused on explicit and implicit minimal surfaces.

Weierstrass [19] gave the explicit representation equations for the minimal surfaces. Lie [13] introduced the algebraic minimal surfaces and presented them in a table. See also [1, 3, 4, 6–11, 14, 17] for the-algebraic-minimal surfaces.

Minimal surfaces isometric to rotational surfaces in 3-dimensional Euclidean space $\mathbb{E}^3$ were introduced by Bour [1] in 1862. All such minimal surfaces are given via the well-known Weierstrass representation for minimal surfaces by choosing suitable data depending on a parameter m, as shown by Schwarz [17]. They are called Bour's minimal surfaces $\mathfrak{B}_m$ of value m. Furthermore, when m is an integer greater than 1, $\mathfrak{B}_m$ becomes algebraic, that is, there is an implicit polynomial equation satisfied by the three coordinates of $\mathfrak{B}_m$, see also [6, 14].

Kobayashi [12] considered an analogous Weierstrass-type representation for conformal spacelike maximal surfaces in Minkowski 3-space $\mathbb{E}^{2,1}$. In generally, contrary to minimal surfaces in Euclidean 3-space $\mathbb{E}^3$, maximal surfaces have singularities. See [5, 18] for the singularities of the maximal surfaces.

We introduce the real parametric minimal surfaces via Weierstrass data (ζ^m, ζ^n) for $\zeta \in \mathbb{C}$, $(m,n) \in \mathbb{N}$. If $(m,n) = (0,0)$, then we get the plane which is a first known minimal surface. Replacing natural numbers (m,n) with its negatives $(-m,-n)$, we then have the same real minimal surfaces, geometrically. Hence we

2020 *Mathematics Subject Classification*. Primary 53A10; Secondary 53C45
Keywords. 3-spaces, minimal-maximal surfaces, Weierstrass representation, degree, class, integral free representation
Received: 22 April 2023; Revised: 03 June 2023; Accepted: 04 June 2023
Communicated by Mića S. Stanković
Email addresses: eguler@bartin.edu.tr (Erhan Güler), okisi@bartin.edu.tr (Ömer Kişi)

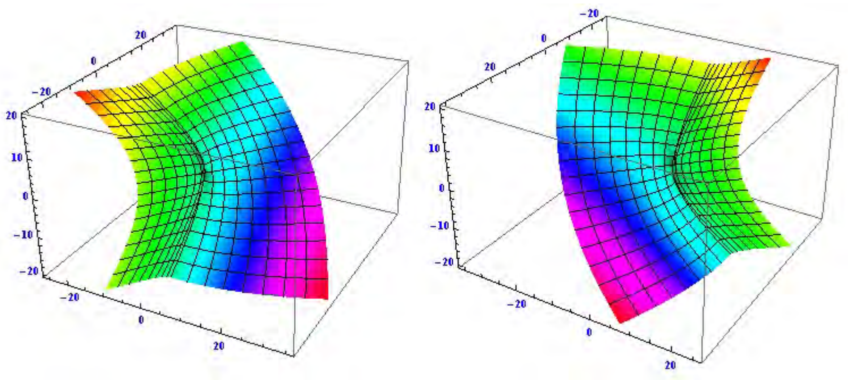

Figure 1: Minimal surfaces $\mathcal{S}_{(-1/2,1/2)}(u,v)$ **Left** $\operatorname{csgn}(i\zeta) = 1$, **Right** $\operatorname{csgn}(i\zeta) = -1$

can get the irreducible algebraic (i.e., implicit) surface equation of the parametric (i.e., explicit) minimal surface equation.

On Weierstrass data (ζ^m, ζ^n), choosing rational numbers, i.e., taking $(m,n) = (p/q, r/s)$, where gcd $(m,n) = 1$, gcd $(r,s) = 1$, we then again have minimal surface equation. See Ribaucour [16] for $m = 1/2$. The surface may not got the algebraic equation, since it has the function $\operatorname{csgn}(i\zeta) = \operatorname{csgn}(-v + iu)$, $\zeta = u + iv$, $i = \sqrt{-1}$. Here, sign function for real and complex expressions is defined by

$$\operatorname{csgn}(\zeta) = \begin{cases} 1, & \operatorname{Re}(\zeta) > 0 \text{ or } \operatorname{Re}(\zeta) = 0 \quad \text{and} \quad \operatorname{Im}(\zeta) > 0, \\ -1, & \operatorname{Re}(\zeta) < 0 \text{ or } \operatorname{Re}(\zeta) = 0 \quad \text{and} \quad \operatorname{Im}(\zeta) < 0. \end{cases}$$

Recalling the signum function (signum) returns the "sign" of a real or complex number, it is defined by $\operatorname{signum}(\zeta) = \zeta/|\zeta|$, for $\zeta \neq 0$.

Computer algorithms are constructed by numbers 0 and 1. But, in fuzzy logic systems may not be 0 and/or 1. Fuzzy logic and soft computing techniques originated by Zadeh [21, 22]. Zadeh considered the following:

"In traditional-hard-computing, the prime desiderata are precision, certainty, and rigor. By contrast, the point of departure in soft computing is the thesis that precision and certainty carry a cost and that computation, reasoning, and decision making should exploit–wherever possihlethe tolerance for imprecision and uncertainty."

