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On the Gaussian curvature of timelike surfaces in Lorentz-Minkowski 3-space

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Abstract. In this study, the various expressions of the Gaussian curvature of timelike surfaces whose parameter curves intersect under any angle are investigated and the Enneper formula is obtained in Lorentz-Minkowski 3-space. By giving an example for these surfaces, the graphs of the surface and its Gaussian curvature are drawn.

1. Introduction

Euclidean geometry, which ruled for thousands of years, left its place to new geometries with the discovery of non-Euclidean geometries that did not provide the parallelism axiom, [1]. Euclidean geometry makes sense for plane geometry, but the shape of the Earth we live on conforms to non-Euclidean geometry. Mathematicians such as Lobachevski, Gauss, Bolyai and Riemann, who set out with this idea, agreed on non-Euclidean geometries. Thus, new geometries emerged, such as Riemann (elliptical) geometry and Lobachevsky (hyperbolic) geometry. These geometries are the most used by NASA today. Even Einstein's theory of relativity is a product of Riemannian geometry. One of the broad areas in the study of Euclidean or non-Euclidean geometry is differential geometry. The most popular subject of differential geometry is the theory of surfaces. One of the tools used when examining the geometry of a surface is the curvature of the surface. The method of calculating the curvature of a surface was defined by Carl Gauss in the 19th century and is therefore called Gaussian curvature. To explain Gauss's method of calculating surface curvature, it is first necessary to understand how the curvature of curves is calculated. Accordingly, the curvature of a curve is inversely proportional to the radius of the circle of curvature at a point on it. Therefore, the smaller the radius, the larger the curvature, while the larger the radius, the smaller the curvature. That is, since the radius of the circle of curvature of the line is infinite, its curvature is zero. The product of the principal curvatures at a point on a surface gives the Gaussian curvature of the surface at that point. If one of the curvatures is zero, it means that the Gaussian curvature of the surface at that point is zero. If a surface has zero Gaussian curvature, the surface is developable. Therefore, surfaces with zero Gaussian curvature can be said to be isomorphic to the plane. The Gaussian curvature of surfaces that have a nonzero curvature are expressed as either positive or negative. In cases where both circles of curvature are on the same side of the surface, there is positive Gaussian curvature at that point, and negative when one is on the opposite side. On a surface, the sum of the interior angles of the triangle in areas with zero Gaussian curvature is 180

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degrees, the sum of the interior angles of the triangle in areas with positive Gaussian curvature is more than 180 degrees, and the sum of the interior angles of the triangle in areas with negative Gaussian curvature is less than 180 degrees, Figure (1).

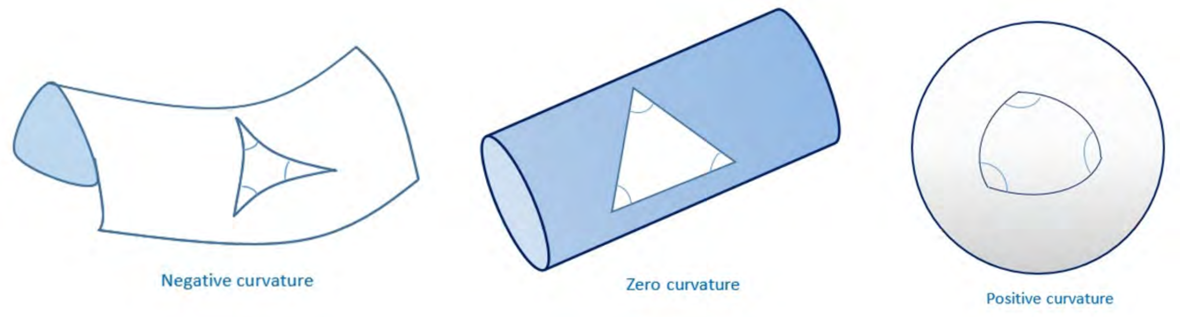

Figure 1: Interpreting the Gaussian curvature's value

Points with zero Gaussian curvature on a surface are called "parabolic points", points with positive Gaussian curvature are called "elliptical points" and points with negative Gaussian curvature are called "hyperbolic points", [2]. Thus, if the Gaussian curvature at each point of a surface is zero, that is the surface is flat, the surface is related to Euclidean geometry, if the curvature at each point of the surface is positive, the surface is related to Riemannian geometry, and if the curvature at each point of the surface is negative, the surface is related to Lobachevsky geometry. The subject of surfaces is also important in architectural studies. Surfaces built in architectural studies may have only zero, only positive and only negative Gaussian curvature, as well as these three situations can be found in different parts of the surface, [3]. Nordpark Train Station (Austria), built by architect Zaha Hadid Architects, can be given as an example of structures with three different Gaussian curvatures in different parts of its surface, Figure (2).

Figure 2: Nordpark Train Station

(Accessed on 13.11.2017, used under Creative Commons CC0 license, https://pixabay.com/en/architecture-modern-zaha-hadid-1618100/)

The Lorentz-Minkowski geometry is one of the best known of the non-Euclidean geometries, [4–8]. There are many studies on curves or surfaces in Lorentz-Minkowski 3-space [9–23], which is named with the defined Lorentz metric

$$\langle , \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}\,, \quad \langle z, y \rangle = z_1 y_1 + z_2 y_2 - z_3 y_3,$$

where $z = (z_1,\ z_2,\ z_3)$ and $y = (y_1,\ y_2,\ y_3) \in \mathbb{R}^3$. Some of the studies on the Gaussian curvature of surfaces in this space are [3, 24–29]. In this study, the different expressions of the Gaussion curvature of timelike surfaces which are examined under six cases in [30] is examined and Enneper formula is obtained for all cases.

2. Preliminaries

The causal character of a vector z in Lorentz-Minkowski 3-space is timelike if $\langle z, z\rangle < 0$, spacelike if $\langle z, z\rangle > 0$ or $z = 0$ and lightlike if $\langle z, z\rangle = 0, z \neq 0$, [6]. Similarly, the causal character of the all tangent vectors of any curve is the same as the causal character of the curve. As for surfaces, they are named according to the characters of their normal vectors at each point. If The normal vector at each point is timelike (spacelike), the surface is spacelike (timelike), respectively. Addition, the norm of the vector z is $\|z\| = \sqrt{|\langle z, z\rangle|}$. If $\|z\| = 1$, z is a unit vector. The unit timelike vectors create the hyperbolic unit sphere $\mathbb{H}_0^2 = \left\{ z \in \mathbb{R}_1^3 \mid \langle z, z\rangle = -1 \right\}$ and the unit spacelike vectors create the Lorentz unit sphere $\mathbb{S}_1^2 = \left\{ z \in \mathbb{R}_1^3 \mid \langle z, z\rangle = 1 \right\}$. For $z, y \in \mathbb{R}_1^3$, if $\langle z, y\rangle = 0$, the vectors z and y are called Lorentzian ortgonal vectors. Besides, just like the inner product function, the vectorial product function in this space is defined differently from that in Euclidean space also:

$$\wedge : \quad \mathbb{R}_1^3 \times \mathbb{R}_1^3 \to \mathbb{R}_1^3 , \quad z \wedge y = -\begin{vmatrix} e_1 & e_2 & -e_3 \\ z_1 & z_2 & z_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

