Filomat 37:28 (2023), 9613–9616
https://doi.org/10.2298/FIL2328613O

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The projectively Hurewicz property is t-invariant

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Abstract. A space X is *projectively Hurewicz* provided every separable metrizable continuous image of X is Hurewicz.

In this paper we prove that the projectively Hurewicz property is t-invariant, i.e., if $C_p(X)$ is homeomorphic to $C_p(Y)$ and X is projectively Hurewicz, then Y is projectively Hurewicz, too.

1. Introduction

Let $\mathcal{P}$ be a topological property. A.V. Arhangel'skii calls X *projectively* $\mathcal{P}$ if every second countable continuous image of X is $\mathcal{P}$ [1, 3]. The projective selection principles were introduced and first time considered in [5]. Lj.D.R. Kočinac characterized the classical covering properties of Menger, Rothberger, Hurewicz and Gerlits-Nagy in term of continuous images in $\mathbb{R}^\omega$. Characterizations of the classical covering properties in terms a selection principle restricted to countable covers by cozero sets are given in [4]. In [8, 9] we obtained the functional characterizations of all projective versions of the selection properties in the Scheepers Diagram.

Let us recall that a topological space is *Hurewicz* if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every n, $\mathcal{V}_n$ is a finite subfamily of $\mathcal{U}_n$ and every point of X is contained in $\bigcup \mathcal{V}_n$ for all but finitely many n's.

Recall that if $C_p(X)$ and $C_p(Y)$ are homeomorphic (linearly homeomorphic, uniform homeomorphic), we say that the spaces X and Y are *t-equivalent (l-equivalent, u-equivalent)*. The properties preserved by t-equivalence (l-equivalence, u-equivalence) we call *t-invariant (l-invariant, u-invariant)* [2].

The following interesting results were obtained:
- (Lj.D.R. Kočinac) A space is Hurewicz if and only if it is Lindelöf and projectively Hurewicz [5].
- (L. Zdomskyy) The Hurewicz property is l-invariant (Corollary 7 in [12]).
- (N.V. Velichko) The Lindelöf property is l-invariant [11].
- (M. Krupski) The projectively Hurewicz property is l-invariant (Theorem 1.5 in [7]).

In this paper we prove that the projectively Hurewicz property is t-invariant.

2020 *Mathematics Subject Classification*. Primary 54C35; Secondary 54D20, 54C05, 54C65
Keywords. Projectively Hurewicz space, selection principles, t-invariant, C_p-spaces
Received: 09 March 2023; Revised: 06 June 2023; Accepted: 07 June 2023
Communicated by Ljubiša D.R. Kočimac
The research is funding from the Ministry of Science and Higher Education of the Russian Federation (Ural Federal University Program of Development within the Priority-2030 Program) is gratefully acknowledged.
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2. Main definitions and notation

Throughout this paper, all spaces are assumed to be Tychonoff. The set of positive integers is denoted by $\mathbb{N}$. Let $\mathbb{R}$ be the real line, we put $\mathbb{I} = [0,1] \subset \mathbb{R}$, and let $\mathbb{Q}$ be the rational numbers. For a space X, we denote by $C_p(X)$ the space of all real-valued continuous functions on X with the topology of pointwise convergence. The symbol $\mathbf{0}$ stands for the constant function to 0. Since $C_p(X)$ is a homogenous space we may always consider the point $\mathbf{0}$ when studying local properties of this space.

A basic open neighborhood of $\mathbf{0}$ is of the form $[F,(-\epsilon,\epsilon)] = \{f \in C(X) : f(F) \subset (-\epsilon,\epsilon)\}$, where F is a finite subset of X and $\epsilon > 0$.

We recall that a subset of X that is the complete preimage of zero for a certain function from $C(X)$ is called a zero-set. A subset $O \subseteq X$ is called a cozero-set (or functionally open) of X if $X \setminus O$ is a zero-set.

