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Dispersive estimates for kinetic transport equation in modulation spaces

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Abstract. In this paper, we obtain the dispersive estimates for the kinetic transport equation in modulation spaces. To realize this goal, we establish a new and fundamental important unit decomposition in modulation space, and the frequency decomposition is exploited as well.

1. Introduction

1.1. Background

Kinetic equation comes from non-equilibrium statistical physics that is used in gas theory, aerodynamics, plasma physics, the theory of the passage of particles through matter, and the theory of radiation transfer. The solution of the kinetic equation determines the distribution function of the dynamical states of a single particle, which usually depends on time, coordinates and velocity. In this paper, we study the dispersive estimates in modulation spaces for the kinetic (transport) equation which is defined as below,

$$\begin{cases} \partial_t u + v \cdot \nabla_x u(t,x,v) = F(t,x,v), (t,x,v) \in (0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n, \ n \geq 1 \\ u(0,x,v) = f(x,v). \end{cases} \tag{1}$$

The solution u to (1) has the form $u = U(t)f + W(t)F$, where

$$U(t)f = f(x - tv, v), \ \ W(t)F = \int_{-\infty}^{t} U(t-s)F(s)ds.$$

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1.2. Some known results

In the rest of our paper, the notation $\lesssim$ means $\leq$ up to a generic positive constant C, depending on the dimension and possibly other non-essential parameters from the context. Now we recall some theorems from [5] as our starting point.

Theorem 1.1. *([5]) Define a=HM(p,r) whenever* $\frac{1}{a}=\frac{1}{2}(\frac{1}{r}+\frac{1}{p})$.
The following estimate holds:

$$\|U(t)f\|_{L_t^q L_x^r L_v^p} \lesssim \|f\|_{L_x^b L_v^c},$$

where

$$\frac{1}{q}+\frac{n}{r}=\frac{n}{b},\ HM(p,r)=HM(b,c)=a,\ q\geq c.$$

Theorem 1.2. *([5])* $\|W(t)F\|_{L_t^q L_x^r L_v^p} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}}$*, where*

$$\frac{1}{q}+\frac{1}{\tilde{q}}=n(1-\frac{1}{r}-\frac{1}{\tilde{r}}),\ HM(p,r)=HM(\tilde{p}',\tilde{r}')=a.$$

These theorems are basic to the estimates in partial differential equations and they are Strichartz estimates of Kinetic equations in L^p spaces. Strichartz estimate is fundamentally important to the study of nonlinear dispersive equations. One is referred to [3], [4] and [6] for some historical work on this subject. For more applications of Strichartz estimates in partial differential equations such as Schr*ö*dinger equation and Klein-Gordon equation in modulation space, see [7]-[11].

1.3. Motivation

Our motivation comes from the following question:

Question 1.1. *Are there analogous results as in [5] for kinetic transport equation in modulation spaces?*

To answer this question, it comes naturally to ask whether we could do the similar procedure as in literature mentioned above to establish an analogous Strichartz estimate for the kinetic equations in modulation space. However, up to now, we even do not have a dispersive estimate for modulation space, not to mention the Strichartz estimates. Thus, we try to move forward to this direction which is the theme in this paper.

2. Dispersive Estimate in Modulation Spaces

2.1. Definition of modulation spaces

Firstly let us recall the definition of Fourier transform of an integrable function f on $\mathbb{R}^n$ and its inverse, which are defined as

$$(\mathcal{F}f)(\xi)=\hat{f}(\xi)=\int_{\mathbb{R}^n} e^{-ix\cdot\xi}f(x)dx, \tag{2}$$

and

$$(\mathcal{F}^{-1}f)(x)=\check{f}(x)=\int_{\mathbb{R}^n} e^{ix\cdot\xi}f(\xi)d\xi. \tag{3}$$