This fuction is called Lorentz vectorial product of z and y vectors. The causal characters of the vectors in $\mathbb{R}_1^3$ also affect the result of the vector products of these vectors with each other, and this effect causes various cases on a timelike surface. Depending on these cases, the Darboux frame is shaped in various ways according to the characters of the elements that creat the frame. For example, let's take the Darboux frame $\{t, g, N\}$ on a timelike surface. Here the normal vector N is a spacelike, by definition of a timelike surface. Let's assume that vector g is also spacelike. So, this vector is defined by $g = -N \wedge t$, here the vector t is timelike. In the present case, Darboux vector of this frame is [21]

$$w = \frac{t}{T_g} + \frac{g}{R_n} - \frac{N}{R_g}. \tag{1}$$

Let the parameter curves (c_1) and (c_2) of a timelike surface be two curves that intersect at any angle. Let any curve (c) pass through the point where (c_1) and (c_2) intersect. Six different cases appear on these surface with different combinations of the characters of the elements of the Darboux frames of (c), (c_1) and (c_2), [30]. Many equations have been obtained for these six cases on the surface. I do not see any harm in adding only the following ones, which will be used in this study, to this section. There is the equation below between the Darboux vectors w, w_1 and w_2 of (c) , (c_1) and (c_2) on the timelike surface, respectively,
Case 1.

$$w = -\frac{\sinh\left(\theta - \phi\right)}{\cosh\theta} w_1 + \frac{\cosh\phi}{\cosh\theta} w_2 - \frac{d\phi}{ds} N. \tag{2}$$

For other cases, you can examine the paper [30]. For radii of principal curvature R_1 and R_2 of (c_1) and (c_2) and radii of geodesic and normal be R_n and T_g of the surface, there are the equations below:
Case 1.

$$\frac{\sinh\left(\theta - \phi\right)}{R_1} = \frac{\sinh\left(\theta - \phi\right)}{R_n} - \frac{\cosh\left(\theta - \phi\right)}{T_g}, \tag{3}$$

$$\frac{\cosh\phi}{R_2} = \frac{\cosh\phi}{R_n} + \frac{\sinh\phi}{T_g}, \tag{4}$$

Case 2.

$$\frac{\cosh\left(\theta - \phi\right)}{R_1} = \frac{\cosh\left(\theta - \phi\right)}{(R_n)_0} + \frac{\sinh\left(\theta - \phi\right)}{\left(T_g\right)_0}, \tag{5}$$

$$\frac{\sinh\phi}{R_2}=\frac{\sinh\phi}{(R_n)_0}-\frac{\cosh\phi}{\left(T_g\right)_0}. \tag{6}$$

For other cases, you can examine the paper [30]. Let radii of normal curvature be $(R_n)_1$, $(R_n)_2$ and radii of geodesic torsion be $\left(T_g\right)_1$, $\left(T_g\right)_2$ of the curves (c_1) and (c_2) on $x = x(u, v)$, respectively. For the geodesic torsion at the direction t, there are following equations:

$$\textit{Case 1.}\ \frac{1}{T_g}=\frac{1}{\cosh\theta}\left[-\frac{\sinh(\theta-\phi)\sinh\phi}{\left(T_g\right)_1}+\frac{\cosh(\theta-\phi)\cosh\phi}{\left(T_g\right)_2}-\frac{\sinh(\theta-\phi)\cosh\phi}{(R_n)_1}+\frac{\sinh(\theta-\phi)\cosh\phi}{(R_n)_2}\right], \tag{7}$$

$$\textit{Case 2.}\ \frac{1}{\left(T_g\right)_0}=\frac{1}{\cosh\theta}\left[\frac{\cosh(\theta-\phi)\cosh\phi}{\left(T_g\right)_1}-\frac{\sinh(\theta-\phi)\sinh\phi}{\left(T_g\right)_2}+\frac{\cosh(\theta-\phi)\sinh\phi}{(R_n)_1}-\frac{\cosh(\theta-\phi)\sinh\phi}{(R_n)_2}\right]. \tag{8}$$

For other cases, you can examine the paper [11]. Let radii of normal curvature be $(R_n)_1$, $(R_n)_2$ and radii of geodesic torsion be $\left(T_g\right)_1$, $\left(T_g\right)_2$ of the curves (c_1) and (c_2) on $x(u, v)$, respectively. For the normal curvature at the direction t of the surface, we get:

$$\textit{Case 1.}\ \frac{1}{R_n}=\frac{1}{\cosh\theta}\left[\left(\frac{1}{\left(T_g\right)_1}+\frac{1}{\left(T_g\right)_2}\right)\sinh(\theta-\phi)\cosh\phi+\frac{\sinh(\theta-\phi)\sinh\phi}{(R_n)_1}+\frac{\cosh(\theta-\phi)\cosh\phi}{(R_n)_2}\right], \tag{9}$$

$$\textit{Case 2.}\ \frac{1}{(R_n)_0}=\frac{1}{\cosh\theta}\left[\left(\frac{1}{\left(T_g\right)_1}+\frac{1}{\left(T_g\right)_2}\right)\cosh(\theta-\phi)\sinh\phi+\frac{\cosh(\theta-\phi)\cosh\phi}{(R_n)_1}+\frac{\sinh(\theta-\phi)\sinh\phi}{(R_n)_2}\right]. \tag{10}$$

For other cases, you can examine the paper [11]. Let radii of principal curvature of (c_1) and (c_2) be R_1 and R_2, the radius of normal curvature be R_n, the radius of geodesic torsion of (c) be T_g and the radius of geodesic torsion of (c_0) perpendicular to (c) be $\left(T_g\right)_0$ on $x(u, v)$, respectively. Then, we get:

$$\textit{Case 1.}\quad \left(\frac{1}{R_n}-\frac{1}{R_1}\right)\left(\frac{1}{R_n}-\frac{1}{R_2}\right)=-\frac{\cosh(\theta-\phi)\sinh\phi}{\sinh(\theta-\phi)\cosh\phi}\,\frac{1}{T_g^2}\,. \tag{11}$$

For other cases, you can examine the paper [11].

3. On Gaussian Curvature of Timelike Surfaces in Lorentz-Minkowski 3-Space

In this section, the various expressions of the Gaussian curvature of the timelike surface are obtained and special cases are examined. Let the Gaussian curvatures of the curves (c) and (c_0) perpendicular to (c) be K and K_0 on the surface $x(u, v)$, respectively. Let the hyperbolic angle between the tangent vector t_1 of the parameter curve (c_1) and the tangent vector t of any curve (c) be ϕ, and the tangent vectors t_1 and t_2 of parameter curves (c_1) and (c_2) intersect under the hyperbolic angle θ on timelike surface $x = x(u, v)$.