Many topological properties are characterized in terms of the following classical selection principles. Let $\mathcal{A}$ and $\mathcal{B}$ be sets consisting of families of subsets of an infinite set X. Then:

$S_{fin}(\mathcal{A},\mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that for each n, $B_n \subseteq A_n$, and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

$U_{fin}(\mathcal{A},\mathcal{B})$ is the selection hypothesis: whenever $\mathcal{U}_1, \mathcal{U}_2, ... \in \mathcal{A}$ and none contains a finite subcover, there are finite sets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\} \in \mathcal{B}$.

In this paper, by a cover we mean a nontrivial one, that is, $\mathcal{U}$ is a cover of X if $X = \bigcup \mathcal{U}$ and $X \notin \mathcal{U}$.

An open cover $\mathcal{U}$ of a space X is:

• an *ω-cover* if every finite subset of X is contained in a member of $\mathcal{U}$.

• a *γ-cover* if it is infinite and each $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$.

• *γ_F-shrinkable* if $\mathcal{U}$ is a cozero γ-cover and there exists a γ-cover $\{F_U : U \in \mathcal{U}\}$ of zero-sets of X with $F_U \subset U$ for every $U \in \mathcal{U}$.

• *ω-groupable* if there is a partition of the cover into finite parts such that for each finite set $F \subseteq X$ and all but finitely many parts $\mathcal{P}$ of the partition, there is a set $U \in \mathcal{P}$ with $F \subseteq U$ [6].

For a topological space X we denote:

• $\mathcal{O}$ — the family of all open covers of X;

• $\mathcal{O}^{\omega}_{cz}$ — the family of all countable cozero covers of the space X;

• Γ — the family of all open γ-covers of the space X;

• Γ_{cz} — the family of all cozero γ-covers of the space X;

• Ω^{gr} — the family of open ω-groupable covers of the space X;

• Γ_F — the family of all cozero γ_F-shrinkable covers of the space X.

Since any infinite part of the γ-cover is also a γ-cover, we further assume that all γ_F-shrinkable covers are countable.

Let us recall that a topological space X is *Hurewicz* if X has the property $U_{fin}(\mathcal{O},\Gamma)$.

3. The projectively Hurewicz property

A space X is *projectively Hurewicz* provided every separable metrizable continuous image of X is Hurewicz.

In ([4], Theorem 30), M. Bonanzinga, F. Cammaroto, M. Matveev proved

Theorem 3.1. *The following conditions are equivalent for a space X:*

1. *X is projectively $U_{fin}(\mathcal{O},\Gamma)$ [$projectivelyHurewicz$];*
2. *Every Lindelöf continuous image of X is Hurewicz;*
3. *for every continuous mapping $f : X \mapsto \mathbb{R}^{\omega}$, $f(X)$ is Hurewicz;*
4. *for every continuous mapping $f : X \mapsto \mathbb{R}^{\omega}$, $f(X)$ is bounded;*
5. *X satisfies $U_{fin}(\mathcal{O}^{\omega}_{cz},\Gamma)$.*

Proposition 3.2. (Proposition 31 in [4])

1. *Every σ-pseudocompact space is projectively Hurewicz.*
2. *Every space of cardinality less than $\mathfrak{b}$ is projectively Hurewicz.*
3. *The projectively Hurewicz property is preserved by continuous images, by countably unions, by C^*-embedded zero-sets, and by cozero sets.*

Definition 3.3. Let $\mathcal{S} = \{S_n : n \in \mathbb{N}\}$ be a family of subsets of a space X and $x \in X$. Then $\mathcal{S}$ *weakly converges* to x if for every neighborhood W of x there is a sequence $(s_n : n \in \mathbb{N})$ such that $s_n \in S_n$ for each $n \in \mathbb{N}$ and there is n' such that $s_n \in W$ for each $n > n'$.

Let us recall that a subset A of X *converges* to x if A is infinite, $x \notin A$, and for each neighborhood U of x, $A \setminus U$ is finite. We write $x = \lim A$ if A converges to x. Consider the following collections:
• $\Gamma_x = \{A \subseteq X : x = \lim A\}$;
• $w\Gamma_x$ = the family of all subsets of X admitting a partition $\mathcal{S} = \{S_n : n \in \mathbb{N}\}$ such that for every n the set S_n is finite and $\mathcal{S}$ weakly converges to x.