Fuzzy logic has similar to above functions, interestingly. For example, $\left(\zeta^{-1/2}, \zeta^{1/2}\right)$ gives the following minimal surface equation

$$\mathcal{S}_{(-1/2,1/2)}(u,v) = \begin{pmatrix} -(1/3)\alpha u + \alpha - (1/3)\operatorname{csgn}(i\zeta)\beta v \\ (1/3)\operatorname{csgn}(i\zeta)\alpha u + \operatorname{csgn}(i\zeta)\beta - (1/3)\alpha v \\ 2u \end{pmatrix},$$

where $\alpha = (2\lambda^{1/2} + 2u)^{1/2}$, $\beta = (2\lambda^{1/2} - 2u)^{1/2}$, $\lambda = u^2 + v^2$. Here, we can choose $\operatorname{csgn}(i\zeta) = -1$ or 1. Therefore, choosing the function $\operatorname{csgn}(\zeta) = -1$ or 1, we can obtain two different (not same, but symmetric) parametric minimal surface eqs. of $\mathcal{S}_{(-1/2,1/2)}(u,v)$. See Figure 1 for two different minimal surfaces $\mathcal{S}_{(-1/2,1/2)}(u,v)$, taking $\operatorname{csgn}(i\zeta) = \pm 1$.

In addition, we can compute two different algebraic equation (like as fuzzy logic, soft computing) of the minimal surface $\mathcal{S}_{(-1/2,1/2)}(u,v)$ by using elimination methods such as Gauss, Sylvester, Gröbner, FGb of Faugere.

In this study, we opt for natural numbers (m,n) instead of rationals $(p/q, r/s)$ to avoid the issue of $\operatorname{csgn}(i\zeta) = \pm 1$.

The aim of this work is to investigate the properties of minimal and spacelike maximal surfaces in Euclidean and Minkowski 3-spaces. The Weierstrass data is used to obtain different surface equations and figures. By utilizing both soft and hard computing techniques, the family of surfaces $\mathcal{S}_{(m,n)}$ is explored, with a focus on determining their classes and degrees. Algebraic functions are defined for these surfaces using the integral-free form of Weierstrass representation. The work also provides explicit parametrizations for spacelike maximal surfaces and computes the implicit equations, degree, and class of specific surfaces, such as $\mathcal{S}_{(0,1)}$, $\mathcal{S}_{(1,1)}$, and $\mathcal{S}_{(2,1)}$ in terms of their cartesian or inhomogeneous tangential coordinates in $\mathbb{E}^{2,1}$.

In section 2, we give the family of minimal surfaces $\mathcal{S}_{(m,n)}$ for natural numbers m and n in $\mathbb{E}^3$. We obtain the classes and degrees of surfaces $\mathcal{S}_{(1,1)}$ and $\mathcal{S}_{(2,1)}$ in $\mathbb{E}^3$ in Section 3. Via the integral free form of Weierstrass, we reveal algebraic functions for $\mathcal{S}_{(m,n)}$ in $\mathbb{E}^3$ in Section 4.

In Section 5 of this work, indicating several maximal surfaces of value (m,n) are algebraic, we recall Weierstrass-type representations for maximal surfaces in $\mathbb{E}^{2,1}$, and give explicit parametrizations for spacelike maximal surfaces. In Section 6, we compute the degree, classe and algebraic equations of the maximal surfaces $\mathcal{S}_{(0,1)}$, $\mathcal{S}_{(1,1)}$ and $\mathcal{S}_{(2,1)}$ in terms of their cartesian or inhomogeneous tangential coordinates in $\mathbb{E}^{2,1}$. With the help of Weierstrass-type integral free form for the maximal surfaces, we reveal some algebraic functions in $\mathbb{E}^{2,1}$ in the last section.

2. The family of minimal surfaces $\mathcal{S}_{(m,n)}$ in $\mathbb{E}^3$

We recall a Euclidean space as follows.

Definition 2.1. $\mathbb{E}^n = \left(\{\overrightarrow{x} = (x_1, \cdots, x_n)^t | x_i \in \mathbb{R}\}, \langle\cdot,\cdot\rangle\right)$ *is the n-dimensional Euclidean space with Euclidean metric*

$$\langle \overrightarrow{x}, \overrightarrow{y} \rangle = x_1 y_1 + \cdots + x_n y_n.$$

We will often identify $\overrightarrow{x}$ and $\overrightarrow{x^t}$ without further comment, and will use (m,n) for natural numbers m and n.

Definition 2.2. *Let $\mathcal{U}$ be an open subset of $\mathbb{C}$. A minimal (or isotropic) curve is an analytic function $\Psi : \mathcal{U} \to \mathbb{C}^n$ such that $\langle \Psi'(\zeta), \Psi'(\zeta) \rangle = 0$, where $\zeta \in \mathcal{U}$, and $\Psi' := \frac{\partial \Psi}{\partial \zeta}$. In addition, if $\left\langle \Psi', \overline{\Psi'} \right\rangle = |\Psi'|^2 \neq 0$, then Ψ is a regular minimal curve.*

We then have minimal surfaces in the associated family of a minimal curve, as given by the following Weierstrass representation theorem for minimal surfaces.