Theorem 3.1. *Let radii of principal curvature of (c_1) and (c_2) be R_1 and R_2, radii of normal curvature and geodesic torsion of (c) be R_n and T_g, radii of normal curvature and geodesic torsion of (c_0) perpendicular to (c) be $(R_n)_0$ and $\left(T_g\right)_0$ on $x(u,v)$, respectively. The Gaussian curvature is given by follows:*
Cases 1 - 2.

$$K = K_0 = \frac{1}{R_1R_2} = \frac{1}{T_g\left(T_g\right)_0} + \frac{1}{R_n\left(R_n\right)_0} = \frac{1}{\left(T_g\right)_1}\frac{1}{\left(T_g\right)_2} + \frac{1}{(R_n)_1}\frac{1}{(R_n)_2} + \frac{\sinh\theta}{\cosh\theta}\left(\frac{1}{\left(T_g\right)_1}\frac{1}{(R_n)_2} + \frac{1}{\left(T_g\right)_2}\frac{1}{(R_n)_1}\right). \quad (12)$$

Cases 3 - 4.

$$K = K_0 = \frac{1}{R_1R_2} = \frac{1}{T_g\left(T_g\right)_0} + \frac{1}{R_n\left(R_n\right)_0} = \frac{1}{\left(T_g\right)_1}\frac{1}{\left(T_g\right)_2} + \frac{1}{(R_n)_1}\frac{1}{(R_n)_2} - \frac{\sinh\theta}{\cosh\theta}\left(\frac{1}{\left(T_g\right)_1}\frac{1}{(R_n)_2} + \frac{1}{\left(T_g\right)_2}\frac{1}{(R_n)_1}\right).$$

Cases 5 - 6.

$$K = K_0 = \frac{1}{R_1R_2} = \frac{1}{T_g\left(T_g\right)_0} + \frac{1}{R_n\left(R_n\right)_0} = -\frac{1}{\left(T_g\right)_1}\frac{1}{\left(T_g\right)_2} + \frac{1}{(R_n)_1}\frac{1}{(R_n)_2} + \frac{\cosh\theta}{\sinh\theta}\left(\frac{1}{\left(T_g\right)_1}\frac{1}{(R_n)_2} + \frac{1}{\left(T_g\right)_2}\frac{1}{(R_n)_1}\right).$$

Proof.
For Cases 1 - 2. Firstly, if we multiply (3) and (6) side by side, we get

$$\frac{\sinh\left(\theta-\phi\right)\sinh\phi}{R_1R_2} = \frac{\sinh\left(\theta-\phi\right)\sinh\phi}{R_n\left(R_n\right)_0} + \frac{\cosh\left(\theta-\phi\right)\cosh\phi}{T_g\left(T_g\right)_0} \quad (13)$$

and if we multiply the expressions (4) and (5) side by side, we get

$$\frac{\cosh\left(\theta-\phi\right)\cosh\phi}{R_1R_2} = \frac{\cosh\left(\theta-\phi\right)\cosh\phi}{R_n\left(R_n\right)_0} + \frac{\sinh\left(\theta-\phi\right)\sinh\phi}{T_g\left(T_g\right)_0}. \quad (14)$$

So, if we add the expressions (13) and (14) side by side, we have

$$\frac{1}{R_1R_2} = \frac{1}{R_n\left(R_n\right)_0} + \frac{1}{T_g\left(T_g\right)_0}. \quad (15)$$

On the other hand, if we multiply the expressions (9) and (10) side by side, we get

$$\begin{aligned}
\frac{1}{R_n\left(R_n\right)_0} &= \frac{1}{\cosh^2\theta}\Bigg[\cosh\left(\theta-\phi\right)\cosh\phi\sinh\left(\theta-\phi\right)\sinh\phi\left(\left(\frac{1}{\left(T_g\right)_1} + \frac{1}{\left(T_g\right)_2}\right)^2 + \frac{1}{(R_n)_1^2} + \frac{1}{(R_n)_2^2}\right)\\
&+ \frac{\cosh\left(\theta-\phi\right)\sinh\left(\theta-\phi\right)\left(\cosh^2\phi+\sinh^2\phi\right)}{(R_n)_1}\left(\frac{1}{\left(T_g\right)_1} + \frac{1}{\left(T_g\right)_2}\right)\\
&+ \frac{\cosh\phi\sinh\phi\left(\cosh^2\left(\theta-\phi\right)+\sinh^2\left(\theta-\phi\right)\right)}{(R_n)_2}\left(\frac{1}{\left(T_g\right)_1} + \frac{1}{\left(T_g\right)_2}\right)\\
&+ \left(\cosh^2\left(\theta-\phi\right)\cosh^2\phi + \sinh^2\left(\theta-\phi\right)\sinh^2\phi\right)\frac{1}{(R_n)_1(R_n)_2}\Bigg] \quad (16)
\end{aligned}$$

and if we multiply the expressions (7) and (8) side by side, we get

$$\begin{aligned}\frac{1}{T_g\left(T_g\right)_0} &= \frac{1}{\cosh^2\theta}\Bigg[-\cosh\left(\theta-\phi\right)\cosh\phi\sinh\left(\theta-\phi\right)\sinh\phi\left(\frac{1}{\left(T_g\right)_1}+\frac{1}{\left(T_g\right)_2}+\left(\frac{1}{(R_n)_1}-\frac{1}{(R_n)_2}\right)^2\right)\\
&- \frac{\cosh\left(\theta-\phi\right)\sinh\left(\theta-\phi\right)\left(\cosh^2\phi+\sinh^2\phi\right)}{\left(T_g\right)_1}\left(\frac{1}{(R_n)_1}-\frac{1}{(R_n)_2}\right)\\
&+ \frac{\cosh\phi\sinh\phi\left(\cosh^2\left(\theta-\phi\right)+\sinh^2\left(\theta-\phi\right)\right)}{\left(T_g\right)_2}\left(\frac{1}{(R_n)_1}-\frac{1}{(R_n)_2}\right)\\
&+ \left(\cosh^2\left(\theta-\phi\right)\cosh^2\phi+\sinh^2\left(\theta-\phi\right)\sinh^2\phi\right)\frac{1}{\left(T_g\right)_1\left(T_g\right)_2}\Bigg]. \qquad (17)\end{aligned}$$

So, if we add the expressions (16) and (17) side by side, we have

$$\frac{1}{R_n\,(R_n)_0}+\frac{1}{T_g\left(T_g\right)_0}=\frac{1}{(R_n)_1}\frac{1}{(R_n)_2}+\frac{1}{\left(T_g\right)_1}\frac{1}{\left(T_g\right)_2}+\frac{\sinh\theta}{\cosh\theta}\left(\frac{1}{\left(T_g\right)_1}\frac{1}{(R_n)_2}+\frac{1}{\left(T_g\right)_2}\frac{1}{(R_n)_1}\right). \qquad (18)$$

From the equality of (15) and (18), the proof is obtained. The proof of other cases is done in a similar way. □

Proposition 3.2. *If t_1 and t_2 are Lorentzian orthogonal vectors in Theorem 3.1, then for the Gaussian curvature, we obtain:*

Special Cases 1 - 2 - 3 and 4.

$$K=K_0=\frac{1}{R_1R_2}=\frac{1}{R_n\,(R_n)_0}+\frac{1}{T_g\left(T_g\right)_0}=\frac{1}{(R_n)_1}\frac{1}{(R_n)_2}+\frac{1}{\left(T_g\right)_1}\frac{1}{\left(T_g\right)_2}.$$

There is no special cases 5 - 6.