Now we give the definition of modulation spaces $M^s_{p,q}(\mathbb{R}^n)$ [8]. For $\xi=(\xi_1,\xi_2,\cdots,\xi_n)$, we define $|\xi|_\infty=\max_{i=1,2,\cdots,n}|\xi_i|$. Let $\rho\in\mathcal{S}(\mathbb{R}^n)$ which is Schwartz function, $\rho:\mathbb{R}^n\to[0,1]$ be a smooth function verifying $\rho(\xi)=1$ for $|\xi|_\infty\leq 1/2$ and $\rho(\xi)=0$ for $|\xi|_\infty\geq 1$. Let $k=(k_1,k_2,\cdots,k_n)\in\mathbb{Z}^n$, ρ_k be the translation of ρ,

$$\rho_k(\xi)=\rho(\xi-k),\quad k\in\mathbb{Z}^n. \tag{4}$$

Denote

$$\varphi_k(\xi) = \rho_k(\xi)\Big(\sum_{k\in\mathbb{Z}^n}\rho_k(\xi)\Big)^{-1}, \quad k \in \mathbb{Z}^n, \tag{5}$$

and

$$\Box_k := \mathcal{F}^{-1}\varphi_k\mathcal{F}, \quad k \in \mathbb{Z}^n, \tag{6}$$

$\{\Box_k\}_{k\in\mathbb{Z}^n}$ are said to be frequency-uniform decomposition operator. For $k \in \mathbb{Z}^n$, we denote

$$|k| = |k_1| + \cdots + |k_n|, \quad \langle k\rangle = 1 + |k|.$$

Let $-\infty < s < \infty, 0 \le p, q \le \infty$, We define

$$M^s_{p,q}(\mathbb{R}^n) = \Big\{f \in \mathcal{S}^{'}(\mathbb{R}^n) : \|f\|_{M^s_{p,q}} < \infty\Big\}, \tag{7}$$

$$\|f\|_{M^s_{p,q}}(\mathbb{R}^n) := \Big(\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{sq}\|\Box_k f\|_p^q\Big)^{1/q}, \tag{8}$$

$M^s_{p,q}(\mathbb{R}^n)$ is said to be the modulation space. For simplicity, we write $M^s_{p,q}(\mathbb{R}^n) = M^s_{p,q}$ and $M^0_{p,q}(\mathbb{R}^n) = M_{p,q}$. With this, we can give the definition of our target space.
Define

$$\|f\|_{M^s_{(r,p),q}} = \Big|\langle k\rangle^s\|\Box_k f\|_{L^r_x L^p_v}\Big|_{l^q_k},$$

in particular,

$$\|f\|_{M^0_{(r,p),q}} = \|f\|_{M_{(r,p),q}} = \Big|\|\Box_k f\|_{L^r_x L^p_v}\Big|_{l^q_k},$$

where

$$\|\xi\|_{l^q} = (\sum_{i=1}^n|\xi_i|^q)^{\frac{1}{q}}, 0 < q < \infty.$$

2.2. *Main result*

Now we are in the position to state our main theorem which is the dispersive estimate for transport equation in $M_{(\infty,1),q}$ spaces.

Theorem 2.1. *(Dispersive Estimate) The kinetic transport evolution group $U(t)$ obeys the following estimates,*

$$\|U(t)f\|_{M_{(\infty,1),q}} \le \frac{1}{|t|^n}\|f\|_{M_{(1,\infty),q}}, \quad 0 < t < 1, \tag{9}$$

and

$$\|U(t)f\|_{M_{(\infty,1),q}} \le \|f\|_{M_{(1,\infty),q}}, \quad t \ge 1, \tag{10}$$

where the generic constants are independent of t.

2.3. Difficulties

The evolution group $\tilde{U}(t) = e^{it\Delta}$ for the Schrödinger equation commutes with $\Box_k$, i.e.,

$$\tilde{U}(t)\Box_k = \Box_k \tilde{U}(t);$$

however the evolution group $U(t)$, $U(t)f = f(x - tv, v)$, for the kinetic equation does not commute with $\Box_k$, i.e.,

$$U(t)\Box_k \neq \Box_k U(t).$$

One could observe this fact by doing some computations . After taking Fourier transforms, we have

$$\mathcal{F}\Box_k U(t)f = \varphi_k(\eta, \xi)\hat{f}(\eta, \xi + t\eta), \tag{11}$$

and

$$\mathcal{F}U(t)\Box_k f = \varphi_k(\eta, \xi + t\eta)\hat{f}(\eta, \xi + t\eta), \tag{12}$$

where we use (η, ξ) to represent the dual variables of (x, v). Note that there is a translation $t\eta$ in the ξ variable. How to deal with this translation and establish the relation between $\Box_k U(t)f$ and $U(t)\Box_k f$ is the most challenging part. This involves careful geometric analysis which is far from obvious. We will explain it in detail in Lemma 2.2.