Theorem 3.4. *The following conditions are equivalent for a space X:*

1. $C_p(X)$ *satisfies* $S_{fin}(\Gamma_{\mathbf{0}}, w\Gamma_{\mathbf{0}})$;
2. X *satisfies* $S_{fin}(\Gamma_F, \Omega^{gr})$;
3. X *satisfies* $U_{fin}(\Gamma_F, \Gamma)$;
4. X *satisfies* $U_{fin}(\mathcal{O}^{\omega}_{cz}, \Gamma)$;
5. X *is projectively Hurewicz.*

Proof. (3) ⇔ (2). By Theorem 3.4 in [10], the equality $U_{fin}(\mathcal{O}, \Gamma) = S_{fin}(\Gamma, \Omega^{gr})$ is true in the class of metric separable spaces. Let X satisfies $U_{fin}(\Gamma_F, \Gamma)$. By Theorem 5.4 in [9] and Theorem 3.1, $U_{fin}(\Gamma_F, \Gamma) = U_{fin}(\mathcal{O}^{\omega}_{cz}, \Gamma)$, i.e., X is projectively Hurewicz.

Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of countable γ_F-shrinkable covers of X. For every $n \in \mathbb{N}$ and every $U \in \mathcal{U}_n$ fix a continuous function $f_U : X \to \mathbb{R}$ such that $U = f_U^{-1}[\mathbb{R} \setminus \{0\}]$. Put $h = \prod\{f_U : U \in \mathcal{U}_n, n \in \mathbb{N}\}$. Then h is a continuous mapping from X onto $h(X) \subseteq \mathbb{R}^{\omega}$, thus $h(X)$ satisfies $U_{fin}(\mathcal{O}, \Gamma) = S_{fin}(\Gamma, \Omega^{gr})$. Let $h(\mathcal{U}_n) = \{h(U) : U \in \mathcal{U}_n\}$. Since $(h(\mathcal{U}_n) : n \in \mathbb{N})$ be a sequence of γ-covers of $h(X)$ we get (2). Since a continuous metrizable image of a space satisfying the property $S_{fin}(\Gamma_F, \Omega^{gr})$ is a space with this property and $S_{fin}(\Gamma_F, \Omega^{gr}) = S_{fin}(\Gamma, \Omega^{gr})$ for metrizable spaces, the implication (2) ⇒ (3) is proved similarly.

(4) ⇔ (5). By Theorem 3.1.

(5) ⇒ (3). By Theorem 5.4 in [9] (or Theorem 4.1 in [8]).

(3) ⇒ (4). Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of countable cozero covers of X. Enumerate $\mathcal{U}_n = \{U^n_m : m \in \mathbb{N}\}$.

For $n, m \in \mathbb{N}$, fix a continuous function $f_{n,m} : X \to [0, 1]$ that witnesses U^n_m being cozero, i.e. $f^{-1}(0, 1] = U^n_m$. For every $n, m, i \in \mathbb{N}$, let us define

$W^n_{m,i} = f^{-1}_{n,m}(\frac{1}{i+1}, 1]$ and $H^n_{m,i} = f^{-1}_{n,m}[\frac{1}{i+1}, 1]$.

Clearly, the set $W^n_{m,i}$ is cozero and $H^n_{m,i}$ is a zero-set. Note that

$W^n_{m,i} \subseteq H^n_{m,i} \subseteq W^n_{m,i+1} \subseteq U^n_m$ and $U^n_m = \bigcup\limits_{i=1}^{\infty} W^n_{m,i}$.