Corollary 3.3. *Another expression of the Gaussian curvature in Theorem 3.1 is as given below:*

Cases 1 - 2.

$$K=K_0=\frac{1}{R_1R_2}=-\left[\left\langle\frac{\partial t_1}{\partial s_1},\frac{\partial t_2}{\partial s_2}\right\rangle+\left\langle\frac{\partial g_1}{\partial s_1},\frac{\partial g_2}{\partial s_2}\right\rangle\right]+\frac{\sinh\theta}{\cosh\theta}\left[\left\langle\frac{\partial t_1}{\partial s_1},\frac{\partial g_2}{\partial s_2}\right\rangle+\left\langle\frac{\partial t_2}{\partial s_2},\frac{\partial g_1}{\partial s_1}\right\rangle\right], \qquad (19)$$

Cases 3 - 4.

$$K=K_0=\frac{1}{R_1R_2}=-\left[\left\langle\frac{\partial t_1}{\partial s_1},\frac{\partial t_2}{\partial s_2}\right\rangle+\left\langle\frac{\partial g_1}{\partial s_1},\frac{\partial g_2}{\partial s_2}\right\rangle\right]-\frac{\sinh\theta}{\cosh\theta}\left[\left\langle\frac{\partial t_1}{\partial s_1},\frac{\partial g_2}{\partial s_2}\right\rangle+\left\langle\frac{\partial t_2}{\partial s_2},\frac{\partial g_1}{\partial s_1}\right\rangle\right],$$

Cases 5 - 6.

$$K=K_0=\frac{1}{R_1R_2}=\left[\left\langle\frac{\partial t_1}{\partial s_1},\frac{\partial t_2}{\partial s_2}\right\rangle+\left\langle\frac{\partial g_1}{\partial s_1},\frac{\partial g_2}{\partial s_2}\right\rangle\right]+\frac{\cosh\theta}{\sinh\theta}\left[\left\langle\frac{\partial t_1}{\partial s_1},\frac{\partial g_2}{\partial s_2}\right\rangle+\left\langle\frac{\partial t_2}{\partial s_2},\frac{\partial g_1}{\partial s_1}\right\rangle\right].$$

Proof.

For Cases 1 - 2. From the Darboux derivative formulas, we know the following equations:

$$\begin{aligned} \frac{\partial t_1}{\partial s_1} &= \frac{1}{\left(R_g\right)_1} g_1 - \frac{1}{(R_n)_1} N, & \frac{\partial t_2}{\partial s_2} &= \frac{1}{\left(R_g\right)_2} g_2 + \frac{1}{(R_n)_2} N, \\ \frac{\partial g_1}{\partial s_1} &= \frac{1}{\left(R_g\right)_1} t_1 + \frac{1}{\left(T_g\right)_1} N, & \frac{\partial g_2}{\partial s_2} &= \frac{1}{\left(R_g\right)_2} t_2 - \frac{1}{\left(T_g\right)_2} N, \end{aligned} \tag{20}$$

[21]. From the expression (20), we write the following equations:

$$\left\langle \frac{\partial t_1}{\partial s_1}, \frac{\partial t_2}{\partial s_2} \right\rangle + \left\langle \frac{\partial g_1}{\partial s_1}, \frac{\partial g_2}{\partial s_2} \right\rangle = \frac{2 \sinh \theta}{\left(R_g\right)_1 \left(R_g\right)_2} - \frac{1}{(R_n)_1 (R_n)_2} - \frac{1}{\left(T_g\right)_1 \left(T_g\right)_2}, \tag{21}$$

$$\left\langle \frac{\partial t_1}{\partial s_1}, \frac{\partial g_2}{\partial s_2} \right\rangle + \left\langle \frac{\partial t_2}{\partial s_2}, \frac{\partial g_1}{\partial s_1} \right\rangle = \frac{2 \cosh \theta}{\left(R_g\right)_1 \left(R_g\right)_2} + \frac{1}{(R_n)_1 \left(T_g\right)_2} + \frac{1}{(R_n)_2 \left(T_g\right)_1}. \tag{22}$$

If (21) and (22) are substituted in (19), the expressions (12) and (19) appear to be equal. So, the proof is completed. The proof of other cases is done in a similar way. □

Proposition 3.4. *If t_1 and t_2 are Lorentzian orthogonal vectors in Corollary 3.3, for the Gaussian curvature, we obtain the equation below:*

Special Cases 1 - 2 - 3 and 4.

$$K = K_0 = \frac{1}{R_1 R_2} = -\left[\left\langle \frac{\partial t_1}{\partial s_1}, \frac{\partial t_2}{\partial s_2} \right\rangle + \left\langle \frac{\partial g_1}{\partial s_1}, \frac{\partial g_2}{\partial s_2} \right\rangle \right].$$

There is no special cases 5 - 6.

Theorem 3.5 (Enneper Formula). *Let radii of geodesic torsion of (c) and (c_0) perpendicular to (c) be T_g and $\left(T_g\right)_0$ on $x(u, v)$, respectively. There is the relation below between the torsion T (or T_0) of the asymptotic lines of the timelike surface passing through a point P and the Gaussian curvature on the surface at the same point:*

Case 1.

$$K = -\frac{\cosh\left(\theta - \phi\right) \sinh \phi}{\sinh\left(\theta - \phi\right) \cosh \phi} \frac{1}{T_g^2} = -\frac{\cosh\left(\theta - \phi\right) \sinh \phi}{\sinh\left(\theta - \phi\right) \cosh \phi} \frac{1}{T^2},$$

Case 2.

$$K_0 = -\frac{\sinh\left(\theta - \phi\right) \cosh \phi}{\cosh\left(\theta - \phi\right) \sinh \phi} \frac{1}{\left(T_g\right)_0^2} = -\frac{\sinh\left(\theta - \phi\right) \cosh \phi}{\cosh\left(\theta - \phi\right) \sinh \phi} \frac{1}{T_0^2},$$

Case 3.

$$K = -\frac{\sinh\left(\theta - \phi\right) \cosh \phi}{\cosh\left(\theta - \phi\right) \sinh \phi} \frac{1}{T_g^2} = -\frac{\sinh\left(\theta - \phi\right) \cosh \phi}{\cosh\left(\theta - \phi\right) \sinh \phi} \frac{1}{T^2},$$

Case 4.

$$K_0 = -\frac{\cosh\left(\theta - \phi\right) \sinh \phi}{\sinh\left(\theta - \phi\right) \cosh \phi} \frac{1}{\left(T_g\right)_0^2} = -\frac{\cosh\left(\theta - \phi\right) \sinh \phi}{\sinh\left(\theta - \phi\right) \cosh \phi} \frac{1}{T_0^2},$$

Case 5.

$$K = -\frac{\cosh\left(\theta-\phi\right)\cosh\phi}{\sinh\left(\theta-\phi\right)\sinh\phi}\frac{1}{T_g^2} = -\frac{\cosh\left(\theta-\phi\right)\cosh\phi}{\sinh\left(\theta-\phi\right)\sinh\phi}\frac{1}{T^2},$$

Case 6.

$$K_0 = -\frac{\sinh\left(\theta-\phi\right)\sinh\phi}{\cosh\left(\theta-\phi\right)\cosh\phi}\frac{1}{\left(T_g\right)_0^2} = -\frac{\sinh\left(\theta-\phi\right)\sinh\phi}{\cosh\left(\theta-\phi\right)\cosh\phi}\frac{1}{T_0^2}.$$

Proof.