Theorem 2.3. *(K. Weierstrass [19]). Let g be a meromorphic function and let ω be a holomorphic function defined on a simply connected open subset $\mathcal{U} \subset \mathbb{C}$ such that ω does not vanish on $\mathcal{U}$. Therefore,*

$$\mathbf{x}(\zeta) = Re \int \begin{pmatrix} \left(1 - g^2\right)\omega \\ i\left(1 + g^2\right)\omega \\ 2g\omega \end{pmatrix} d\zeta$$

is a conformal immersion with mean curvature identically 0 (i.e., conformal minimal surface). Conversely, any conformal minimal surface can be described in this manner.

Definition 2.4. *A pair of a meromorphic function g and a holomorphic function ω, (ω, g) is called Weierstrass data for a minimal surface.*

Lemma 2.5. *The curve*

$$\mathfrak{c}_{(m,n)}(\zeta) = \begin{pmatrix} \frac{1}{m+1}\zeta^{m+1} - \frac{1}{m+2n+1}\zeta^{m+2n+1} \\ \frac{i}{m+1}\zeta^{m+1} + \frac{i}{m+2n+1}\zeta^{m+2n+1} \\ \frac{2}{m+n+1}\zeta^{m+n+1} \end{pmatrix} \tag{1}$$

is a minimal curve in $\mathbb{C}^3$, $\zeta \in \mathbb{C}$, $i = \sqrt{-1}$.

We checked $\left\langle \mathfrak{c}'_{(m,n)}, \mathfrak{c}'_{(m,n)} \right\rangle = 0$, and then the minimal surface of value (m, n) in $\mathbb{E}^3$ is stated by

$$\mathcal{S}_{(m,n)}(\zeta) = \operatorname{Re} \int \mathfrak{c}'_{(m,n)}(\zeta)\, d\zeta. \tag{2}$$

Lemma 2.6. *The minimal surface $\mathcal{S}_{(m,n)}$is constructed by the Weierstrass data given by*

$$(\omega, g) = (\zeta^m, \zeta^n).$$

Therefore, the associated family of minimal surfaces is described by

$$\begin{aligned} \mathcal{S}(r,\theta;\beta) &= \operatorname{Re} \int e^{-i\beta} \mathfrak{c}'_{(m,n)} \\ &= \cos(\beta) \ \operatorname{Re} \int \mathfrak{c}'_{(m,n)} + \sin(\beta) \ \operatorname{Im} \int \mathfrak{c}'_{(m,n)} \\ &= \cos(\beta) \ \mathcal{S}_{(m,n)}(r,\theta) + \sin(\beta) \ \mathcal{S}^*_{(m,n)}(r,\theta). \end{aligned}$$

When $\beta = 0$ (resp., $\beta = \pi/2$), we have the surface $\mathcal{S}_{(m,n)}$ (resp., the conjugate surface $\mathcal{S}^*_{(m,n)}$).

Taking $\zeta = re^{i\theta}$, we obtain the following parametric equation of $\mathcal{S}_{(m,n)}$:

$$\mathcal{S}_{(m,n)}(r,\theta) = \begin{pmatrix} \frac{r^{m+1}\cos[(m+1)\theta]}{m+1} - \frac{r^{m+2n+1}\cos[(m+2n+1)\theta]}{m+2n+1} \\ -\frac{r^{m+1}\sin[(m+1)\theta]}{m+1} - \frac{r^{m+2n+1}\sin[(m+2n+1)\theta]}{m+2n+1} \\ \frac{2r^{m+n+1}\cos[(m+n+1)\theta]}{m+n+1} \end{pmatrix}. \tag{3}$$

Via the binomial formula, we find $\mathcal{S}_{m,n}(u, v)$:

$$\begin{aligned} x &= \operatorname{Re}\left\{ \begin{array}{c} \frac{1}{m+1}\left[\sum_{k=0}^{m+1}\binom{m+1}{k}u^{m+1-k}(iv)^k\right] \\ -\frac{1}{m+2n+1}\left[\sum_{k=0}^{m+2n+1}\binom{m+2n+1}{k}u^{m+2n+1-k}(iv)^k\right] \end{array} \right\}, \\ y &= \operatorname{Re}\left\{ \begin{array}{c} \frac{i}{m+1}\left[\sum_{k=0}^{m+1}\binom{m+1}{k}u^{m+1-k}(iv)^k\right] \\ +\frac{i}{m+2n+1}\left[\sum_{k=0}^{m+2n+1}\binom{m+2n+1}{k}u^{m+2n+1-k}(iv)^k\right] \end{array} \right\}, \\ z &= \operatorname{Re}\left\{ \frac{2}{m+n+1}\left[\sum_{k=0}^{m+n+1}\binom{m+n+1}{k}u^{m+n+1-k}(iv)^k\right] \right\}, \end{aligned} \tag{4}$$

with Gauss map

$$g = \left(\frac{2\operatorname{Re}(\zeta^n)}{\lambda^n+1}, \frac{2\operatorname{Im}(\zeta^n)}{\lambda^n+1}, \frac{\lambda^n-1}{\lambda^n+1} \right), \tag{5}$$

where $\zeta = u + iv$, $|\zeta| = \lambda = u^2 + v^2$.

It is known that the surface of $\mathcal{S}_{(0,1)}$ has class number 6, degree number 9, it is also an algebraic minimal surface. See [3, 14] for expanded results.

3. Degree and class of surface $\mathcal{S}_{(m,n)}$ in $\mathbb{E}^3$

By using polynomial elimination methods, we calculate the implicit equations, degree and class of $\mathcal{S}_{(1,1)}$ and $\mathcal{S}_{(2,1)}$. Let us now see some definitions for these surfaces.