For $k \in \mathbb{N}$, write $W^n_k = \bigcup\{W^n_{m,i+1} : i, m \le k\}$ and let $\mathcal{W}_n = \{W^n_1, W^n_2, ...\}$. Observe that $\mathcal{W}_n \in \Gamma_F$ because $H^n_k = \bigcup\{H^n_{m,i} : i, m \le k\}$ is a zero-set contained in W^n_k. Moreover the family $\{H^n_k : k \in \mathbb{N}\}$ is a γ-cover of X since one readily checks that the family $\{\bigcup\{W^n_{m,i} : i, m \le k\} : k \in \mathbb{N}\}$ is a γ-cover and $\bigcup\{W^n_{m,i} : i, m \le k\} \subseteq H^n_k$. Now apply the property $U_{fin}(\Gamma_F, \Gamma)$ to the sequence $(\mathcal{W}_n : n \in \mathbb{N})$ together with the fact that $\mathcal{W}_n$ is a finer cover that $\mathcal{U}_n$ for all n.

(1) ⇒ (2). Let $\{\mathcal{V}_i : i \in \mathbb{N}\} \in [\Gamma_F]^{\omega}$. Note that we assume that all γ_F-shrinkable covers are countable.

Since $\mathcal{V}_i = \{V_{i,j} : j \in \mathbb{N}\} \in \Gamma_F$, there is $\{F_{i,j} : j \in \mathbb{N}\} \in \Gamma$ such that $F_{i,j}$ is a zero-set in X and $F_{i,j} \subset V_{i,j} \in \mathcal{V}_i$ for each $j \in \mathbb{N}$. Let $T_i = \{f_{i,j} \in C_p(X) : f_{i,j}(F_{i,j}) = 0$ and $f_{i,j}(X \setminus V_{i,j}) = 1$ for each $i, j \in \mathbb{N}\}$. Since $\{F_{i,j} : j \in \mathbb{N}\}$ is a γ-cover, we have $\lim\limits_{j\to\infty} T_i = \mathbf{0}$ for each $i \in \mathbb{N}$. By (1), there are finite subsets T'_i of T_i and a partition of the set $\bigcup T'_i$ into finite parts such that for each neighborhood $O = [K, (-\epsilon, \epsilon)]$ of the function $\mathbf{0}$ where K is a finite subset of X and $\epsilon > 0$, and all but finitely many parts $\mathcal{P}$ of the partition, there is a function $g \in \mathcal{P}$ with $g \in O$.

Let $\mathcal{P} = \{\{g_{l,1}, ..., g_{l,k_l}\} : l \in \mathbb{N}\}$. Since $g_{l,m} = f_{i_s,j_s}$ for some $i_s, j_s \in \mathbb{N}$, we can consider $Q = \{V_{l,m} : V_{l,m} = V_{i_s,j_s}, f_{i_s,j_s}(X \setminus V_{i_s,j_s}) = 1, f_{i_s,j_s} = g_{l,m}, l \in \mathbb{N}\}$. Then Q has a partition $\mathcal{Q} = \{\{V_{l,1}, ..., V_{l,k_l}\} : l \in \mathbb{N}\}$ and, for any finite subset K of X all but finitely many parts $\mathcal{Q}$ of the partition, there is $V_{l,k}$ with $K \subseteq V_{l,k}$. Thus, $Q \in \Omega^{gr}$.

(2) ⇒ (1). Let $T_i \in \Gamma_{\mathbf{0}}$ for each $i \in \mathbb{N}$. By passing to a countable infinite subset, we can without loss of generality assume that each T_i is countable. Enumerate $T_i = \{f_{i,j} \in C_p(X) : j \in \mathbb{N}\}$.

For i, j define $V_{i,j} = f_{i,j}^{-1}((-\frac{1}{i}, \frac{1}{i}))$ (we can without loss of generality assume that each $V_{i,j}$ is non-empty), and let $\mathcal{V}_i = \{V_{i,j} : j \in \mathbb{N}\}$.

Note that $V_{i,j}$ is a cozero-set in X for each $i, j \in \mathbb{N}$.

Thus we have a mapping $\Phi : \bigcup \mathcal{V}_i \to \bigcup T_i$ such that $\Phi(V_{i,j}) = f_{i,j}$ for $i, j \in \mathbb{N}$.