Case 1. We know that $\frac{1}{R_n} = 0$ on the asymptotic lines. Besides, since the hyperbolic angle between the principal normal of an asymptotic line and the normal of the timelike surface is always constant, $\frac{1}{T_g} = \frac{1}{T}$ is obtained, [21]. If these values are substituted in (11), the proof is obtained. The proof of other cases is done in a similar way. □

Proposition 3.6 (Enneper Formula). *If t_1 and t_2 are Lorentzian orthogonal vectors in Theorem 3.5, that is $\theta = 0$, then, for the Gaussian curvature, we obtain follows:*

Special Cases 1 - 3.

$$K = \frac{1}{T_g^2} = \frac{1}{T^2},$$

Special Cases 2 - 4.

$$K_0 = \frac{1}{\left(T_g\right)_0^2} = \frac{1}{T_0^2}.$$

There is no special cases 5 - 6.

Theorem 3.7. *Let the Darboux vectors of (c), (c_1), (c_2) and (c_0) perpendicular to (c) be w, w_1, w_2 and w_0 on $x(u,v)$, respectively. If normal vector at any point P of the surface is N, the Gaussian curvature is given as below:*

Cases 1 - 2 - 3 and 4.

$$K = K_0 = (N, w_0, w)\cosh^2\theta = (N, w_1, w_2)\cosh\theta,$$

Cases 5 - 6.

$$K = K_0 = (N, w, w_0)\sinh^2\theta = (N, w_2, w_1)\sinh\theta.$$

Proof.

Cases 1 - 2. From the following vectors

$$N = \frac{t_1 \wedge t_2}{\cosh\theta}, \tag{23}$$

$$w_1 = -\left(\frac{1}{\left(T_g\right)_1} + \frac{\sinh\theta}{\cosh\theta\ (R_n)_1}\right)t_1 + \frac{t_2}{\cosh\theta\ (R_n)_1} + \frac{N}{\left(R_g\right)_1},$$

and

$$w_2 = \frac{t_1}{\cosh\theta\ (R_n)_2} + \left(\frac{1}{\left(T_g\right)_2} + \frac{\sinh\theta}{\cosh\theta\ (R_n)_2}\right)t_2 - \frac{N}{\left(R_g\right)_2},$$

[30], we get

$$\langle N, w_2 \wedge w_1\rangle = \frac{1}{\cosh\theta}\left[\frac{1}{\left(T_g\right)_1}\frac{1}{\left(T_g\right)_2} + \frac{1}{(R_n)_1}\frac{1}{(R_n)_2} + \frac{\sinh\theta}{\cosh\theta}\left(\frac{1}{\left(T_g\right)_1}\frac{1}{(R_n)_2} + \frac{1}{\left(T_g\right)_2}\frac{1}{(R_n)_1}\right)\right]. \tag{24}$$

Considering the expression (12) in the expression (24), we obtain

$$\langle N, w_2 \wedge w_1\rangle = \frac{K}{\cosh\theta} = \frac{K_0}{\cosh\theta}. \tag{25}$$

Besides, from the expression (23) and the vectors

$$w = -\frac{\sinh\left(\theta-\phi\right)}{\cosh\theta}w_1 + \frac{\cosh\phi}{\cosh\theta}w_2 - \frac{d\phi}{ds}N, \tag{26}$$

$$w_0 = \frac{\cosh\left(\theta-\phi\right)}{\cosh\theta}w_1 + \frac{\sinh\phi}{\cosh\theta}w_2 - \frac{d\phi}{ds}N, \tag{27}$$

[30], we get

$$\langle N, w \wedge w_0\rangle = \frac{1}{\cosh^2\theta}\left[\frac{1}{\left(T_g\right)_1}\frac{1}{\left(T_g\right)_2} + \frac{1}{(R_n)_1}\frac{1}{(R_n)_2} + \frac{\sinh\theta}{\cosh\theta}\left(\frac{1}{\left(T_g\right)_1}\frac{1}{(R_n)_2} + \frac{1}{\left(T_g\right)_2}\frac{1}{(R_n)_1}\right)\right]. \tag{28}$$

Considering the expression (12) in the expression (28), we have

$$\langle N, w \wedge w_0\rangle\cosh\theta = \frac{K}{\cosh\theta} = \frac{K_0}{\cosh\theta}. \tag{29}$$

From the equation of (25) and (29), the proof is obtained. On the other hand, from vectorial product of vectors in (26) and (27), we get

$$w_0 \wedge w = \frac{w_1 \wedge w_2}{\cosh\theta} - \left[\left(\frac{\cosh(\theta-\phi)+\sinh(\theta-\phi)}{\cosh\theta}\right)w_1 \wedge N + \left(\frac{\sinh\phi - \cosh\phi}{\cosh\theta}\right)w_2 \wedge N\right]\frac{d\phi}{ds}. \tag{30}$$

And if we inner product both sides of (30) with N, we have the following equation:

$$\langle N, w_0 \wedge w\rangle\cosh\theta = \langle N, w_1 \wedge w_2\rangle. \tag{31}$$

From the equation of (29) and (31), the proof is obtained. The proof of other cases is done in a similar way. □

Proposition 3.8. *If t_1 and t_2 are Lorentzian orthogonal vectors in Theorem 3.7, then for the Gaussian curvature, we get:*

Special Cases 1 - 2 - 3 and 4.

$$K = K_0 = (N, w_0, w) = (N, w_1, w_2).$$

There is no special cases 5 - 6.

Corollary 3.9. *If normal vector at the point P of timelike surface is N, then there is the following relationship between of the normal vector, its derivatives and the Gaussian curvature:*

Cases 1 - 2 - 3 and 4.

$$K = K_0 = -\left(N, \frac{\partial N}{\partial s_1}, \frac{\partial N}{\partial s_2}\right)\cosh\theta,$$

Cases 5 - 6.

$$K = K_0 = \left(N, \frac{\partial N}{\partial s_1}, \frac{\partial N}{\partial s_2}\right)\sinh\theta.$$

Proof.

Cases 1 - 2 - 3 and 4. We know the following equations

$$\frac{\partial N}{\partial s_1} = w_1 \wedge N \quad and \quad \frac{\partial N}{\partial s_2} = w_2 \wedge N, \tag{32}$$

[30]. From the expressions (25) and (32), we write

$$\frac{\partial N}{\partial s_1} \wedge \frac{\partial N}{\partial s_2} = (w_1 \wedge N) \wedge (w_1 \wedge N) = (N, w_1, w_2)N = \frac{K}{\cosh\theta}N = \frac{K_0}{\cosh\theta}N. \tag{33}$$

If we inner product both sides of (33) by N, the proof is completed. The proof of other cases is done in a similar way. □

Proposition 3.10. *If t_1 and t_2 are Lorentzian orthogonal vectors in Corollary 3.9, that is $\theta = 0$, for the Gaussian curvature, we obtain:*

Cases 1 - 2 - 3 and 4.

$$K = K_0 = -\left(N, \frac{\partial N}{\partial s_1}, \frac{\partial N}{\partial s_2}\right).$$

There is no special cases 5 - 6.