Definition 3.1. *An algebraic function is a function $z = h(x, y)$ which satisfies $Q(x, y, h(x, y)) = 0$, where $Q(x, y, z)$ is a polynomial in x, y, and z with integer coefficients. An algebraic function is a function that can be defined as the root of a polynomial equation.*

Definition 3.2. *A polynomial is named irreducible if it cannot be factored into nontrivial polynomials over the same field.*

Definition 3.3. *The set of roots of a polynomial $Q(x, y, z) = 0$ gives an irreducible algebraic surface. An irreducible algebraic surface $Q(x, y, z) = 0$ of surface*

$$\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))$$

is said to be of degree number $\mathbf{n}$, when $\mathbf{n} = \deg(Q)$.

Definition 3.4. *The tangent plane at a point (u, v) on a surface $\mathbf{x}(u, v)$ is given by*

$$Xx + Yy + Zz + P = 0, \tag{6}$$

with the function $P = P(u, v)$, and the following Gauss map

$$g = (X(u, v), Y(u, v), Z(u, v)).$$

Definition 3.5. *By using the following*

$$a = X/P,\ b = Y/P,\ c = Z/P,$$

the surface in the inhomogeneous tangential coordinates is defined by

$$\widehat{\mathbf{x}}(u, v) = (a(u, v), b(u, v), c(u, v)).$$

Eliminating u, v, we can obtain an irreducible implicit equation $\hat{Q}(a, b, c) = 0$ of $\widehat{\mathbf{x}}(u, v)$ in inhomogeneous tangential coordinates. See [2] for elinimation methods.

Definition 3.6. *The maximum degree of the $\hat{Q}(a, b, c) = 0$ gives the class number of surface $\widehat{\mathbf{x}}(u, v)$.*

See [14], for details.

3.1. *Degree and class of surface $\mathcal{S}_{(1,1)}$ in $\mathbb{E}^3$*

The simplest Weierstrass representation $(\omega, g) = (\zeta, \zeta)$ gives the minimal surface of value $(1, 1)$. In polar coordinates, the parametric equations of $\mathcal{S}_{(1,1)}$ are

$$\mathcal{S}_{(1,1)}(r, \theta) = \begin{pmatrix} \frac{r^2}{2}\cos(2\theta) - \frac{r^4}{4}\cos(4\theta) \\ -\frac{r^2}{2}\sin(2\theta) - \frac{r^4}{4}\sin(4\theta) \\ \frac{2}{3}r^3\cos(3\theta) \end{pmatrix}, \tag{7}$$

where $r \in I \subset \mathbb{R}$, $\theta \in [0, 2\pi)$. The parametric form of the surface in (u, v) coordinates, is given by

$$\mathcal{S}_{(1,1)}(u, v) = \begin{pmatrix} -\frac{u^4}{4} - \frac{v^4}{4} + \frac{3}{2}u^2v^2 + \frac{u^2}{2} - \frac{v^2}{2} \\ -u^3v + uv^3 - uv \\ \frac{2}{3}u^3 - 2uv^2 \end{pmatrix} = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}, \tag{8}$$

where $u, v \in \mathbb{R}$. Eliminating (u, v) of (8), we find the irreducible implicit equation of surface $\mathcal{S}_{(1,1)}$ as follows

$$\begin{aligned} Q_{(1,1)}(x, y, z) \quad = \quad & 3^{16}z^{16} - 2^{17}3^8x^4z^6 - 2^{20}3^6x^4y^2z^4 \\ & -2^{18}3^8x^2y^2z^6 + 2^{21}3^5x^2y^4z^4 \\ & +69 \text{ other lower degree terms.} \end{aligned}$$

Its degree number is 16. Therefore, $Q_{(1,1)}(x, y, z) = 0$ is an algebraic minimal surface.

Finding the class of surface $\mathcal{S}_{(1,1)}$, we obtain the following function

$$P(u,v) = \frac{(\lambda+2)(3uv^2-u^3)}{6(\lambda+1)},$$

where $\lambda = u^2+v^2$. Then, in inhomogeneous tangential coordinates a,b,c, we find the following surface

$$\widehat{\mathcal{S}}_{(1,1)}(u,v) = \frac{6}{(\lambda+2)(3uv^2-u^3)}\begin{pmatrix} 2u \\ 2v \\ \lambda-1 \end{pmatrix} = \begin{pmatrix} a(u,v) \\ b(u,v) \\ c(u,v) \end{pmatrix}.$$

Hence, the irreducible implicit equation of $\widehat{\mathcal{S}}_{(1,1)}(u,v)$ is given by

$$\begin{aligned} \hat{Q}_{(1,1)}(a,b,c) \;=\; & 9a^8+72a^7+144a^6+288a^5c^2+192a^3c^4 \\ & +8a^6c^2-48a^4b^2c^2-576ab^2c^4+81a^2b^6 \\ & +432a^4b^2-45a^6b^2-72a^5b^2+432a^2b^4 \\ & -360a^3b^4-216ab^6+27a^4b^4+144b^6 \\ & -576a^3b^2c^2+72a^2b^4c^2-864ab^4c^2. \end{aligned}$$

Therefore, the class number of the algebraic minimal surface $\hat{Q}_{(1,1)}(a,b,c)=0$ is 8.