2.4. Strategies

In this subsection, we would like to illustrate our strategies. As we stated in Section 2.3, the difficulty lies in establishing the relation between $\Box_k U(t)f$ and $U(t)\Box_k f$. With (11) and (12) in mind and note that

$$\sum_{l\in\mathbb{Z}^{2n}} \varphi_l(\eta, \xi) \equiv 1, \quad \forall\ (\eta, \xi) \in \mathbb{R}^{2n}, \tag{13}$$

we have

$$\varphi_k(\eta, \xi)\hat{f}(\eta, \xi + t\eta) = \sum_{l\in\mathbb{Z}^{2n}} \varphi_k(\eta, \xi)\varphi_{k+l}(\eta, \xi + t\eta)\hat{f}(\eta, \xi + t\eta). \tag{14}$$

Note that

$$\varphi_{k+l}(\eta, \xi + t\eta)\hat{f}(\eta, \xi + t\eta)$$

in (14) is the same as the right hand side in (12), and

$$\varphi_k(\eta, \xi)\varphi_{k+l}(\eta, \xi + t\eta) = 0$$

leads to

$$\varphi_k(\eta, \xi)\varphi_{k+l}(\eta, \xi + t\eta)\hat{f}(\eta, \xi + t\eta) = 0.$$

Thus, we only need to consider the term in l for any fixed k such that

$$\varphi_k(\eta, \xi)\varphi_{k+l}(\eta, \xi + t\eta) \neq 0.$$

So the question is:

Question 2.1. *Does the cardinality of l such that $\{l : \varphi_k(\eta, \xi)\varphi_{k+l}(\eta, \xi + t\eta) \neq 0\}$ depends on k, and whether the series converges in l?*

Fortunately, it turns out l in Question 2.1 only depends on t, not on k, this fact needs geometric analysis which could be translated into some inequalities. To be precise, we will prove a relation

$$\Box_k U(t)f = \mathcal{F}^{-1}\varphi_k(\eta, \xi) * \Big(\sum_{l\in E, \#E\leq 5^n\cdot(5+2t)^n} U(t)\Box_{k+l} f \Big), \tag{15}$$

where $E = \{l : supp\ \varphi_k(\eta, \xi) \cap supp\ \varphi_{k+l}(\eta, \xi + t\eta) \neq \emptyset\}$, and $\#E$ denotes the cardinality of the set E.

2.5. New unit decomposition

To achieve (15), we endeavor to prove a key lemma which translates the geometric structure in frequency space to some elementary inequalities.

Lemma 2.2. *For any* $\eta = (\eta_1, \eta_2, \cdots, \eta_n), \xi = (\xi_1, \xi_2, \cdots, \xi_n), k = (k_1, k_2, \cdots, k_{2n}), l = (l_1, l_2, \cdots, l_{2n}), t \in (0, \infty)$, *we have*

$$\sum_{l \in E, \#E \leq 5^n \cdot (5+2t)^n} \varphi_k(\eta, \xi)\varphi_{k+l}(\eta, \xi + t\eta) \equiv 1. \tag{16}$$

Remark 2.3. $k, l \in \mathbb{Z}^{2n}$ *since* $(\eta, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$.

Proof. Note that

$$\varphi_k(\eta, \xi) \cdot \varphi_{k+l}(\eta, \xi + t\eta) \neq 0 \tag{17}$$

leads to

$$\begin{cases} |(\eta, \xi) - k|_\infty \leq 1, \\ |(\eta, \xi + t\eta) - k - l|_\infty \leq 1, \end{cases} \tag{18}$$

which implies that for any $i = 1, 2, \cdots, n$,

$$\begin{cases} |\eta_i - k_i| \leq 1 \\ |\xi_i - k_{i+n}| \leq 1 \\ |\eta_i - k_i - l_i| \leq 1 \\ |\xi_i + t\eta_i - k_{i+n} - l_{i+n}| \leq 1. \end{cases} \tag{19}$$