Theorem 3.11. *The Gaussian curvature is expressed by E, F and their derivatives on $x(u,v)$ as follows:*

Cases 1 - 2.

$$K = K_0 = \frac{D(t_1,t_2)}{D(s_1,s_2)} = \frac{1}{\sqrt{EG}}\left[\frac{\partial}{\partial u}\left(\frac{\left(\sqrt{G}\right)_u + \sinh\theta\left(\sqrt{E}\right)_v}{\sqrt{E}}\right) - \frac{\partial}{\partial v}\left(\frac{\left(\sqrt{E}\right)_v - \sinh\theta\left(\sqrt{G}\right)_u}{\sqrt{G}}\right)\right],$$

Cases 3 - 4.

$$K = K_0 = \frac{D(t_1,t_2)}{D(s_1,s_2)} = \frac{1}{\sqrt{EG}}\left[-\frac{\partial}{\partial u}\left(\frac{\left(\sqrt{G}\right)_u - \sinh\theta\left(\sqrt{E}\right)_v}{\sqrt{E}}\right) + \frac{\partial}{\partial v}\left(\frac{\left(\sqrt{E}\right)_v + \sinh\theta\left(\sqrt{G}\right)_u}{\sqrt{G}}\right)\right],$$

Cases 5 - 6.

$$K = K_0 = -\frac{D(t_1,t_2)}{D(s_1,s_2)} = -\frac{1}{\sqrt{EG}}\left[\frac{\partial}{\partial u}\left(\frac{\left(\sqrt{G}\right)_u - \cosh\theta\left(\sqrt{E}\right)_v}{\sqrt{E}}\right) + \frac{\partial}{\partial v}\left(\frac{\left(\sqrt{E}\right)_v - \cosh\theta\left(\sqrt{G}\right)_u}{\sqrt{G}}\right)\right].$$

Proof.

Cases 1 - 2. We know the following equality

$$\frac{\partial\, t_i}{\partial\, s_j} = w_j \wedge t_i, \quad i,\ j = 1, 2, \tag{34}$$

[30]. From the expressions (25) and (34), we get

$$K = K_0 = \langle t_1 \wedge t_2, w_2 \wedge w_1 \rangle = -\langle t_1, w_2 \rangle \langle t_2, w_1 \rangle + \langle t_1, w_1 \rangle \langle t_2, w_2 \rangle = \begin{vmatrix} w_1 \wedge t_1 & w_1 \wedge t_2 \\ w_2 \wedge t_1 & w_2 \wedge t_2 \end{vmatrix},$$

$$K = K_0 = \left\langle \frac{\partial t_1}{\partial s_1}, \frac{\partial t_2}{\partial s_2} \right\rangle - \left\langle \frac{\partial t_2}{\partial s_1}, \frac{\partial t_1}{\partial s_2} \right\rangle = \frac{D(t_1, t_2)}{D(s_1, s_2)}. \tag{35}$$

Besides, we write the following equalities

$$\frac{\partial t_1}{\partial s_1} = \frac{\partial t_1}{\partial u} \frac{1}{\sqrt{E}}, \qquad \frac{\partial t_2}{\partial s_1} = \frac{\partial t_2}{\partial u} \frac{1}{\sqrt{E}},$$

$$\frac{\partial t_1}{\partial s_2} = \frac{\partial t_1}{\partial v} \frac{1}{\sqrt{G}}, \qquad \frac{\partial t_2}{\partial s_2} = \frac{\partial t_2}{\partial v} \frac{1}{\sqrt{G}},$$

[30]. If these values are substituted in (35), considering the equation

$$\frac{\partial}{\partial v}\left(\left\langle t_1, \frac{\partial t_2}{\partial u}\right\rangle\right) - \frac{\partial}{\partial u}\left(\left\langle \langle t_1, \frac{\partial t_2}{\partial v}\right\rangle\right) = \frac{\partial}{\partial v}\left(\frac{\left(\sqrt{E}\right)_v - \sinh\theta\left(\sqrt{G}\right)_u}{\sqrt{G}}\right) + \frac{\partial}{\partial u}\left(\frac{-\left(\sqrt{G}\right)_u - \sinh\theta\left(\sqrt{E}\right)_v}{\sqrt{E}}\right),$$

[30], the proof is obtained. The proof of other cases is done in a similar way. □

Proposition 3.12. *If t_1 and t_2 are Lorentzian orthogonal vectors in Theorem 3.11, that is $\theta = 0$, for the Gaussian curvature, we have:*

Cases 1 - 2.

$$K = K_0 = \frac{D(t_1, t_2)}{D(s_1, s_2)} = \frac{1}{\sqrt{EG}}\left[\frac{\partial}{\partial u}\left(\frac{\left(\sqrt{G}\right)_u}{\sqrt{E}}\right) - \frac{\partial}{\partial v}\left(\frac{\left(\sqrt{E}\right)_v}{\sqrt{G}}\right)\right],$$

Cases 3 - 4.

$$K = K_0 = \frac{D(t_1, t_2)}{D(s_1, s_2)} = \frac{1}{\sqrt{EG}}\left[-\frac{\partial}{\partial u}\left(\frac{\left(\sqrt{G}\right)_u}{\sqrt{E}}\right) + \frac{\partial}{\partial v}\left(\frac{\left(\sqrt{E}\right)_v}{\sqrt{G}}\right)\right].$$

There is no special cases 5 - 6.

Corollary 3.13. *Let the Darboux vectors of (c_1), (c_2), (c) and (c_0) perpendicular to (c) be w_1 , w_2, w and w_0 on $x(u, v)$, respectively. There is the following relation between of the Darboux vectors and the Gaussian curvature:*

Cases 1 - 2.

$$K = K_0 = -(w, w_1, w_2)\cosh\theta \frac{ds}{d\phi} = -(w_0, w_1, w_2)\cosh\theta \frac{ds}{d\phi},$$

Cases 3 - 4.

$$K = K_0 = (w, w_1, w_2)\cosh\theta \frac{ds}{d\phi} = (w_0, w_1, w_2)\cosh\theta \frac{ds}{d\phi},$$

Cases 5 - 6.

$$K = K_0 = (w, w_1, w_2) \sinh\theta \frac{ds}{d\phi} = (w_0, w_1, w_2) \sinh\theta \frac{ds}{d\phi}.$$

Proof.

Cases 1 - 2. From (26), we get

$$(w, w_1, w_2) = -\left\langle -\frac{\sinh\left(\theta - \phi\right)}{\cosh\theta} w_1 + \frac{\cosh\phi}{\cosh\theta} w_2 - \frac{d\phi}{ds} N, w_1 \wedge w_2 \right\rangle = \frac{d\phi}{ds} \langle N, w_1 \wedge w_2 \rangle \tag{36}$$

and from the expression (27), we get

$$(w_0, w_1, w_2) = -\left\langle \frac{\cosh\left(\theta - \phi\right)}{\cosh\theta} w_1 + \frac{\sinh\phi}{\cosh\theta} w_2 - \frac{d\phi}{ds} N, w_1 \wedge w_2 \right\rangle = \frac{d\phi}{ds} \langle N, w_1 \wedge w_2 \rangle . \tag{37}$$

Considering (25) in (36) and (37), the proof is obtained. The proof of other cases is done in a similar way. □

Proposition 3.14. *If t_1 and t_2 are Lorentzian orthogonal vectors in Corollary 3.13, then for the Gaussian curvature, we get equation below:*

Cases 1 - 2.

$$K = K_0 = -(w, w_1, w_2) \frac{ds}{d\phi} = -(w_0, w_1, w_2) \frac{ds}{d\phi},$$

Cases 3 - 4.

$$K = K_0 = (w, w_1, w_2) \frac{ds}{d\phi} = (w_0, w_1, w_2) \frac{ds}{d\phi}.$$

There is no special cases 5 - 6.