3.2. *Degree and class of surface $\mathcal{S}_{(2,1)}$ in $\mathbb{E}^3$*

The parametric form of $\mathcal{S}_{(2,1)}$ is given by

$$\mathcal{S}_{(2,1)}(r,\theta) = \begin{pmatrix} \frac{r^3}{3}\cos(3\theta) - \frac{r^5}{5}\cos(5\theta) \\ -\frac{r^3}{3}\sin(3\theta) - \frac{r^5}{5}\sin(5\theta) \\ \frac{1}{2}r^4\cos(4\theta) \end{pmatrix}, \tag{9}$$

where $r \in I \subset \mathbb{R}$, $\theta \in [0,2\pi)$. In (u,v) coordinates, $\mathcal{S}_{(2,1)}$ has the form as follows

$$\mathcal{S}_{(2,1)}(u,v) = \begin{pmatrix} \frac{1}{3}u^3 - uv^2 - \frac{1}{5}u^5 + 2u^3v^2 - uv^4 \\ -u^2v + \frac{1}{3}v^3 - u^4v + 2u^2v^3 - \frac{1}{5}v^5 \\ \frac{1}{2}u^4 - 3u^2v^2 + \frac{1}{2}v^4 \end{pmatrix}, \tag{10}$$

where $u,v \in \mathbb{R}$. Eliminating u and v, we reveal the implicit equation of $\mathcal{S}_{(2,1)}(u,v)$ as follows

$$\begin{aligned} Q_{(2,1)}(x,y,z) \;=\; & 2^{50}3^{16}z^{25} - 2^{40}3^{16}5^5x^4z^{20} + 2^{41}3^{17}5^5x^2y^2z^{20} \\ & -2^{40}3^{16}5^5y^4z^{20} - 2^{28}3^{18}5^9 13x^8z^{15} \\ & +233 \text{ other lower degree terms.} \end{aligned}$$

Hence, $Q_{(2,1)}(x,y,z)=0$ is an algebraic minimal surface. Its degree number is 25.

Obtaining the class of surface $\mathcal{S}_{(2,1)}$, we have the following function

$$P(u,v) = \frac{(3\lambda+5)\left(v^4+6u^2v^2-u^4\right)}{30(\lambda+1)}.$$

Then, we compute the following surface in inhomogeneous tangential coordinates a,b,c,

$$\widehat{\mathcal{S}}_{(2,1)}(u,v) = \frac{30}{(3\lambda+5)(v^4+6u^2v^2-u^4)}\begin{pmatrix} 2u \\ 2v \\ \lambda-1 \end{pmatrix},$$

where $\lambda = u^2 + v^2$.

Eliminating u and v, we have the following irreducible implicit equation of $\widehat{\mathcal{S}}_{(2,1)}(u,v)$ in inhomogeneous tangential coordinates a, b, c,

$$\begin{aligned}
\hat{Q}_{(2,1)}(a,b,c) \quad = \quad & 16a^{10} + 16b^{10} + 900a^8c + 3600a^6b^2c \\
& +15b^8c^2 - 180a^2c^2 + 416a^4b^6 - 900a^8 \\
& -3600a^2b^6 - 3600a^6b^2 + 8640a^2b^2c^5 \\
& -176a^2b^8 - 5400a^4b^4 + 416a^6b^4 \\
& -900b^8c - 900b^8 + 3600a^2b^6c + 15a^8c^2 \\
& -1440b^4c^5 - 1440a^4c^5 - 2400b^6c^3 \\
& +12000a^4b^2c^3 - 176a^8b^2 - 180a^6b^2c^2 \\
& -2400a^6c^3 + 12000a^2b^4c^3 - 180a^2b^6c^2 \\
& -9000a^4b^4c + 570a^4b^4c^2.
\end{aligned}$$

Hence, the class number of the algebraic minimal surface $\hat{Q}_{(2,1)}(a,b,c) = 0$ is 10.

4. Integral free form in $\mathbb{E}^3$

We recall the following integral free form of Weierstrass [20].

Theorem 4.1. *Integral free form of the Weierstrass representation is defined by*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathrm{Re} \begin{pmatrix} \left(1 - w^2\right)\varkappa''(w) + 2w\varkappa'(w) - 2\varkappa(w) \\ i\left[\left(1 + w^2\right)\varkappa''(w) - 2w\varkappa'(w) + 2\varkappa(w)\right] \\ 2w\varkappa''(w) - 2\varkappa'(w) \end{pmatrix} \equiv \mathrm{Re} \begin{pmatrix} \rho_1(w) \\ \rho_2(w) \\ \rho_3(w) \end{pmatrix}, \tag{11}$$

where the algebraic function $\varkappa = \varkappa(w)$ and the functions $\rho_i = \rho_i(w)$ are connected by the following relation

$$4\varkappa = \left(w^2 - 1\right)\rho_1 - i\left(w^2 + 1\right)\rho_2 - 2w\rho_3 \tag{12}$$

for $w \in \mathbb{C}$.