Since $\lim_{j\to\infty} T_i = \mathbf{0}$, for any finite subset F of X and $\epsilon > 0$ (we can assume that $\epsilon < \frac{1}{i}$), there is $j' \in \mathbb{N}$ such that $f_{i,j} \in [F, (-\epsilon, \epsilon)]$ for each $j > j'$. Thus, $F \subset V_{i,j}$ for each $j > j'$. Thus, $\mathcal{V}_i \in \Gamma_{cz}$.

For i, j define $F_{i,j} = f_{i,j}^{-1}([-\frac{1}{i+1}, \frac{1}{i+1}])$, and let $\mathcal{F}_i = \{F_{i,j} : j \in \mathbb{N}\}$.

Then $F_{i,j} \subset V_{i,j}$ for each $j \in \mathbb{N}$ and $\mathcal{F}_i \in \Gamma$. Note also that $F_{i,j}$ is a zero-set and $V_{i,j}$ is a cozero-set in X for each $j \in \mathbb{N}$. It follows that $\mathcal{V}_i \in \Gamma_F$.

By (2), there are finite subsets $D_i \subset \mathcal{V}_i$ for each $i \in \mathbb{N}$ such that $\bigcup D_i$ is a cozero ω-groupable cover of the space X.

Let $\mathcal{P} = \{\mathcal{P}_k : k \in \mathbb{N}\}$ be a partition of the cover $\bigcup D_i$ into finite parts such that for each finite set $F \subset X$ and all but finitely many parts $\{\mathcal{P}_k : k \in \mathbb{N}\}$ of the partition, there is a set $V_{i(k),j(k)} \in \mathcal{P}_k$ with $F \subset V_{i(k),j(k)}$.

For each k define $S_k = \{f_V : \Phi(V) = f_V, V \in \mathcal{P}_k\}$. The family $\mathcal{S} = \{S_k : k \in \mathbb{N}\}$ is a partition of $\bigcup\{f_{i,j} : V_{i,j} \in D_i, i \in \mathbb{N}\}$. Then, for each finite set $F \subset X$ and $\epsilon > 0$, and all but finitely many parts of the partition $\mathcal{S}$, there is a function $f_{i(k),j(k)} \in \mathcal{S}_k$ with $f_{i(k),j(k)} \in [F, (-\epsilon, \epsilon)]$. Thus, $\bigcup\{f_{i,j} : f_{i,j} \in T_i, V_{i,j} \in D_i, i \in \mathbb{N}\} \in w\Gamma_{\mathbf{0}}$ and $C_p(X)$ satisfies $S_{fin}(\Gamma_{\mathbf{0}}, w\Gamma_{\mathbf{0}})$. □

Note that the property $S_{fin}(\Gamma_x, w\Gamma_x)$ is a *topological* property. Thus, if $C_p(X)$ is homeomorphic to $C_p(Y)$ and $C_p(X)$ satisfies $S_{fin}(\Gamma_{\mathbf{0}}, w\Gamma_{\mathbf{0}})$, then $C_p(Y)$ satisfies $S_{fin}(\Gamma_g, w\Gamma_g)$ for each $g \in C_p(Y)$.

Theorem 3.5. *Suppose that $C_p(X)$ and $C_p(Y)$ are homeomorphic. Then X has the projectively Hurewicz property if and only if Y has the projectively Hurewicz property.*

Problem 3.6. *Let $\mathcal{P} \in \{Menger, Rothberger, Scheepers, S_1(\Gamma, \mathcal{O})\}$. Will the projectively $\mathcal{P}$ property be t-invariant?*

Conjecture 3.7. *The projectively Scheepers Diagram is t-invariant, i.e., each projectively selection property in the Scheepers Diagram is t-invariant.*

If the conjecture is true, then, applying Velichko's result, the Scheepers Diagram is *l*-invariant.

Acknowledgements

The author would like to thank the referee for careful reading and valuable suggestions.

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