4. Example

Let's consider the timelike surface

$$\begin{aligned} x(v,u) \;=\; & \left(\frac{\cos[(n-1)u]\cos[(n-1)v]}{n-1} - \frac{\cos[(n+1)u]\cos[(n+1)v]}{n+1}, \right. \\ & \left. \frac{\cos[(n-1)u]\sin[(n-1)v]}{n-1} - \frac{\cos[(n+1)u]\sin[(n+1)v]}{n+1}, \sin u \sin v \right). \end{aligned}$$

The tangent vector of the parameter curve $v = constant$ of the surface is

$$\begin{aligned} x_u \;=\; & (-\sin[(n-1)u]\cos[(n-1)v] + \sin[(n+1)u]\cos[(n+1)v], \\ & -\sin[(n-1)u]\sin[(n-1)v] + \sin[(n+1)u]\sin[(n+1)v], \cos u \sin v), \end{aligned} \tag{38}$$

and the tangent vector of the parameter curve $u = constant$ of the surface is

$$\begin{aligned} x_v \;=\; & (-\cos[(n-1)u]\sin[(n-1)v] + \cos[(n+1)u]\sin[(n+1)v], \\ & \cos[(n-1)u]\cos[(n-1)v] - \cos[(n+1)u]\cos[(n+1)v], \sin u \cos v). \end{aligned} \tag{39}$$

From the inner product of the expressions (38) and (39),

$$\langle x_u, x_v \rangle \;=\; 3\cos u \cos v \sin u \sin v \neq 0$$

is obtained. Thus, it seen that the parameter curves of the timelike surface $x(v,u)$ do not intersect perpendicularly. The graphs of its parameter curves and this surface for certain intervals of the parameters u and v are given in Figures 3, 4, 5 and 6.

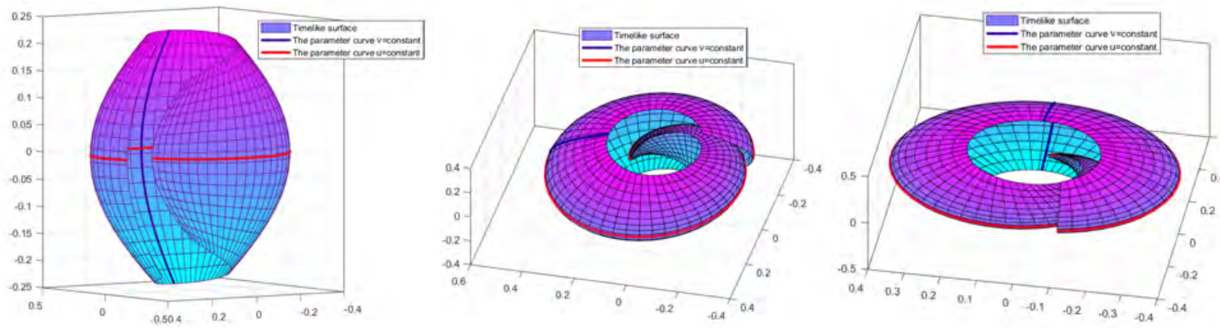

Figure 3: Timelike Surface $x(u,v)$ for $u = -\pi/64n : \pi/120 : \pi/64n$ and $v = -\pi/24n : \pi/120 : \pi/24n$

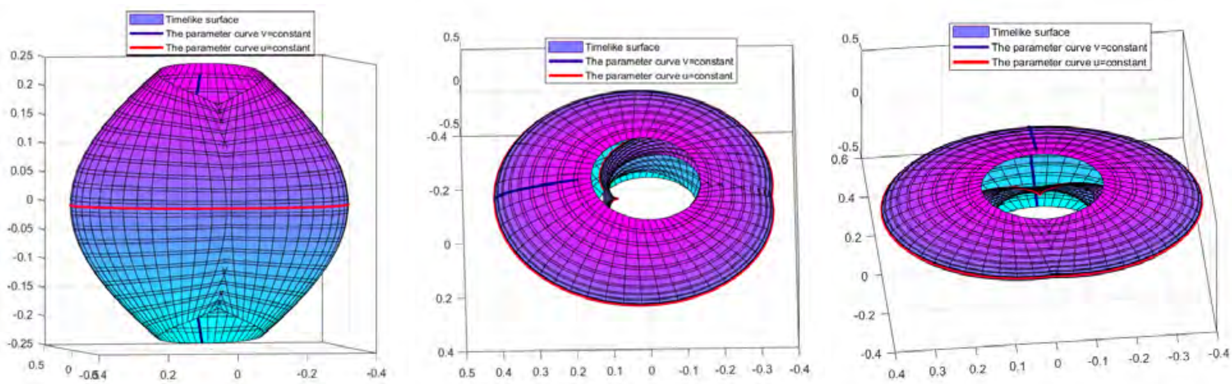

Figure 4: Timelike Surface $x(u,v)$ for $u = -\pi/64n : \pi/120 : \pi/64n$ and $v = -\pi/8n : \pi/120 : \pi/8n$

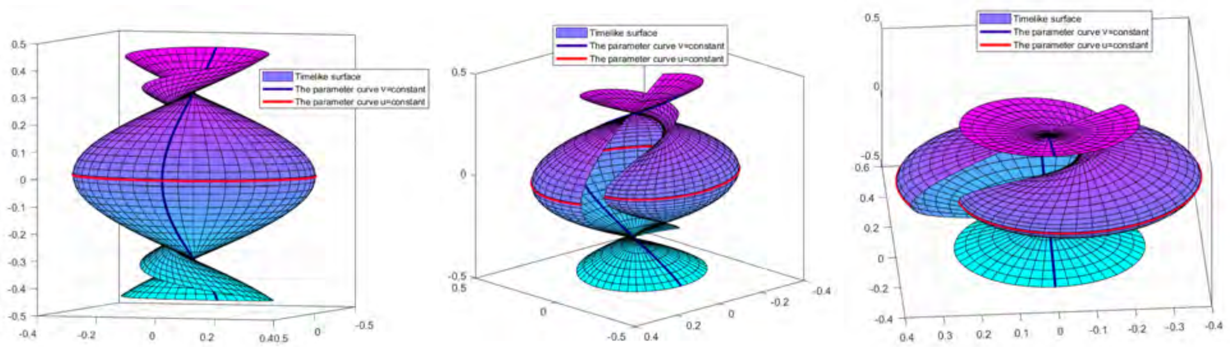

Figure 5: Timelike Surface $x(u,v)$ for $u = -\pi/32n : \pi/120 : \pi/32n$ and $v = -\pi/24n : \pi/120 : \pi/24n$