On the other hand, integral free form equations (11) and (12) are suitable for algebraic minimal surfaces. Then, we present the following:

Corollary 4.2. *The algebraic function $\varkappa(w) = \frac{1}{6}w^3$ gives rise to minimal surface $\mathcal{S}_{(0,1)}$.*

Corollary 4.3. *The algebraic function $\varkappa(w) = \frac{1}{24}w^4$ leads to minimal surface $\mathcal{S}_{(1,1)}$.*

Corollary 4.4. *The algebraic function $\varkappa(w) = \frac{1}{60}w^5$ gives the minimal surface $\mathcal{S}_{(2,1)}$.*

Finally we give the following.

Conjecture 4.5. *The algebraic function $\varkappa(w) = \frac{1}{(m+n+2)!}w^{m+n+2}$ describes the minimal surface $\mathcal{S}_{(m,n)}$.*

5. Spacelike maximal surfaces $\mathcal{S}_{(m,n)}$ in $\mathbb{E}^{2,1}$

We briefly provide some notions of Minkowski space as follows.

Definition 5.1. $\mathbb{E}^{n,1} = \left(\{\overrightarrow{x} = (x_1, \cdots, x_n, x_0)^t | x_i \in \mathbb{R}\}, \langle\cdot,\cdot\rangle\right)$ *is the $(n+1)$-dimensional Lorentz-Minkowski (for short, Minkowski) space with Lorentz metric*

$$\langle\overrightarrow{x}, \overrightarrow{y}\rangle = x_1y_1 + \cdots + x_ny_n - x_0y_0.$$

Definition 5.2. *A vector $\overrightarrow{x} \in \mathbb{E}^{n,1}$ is called*

$$\begin{array}{lll} \textit{spacelike} & \textit{if} & \left\langle \overrightarrow{x}, \overrightarrow{x} \right\rangle > 0, \textit{ or } \overrightarrow{x} = 0, \\ \textit{timelike} & \textit{if} & \left\langle \overrightarrow{x}, \overrightarrow{x} \right\rangle < 0, \\ \textit{lightlike} & \textit{if} & \left\langle \overrightarrow{x}, \overrightarrow{x} \right\rangle = 0 \textit{ and } \overrightarrow{x} \neq 0. \end{array}$$

Definition 5.3. *A surface in $\mathbb{E}^{n,1}$ is called spacelike (resp. timelike, lightlike) if the induced metric on the tangent planes is a positive definite Riemannian (resp. Lorentzian, degenerate) metric.*

See [15] for details.

Kobayashi [12] found a Weierstrass-type representation for spacelike conformal maximal surfaces in $\mathbb{E}^{2,1}$.

Theorem 5.4. *Let g be a meromorphic function and let ω be a holomorphic function defined on a simply connected open subset $\mathcal{U} \subset \mathbb{C}$ such that ω does not vanish on $\mathcal{U}$. Then*

$$\mathbf{x}(\zeta) = Re \int \begin{pmatrix} \left(1 + g^2\right)\omega \\ i\left(1 - g^2\right)\omega \\ 2g\omega \end{pmatrix} d\zeta$$

is a spacelike conformal immersion with mean curvature identically 0 (i.e. spacelike conformal maximal surface). Conversely, any spacelike conformal maximal surface can be described in this manner.

Definition 5.5. *A pair of a meromorphic function g and a holomorphic function ω, (ω, g) is called Weierstrass data for a maximal surface.*

We call maximal surfaces $\mathcal{S}_{(m,n)}$ $(m, n \in \mathbb{N})$ given by Weierstrass data $(\omega, g) = (\zeta^m, \zeta^n)$ the spacelike maximal surfaces of value (m, n). The parametrization of spacelike $\mathcal{S}_{(m,n)}$ is given by

$$\begin{aligned} x &= \operatorname{Re}\left\{ \begin{array}{c} \frac{1}{m+1}\left[\sum_{k=0}^{m+1} \binom{m+1}{k} u^{m+1-k} (iv)^k\right] \\ +\frac{1}{m+2n+1}\left[\sum_{k=0}^{m+2n+1} \binom{m+2n+1}{k} u^{m+2n+1-k} (iv)^k\right] \end{array} \right\}, \\ y &= \operatorname{Re}\left\{ \begin{array}{c} \frac{i}{m+1}\left[\sum_{k=0}^{m+1} \binom{m+1}{k} u^{m+1-k} (iv)^k\right] \\ -\frac{i}{m+2n+1}\left[\sum_{k=0}^{m+2n+1} \binom{m+2n+1}{k} u^{m+2n+1-k} (iv)^k\right] \end{array} \right\}, \\ z &= \operatorname{Re}\left\{ \frac{2}{m+n+1}\left[\sum_{k=0}^{m+n+1} \binom{m+n+1}{k} u^{m+n+1-k} (iv)^k\right] \right\} \end{aligned} \tag{13}$$

with Gauss map

$$g = \left(\frac{2\operatorname{Re}(\zeta^n)}{1 - \lambda^n}, \frac{2\operatorname{Im}(\zeta^n)}{1 - \lambda^n}, \frac{1 + \lambda^n}{1 - \lambda^n} \right),$$

where $\zeta = u + iv$, $|\zeta| = \lambda = u^2 + v^2$.

6. Degree and class of surfaces $\mathcal{S}_{(m,n)}$ in $\mathbb{E}^{2,1}$

In $\mathbb{E}^{2,1}$, the tangent plane at a point (u, v) on a surface $\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))$ given by

$$Xx + Yy - Zz + P = 0, \tag{14}$$

where $P = P(u, v)$, and the Gauss map of the surface $\mathbf{x}$ indicated by

$$g = (X(u, v), Y(u, v), Z(u, v)).$$

Next, we calculate the explicit and implicit equations, degree and class numbers of the spacelike maximal surfaces $\mathcal{S}_{(0,1)}$, $\mathcal{S}_{(1,1)}$, and $\mathcal{S}_{(2,1)}$.