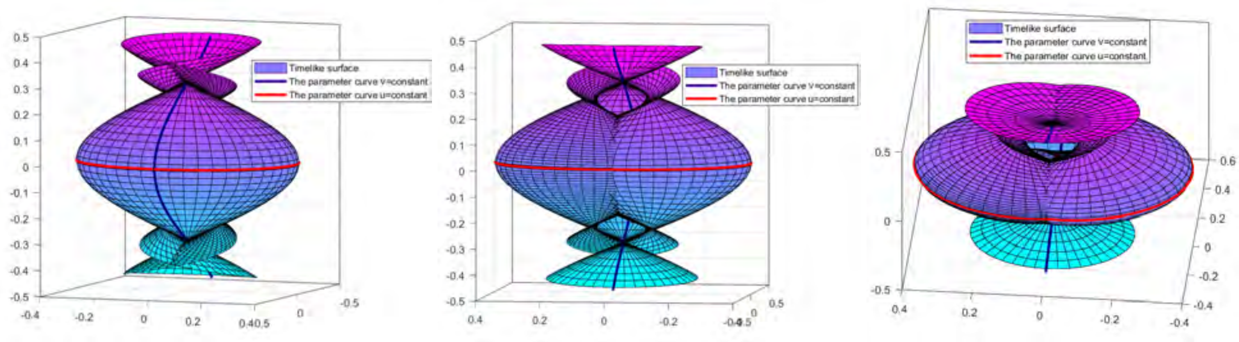

Figure 6: Timelike Surface $x(u, v)$ for $u = -\pi/32n : \pi/120 : \pi/32n$ and $v = -\pi/8n : \pi/120 : \pi/8n$

The Gaussian curvature of the surface is

$$K = \frac{\begin{pmatrix} 4(2A+C)^2\cos(nu)\sin(nu)(B\cos u\sin u + D\cos(nu)\sin(nu)) \\ +4B\cos u\sin u(BD\cos u\sin u + (2A+C)^2\cos(nu)\sin(nu)) \\ -16nAC\cos^2(nu)\cos u\sin u(B\cos u\sin u + (2A+C)\cos(nu)\sin(nu)) + 16n^2A^3\cos^4(nu) \end{pmatrix}}{16A^4\cos^4(nu)(4\sin^2(nu)-1)^2},$$

where $A = \cos^2 u\sin^2 v - \sin^2 u\cos^2 v$, $B = \cos(2nv) - \cos(2nu)$, $C = cos(2v)$, $D = \sin^2 u\sin^2 v - \cos^2 u\cos^2 v$. The graphs of the Gaussian curvature of the timelike surface $x(u, v)$ are shown in Figure 7 and 8 in cases where $u = constant$ and $v = constant$ for various n values, respectively. In addition, the distribution function of the curvature is given in Figure 9.

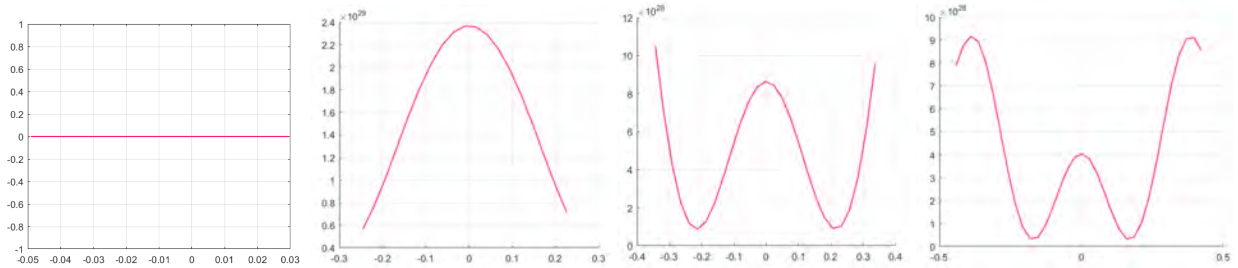

Figure 7: The Gaussian curvature of $x(u, v)$ for $u = constant$ and $n = 0, 5, 7, 9$, respectively

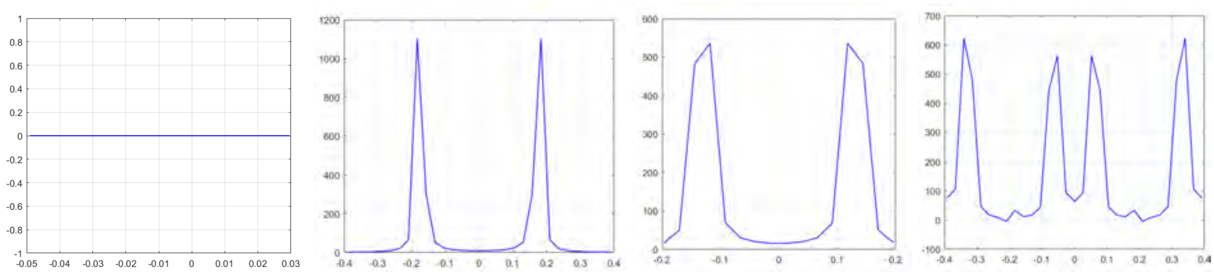

Figure 8: The Gaussian curvature of $x(u, v)$ for $v = constant$ and $n = 0, 3, 4, 8$, respectively

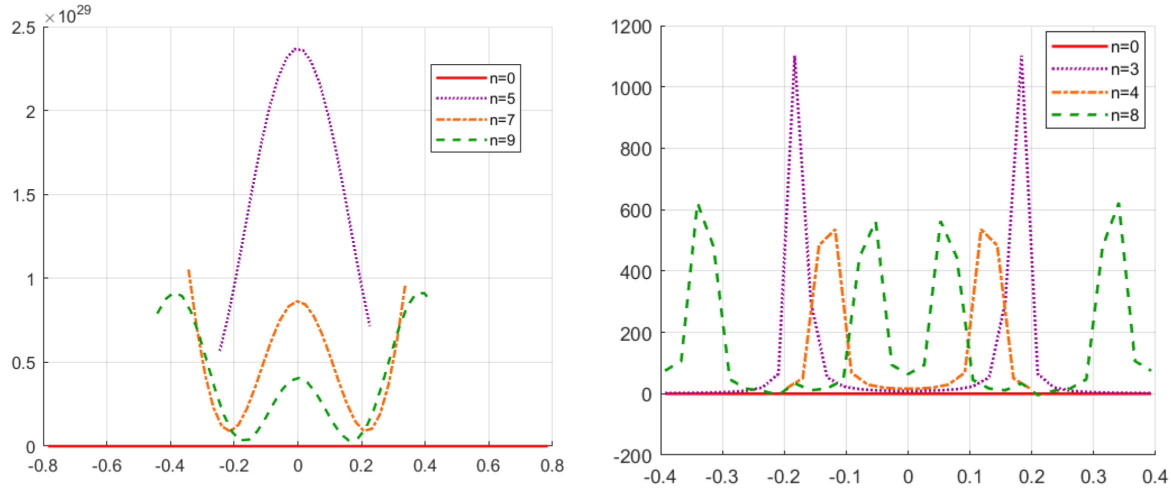

Figure 9: The distribution functions of the Gaussian curvature for $u = constant$ and $v = constant$, respectively

5. Conclusions

We have shown in our [30] study that six different situations occur according to the causal character of parameter curves that intersect at any angle on a time-like surface. In this paper, various expressions of the Gaussian curvature on the timelike surface are examined, new theorems and new equivalents of well-known formulas are given. This study can also be studied in spaces other than the Lorentz-Minkowski 3-space. At the same time, this paper can be studied by including studies in other related disciplines such as architecture, physics, astronomy, singularity theory and submanifold theory, and interesting results can be obtained, [15, 16, 31–35].

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