6.1. Degree and class of spacelike $\mathcal{S}_{(0,1)}$, $\mathcal{S}_{(1,1)}$, $\mathcal{S}_{(2,1)}$ in $\mathbb{E}^{2,1}$

From (13), the parametrization of spacelike maximal surface $\mathcal{S}_{(0,1)}$ is given by

$$\mathcal{S}_{(0,1)}(u,v) = \begin{pmatrix} \frac{1}{3}u^3 - uv^2 + u \\ u^2v - \frac{1}{3}v^3 - v \\ u^2 - v^2 \end{pmatrix},$$

where $u, v \in \mathbb{R}$. $Q_{(m,n)}(x,y,z) = 0$ denotes the irreducible implicit equation that spacelike $\mathcal{S}_{(m,n)}$ will satisfy.

We get the following algebraic eq. of the spacelike maximal surface $\mathcal{S}_{(0,1)}(u,v)$:

$$\begin{aligned} Q_{(0,1)}(x,y,z) \;=\;& 64z^9 - 432x^2z^6 + 432y^2z^6 - 1215x^4z^3 \\ & -6318x^2y^2z^3 + 3888x^2z^5 - 1215y^4z^3 \\ & +3888y^2z^5 - 1152z^7 - 729x^6 + 2187x^4y^2 \\ & +4374x^4z^2 - 2187x^2y^4 - 6480x^2z^4 + 729y^6 \\ & -4374y^4z^2 + 6480y^2z^4 + 729x^4z - 1458x^2y^2z \\ & -3888x^2z^3 + 729y^4z - 3888y^2z^3 + 5184z^5. \end{aligned}$$

Its degree number is 9. Therefore, $Q_{(0,1)}(x,y,z) = 0$ is an algebraic maximal surface.

Revealing the class number of $\mathcal{S}_{(0,1)}$, we obtain

$$P_{(0,1)}(u,v) = -\frac{(\lambda-3)(u^2-v^2)}{3(\lambda-1)},$$

where $P_{(0,1)}(u,v)$ indicates the function as in equation (14). The surface in inhomogeneous tangential coordinates a, b, c, is given by

$$\widehat{\mathcal{S}}_{(0,1)}(u,v) = \frac{3}{(\lambda-3)(u^2-v^2)} \begin{pmatrix} 2u \\ 2v \\ \lambda+1 \end{pmatrix},$$

where $\lambda = u^2 + v^2$. In the inhomogeneous tangential coordinates a, b, c, we reveal the following algebraic surface

$$\begin{aligned} \hat{Q}_{(0,1)}(a,b,c) \;=\;& 4a^6 + 9a^4 + 9b^4 + 6a^2b^2c^2 \\ & -3b^4c^2 - 18b^4c - 4a^4b^2 \\ & -12a^2c^3 - 4a^2b^4 - 3a^4c^2 \\ & +18a^2b^2 - 4a^4b^2 + 4b^6 \\ & +12b^2c^3 + 18a^4c. \end{aligned}$$

Here, $\hat{Q}_{(0,1)}(a,b,c) = 0$ shows the irreducible algebraic equation for spacelike $\mathcal{S}_{(m,n)}$ in terms of inhomogeneous tangential coordinates. Then, the class number of the algebraic surface $\hat{Q}_{(0,1)}(a,b,c) = 0$ is 6.

By using the similar techniques, we get the following parametric eqs.

$$\begin{aligned} \mathcal{S}_{(1,1)}(u,v) &= \begin{pmatrix} \frac{u^4}{4} + \frac{v^4}{4} - \frac{3}{2}u^2v^2 + \frac{u^2}{2} - \frac{v^2}{2} \\ u^3v - uv^3 - uv \\ \frac{2}{3}u^3 - 2uv^2 \end{pmatrix}, \\ \mathcal{S}_{(2,1)}(u,v) &= \begin{pmatrix} \frac{1}{3}u^3 - uv^2 + \frac{1}{5}u^5 - 2u^3v^2 + uv^4 \\ -u^2v + \frac{1}{3}v^3 + u^4v - 2u^2v^3 + \frac{1}{5}v^5 \\ \frac{1}{2}u^4 - 3u^2v^2 + \frac{1}{2}v^4 \end{pmatrix}, \end{aligned}$$

and the following algebraic eqs.

$$\begin{aligned}
Q_{(1,1)}(x,y,z) &= -3^{16}z^{16} \\
&\quad +2^{9}3^{12}x^{3}z^{12} \\
&\quad -2^{9}3^{13}xy^{2}z^{12} \\
&\quad +2^{16}3^{8}7x^{6}z^{8} \\
&\quad +2^{15}3^{9}23x^{4}y^{2}z^{8} \\
&\quad +69 \text{ other lower degree terms,}
\end{aligned}$$

$$\begin{aligned}
Q_{(2,1)}(x,y,z) &= -2^{45}3^{16}z^{25} \\
&\quad +2^{36}3^{16}5^{5}x^{4}z^{20} \\
&\quad -2^{7}3^{16}5^{18}1151x^{4}y^{12}z^{5} \\
&\quad -2^{37}3^{17}5^{5}x^{2}y^{2}z^{20} \\
&\quad -2^{6}3^{15}5^{21}7x^{12}y^{6}z^{2} \\
&\quad +233 \text{ other lower degree terms.}
\end{aligned}$$

Their degree numbers are 16, 25, respectively. Therefore, $Q_{(1,1)}(x,y,z) = 0$ and $Q_{(2,1)}(x,y,z) = 0$ are the algebraic surfaces.

Furthermore, we have the following functions

$$\begin{aligned}
P_{(1,1)}(u,v) &= \frac{(\lambda-2)(u^2-3v^2)}{(\lambda-1)}, \\
P_{(2,1)}(u,v) &= \frac{(3\lambda-5)(\left(u^2-v^2\right)^2-4u^2v^2)}{30(\lambda-1)}.
\end{aligned}$$

After some computations, in the inhomogeneous tangential coordinates, we reveal the following surfaces

$$\begin{aligned}
\widehat{\mathcal{S}}_{(1,1)}(u,v) &= \frac{6}{u(\lambda-2)(u^2-3v^2)}\begin{pmatrix} 2u \\ 2v \\ \lambda+1 \end{pmatrix}, \\
\widehat{\mathcal{S}}_{(2,1)}(u,v) &= \frac{30}{(3\lambda-5)((u^2-v^2)^2-4u^2v^2)}\begin{pmatrix} 2u \\ 2v \\ \lambda+1 \end{pmatrix},
\end{aligned}$$

where $\lambda = u^2 + v^2$.

Hence, we find the following algebraic surfaces, respectively,

$$\begin{aligned}
\hat{Q}_{(1,1)}(a,b,c) &= 9a^8 + 72a^6b^2 - 8a^6c^2 + 144a^4b^4 - 168a^4b^2c^2 \\
&\quad -96a^2b^4c^2 + 96a^2b^2c^4 + 64b^6c^2 - 48b^4c^4 \\
&\quad -72a^7 - 288a^5b^2 + 288a^5c^2 + 288a^3b^2c^2 \\
&\quad -192a^3c^4 + 144a^6,
\end{aligned}$$

$$\begin{aligned}
\hat{Q}_{(2,1)}(a,b,c) &= -16a^{10} - 8640a^2b^2c^5 - 9000a^4b^4c - 3600a^2b^6c \\
&\quad +12000a^2b^4c^3 + 570a^4b^4c^2 - 180a^2b^6c^2 + 15b^8c^2 \\
&\quad -900b^8 + 1440a^4c^5 + 1440b^4c^5 - 5400a^4b^4 \\
&\quad -3600a^2b^6 + 900b^8c - 2400b^6c^3 - 416a^6b^4 \\
&\quad -416a^4b^6 + 176a^2b^8 - 16b^{10} + 12000a^4b^2c^3 \\
&\quad -3600a^6b^2c - 180a^6b^2c^2 - 3600a^6b^2 + 176a^8b^2 \\
&\quad -2400a^6c^3 + 900a^8c + 15a^8c^2 - 900a^8.
\end{aligned}$$

Therefore, we have the following class numbers 8 and 10, respectively, for the algebraic surfaces $\hat{Q}_{(1,1)}(a,b,c) = 0$ and $\hat{Q}_{(2,1)}(a,b,c) = 0$, respectively.

7. Integral free form in $\mathbb{E}^{2,1}$

We consider the integral free form in $\mathbb{E}^{2,1}$.

Theorem 7.1. *Integral free form of the Weierstrass representation for the surfaces is defined by*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \operatorname{Re}\begin{pmatrix} \left(1+w^2\right)\sigma''(w) - 2w\sigma'(w) + 2\sigma(w) \\ i\left(1-w^2\right)\sigma''(w) + 2iw\sigma'(w) - 2i\sigma(w) \\ 2w\sigma''(w) - 2\sigma'(w) \end{pmatrix} \equiv \operatorname{Re}\begin{pmatrix} \tau_1(w) \\ \tau_2(w) \\ \tau_3(w) \end{pmatrix}. \tag{15}$$

Here, the algebraic function $\sigma = \sigma(w)$ and the functions $\tau_i = \tau_i(w)$ are related by

$$4\sigma = \left(w^2+1\right)\tau_1 - i\left(w^2-1\right)\tau_2 - 2w\tau_3 \tag{16}$$

for $w \in \mathbb{C}$.

On the other hand, integral free form equations (15) and (16) are suitable for algebraic spacelike maximal surfaces.

Then, we have the following results.

Corollary 7.2. *The algebraic function $\sigma(w) = \frac{1}{6}w^3$ gives maximal surface $\mathcal{S}_{(0,1)}$.*

Corollary 7.3. *The algebraic function $\sigma(w) = \frac{1}{24}w^4$ gives rise to maximal surface $\mathcal{S}_{(1,1)}$.*

Corollary 7.4. *The algebraic function $\sigma(w) = \frac{1}{60}w^5$ leads to the maximal surface $\mathcal{S}_{(2,1)}$.*

Finally, we serve the following.

Conjecture 7.5. *The algebraic function $\sigma(w) = \frac{1}{(m+n+2)!}w^{m+n+2}$ determines the maximal surface $\mathcal{S}_{(m,n)}$.*

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