From the first and the third inequalities in (19), we have $-2 \leq l_i \leq 2$. Combining the first, the second and the last inequalities in (19) yields that

$$l_{i+n} \leq \xi_i + t\eta_i - k_{i+n} + 1 \leq t\eta_i + 2 \leq t(k_i + 1) + 2, \tag{20}$$

and

$$l_{i+n} \geq -1 + \xi_i + t\eta_i - k_{i+n} \geq -2 + t\eta_i \geq -2 + t(k_i - 1), \tag{21}$$

which means that for any $(\eta, \xi) \in \mathbb{R}^{2n}$, and $t \in (0, \infty)$,

$$-2 + t(k_i - 1) \leq l_{i+n} \leq t(k_i + 1) + 2.$$

Consequently, we prove that the number of l which enables (17) to be true is at most $5^n \cdot (5 + 2t)^n$. Thus Lemma 2.2 is complete. □

Remark 2.4. *We could not obtain* $\#E \leq 5$ *directly from* $-2 + t\eta_i \leq l_{i+n} \leq 2 + t\eta_i$ *in Lemma 2.2 since for any chosen* i, η_i *can vary depending on* k_i. *Also, there is an interesting geometric picture hidden in this proof.*

2.6. Proof of main theorem

With this fundamental important unit decomposition obtained in Lemma 2.2, we are ready to prove Theroem 2.1.

Proof. Since $U(t)f = f(x - tv, v)$, taking the Fourier transform with respect to x and v, we have

$$\begin{aligned} \mathcal{F}U(t)f &= \mathcal{F}f(x - tv, v) \\ &= \int_{\mathbb{R}^{2n}} e^{-i(x\cdot\eta + v\cdot\xi)} f(x - tv, v)dxdv \\ &= \int_{\mathbb{R}^{2n}} e^{-i((\tilde{x}+tv)\cdot\eta + v\cdot\xi)} f(\tilde{x}, v)d\tilde{x}dv \\ &= \int_{\mathbb{R}^{2n}} e^{-i(\tilde{x}\cdot\eta + (\xi + t\eta)\cdot v)} f(\tilde{x}, v)d\tilde{x}dv \\ &= \hat{f}(\eta, \xi + t\eta), \end{aligned} \tag{22}$$

where we applied the change of the variable $x - tv = \tilde{x}$ in the third line.
Note

$$\mathcal{F}\Box_k U(t)f = \sum_{l\in E} \varphi_k(\eta, \xi)\varphi_{k+l}(\eta, \xi + t\eta)\hat{f}(\eta, \xi + t\eta), \tag{23}$$

we have

$$\begin{aligned} \Box_k U(t)f &= \sum_{l\in E} \mathcal{F}^{-1}(\varphi_k(\eta, \xi)\varphi_{k+l}(\eta, \xi + t\eta)\hat{f}(\eta, \xi + t\eta)) \\ &= \mathcal{F}^{-1}\varphi_k(\eta, \xi) * (\sum_{l\in E} \mathcal{F}^{-1}[(\varphi_{k+l}\hat{f})(\eta, \xi + t\eta)]) \\ &= \mathcal{F}^{-1}\varphi_k(\eta, \xi) * (\sum_{l\in E} U(t)(\Box_{k+l}f)). \end{aligned} \tag{24}$$

Consequently,

$$\begin{aligned} \|\Box_k U(t)f\|_{L^\infty_x L^1_v} &\le \|\mathcal{F}^{-1}\varphi_k(\eta, \xi)\|_{L^1_{x,v}} \sum_{l\in E} \|U(t)\Box_{k+l}f\|_{L^\infty_x L^1_v} \\ &\le \sum_{l\in E} \frac{1}{|t|^n} \|\Box_{k+l}f\|_{L^1_x L^\infty_v}, \end{aligned} \tag{25}$$

where we applied Young's inequality [1] and Lemma 4.1 in [5]. Taking the l^q norm on both sides of (25) and applying the resut in Lemma 2.2, we have

$$\begin{aligned} \Big| \|\Box_k U(t)f\|_{L^\infty_x L^1_v} \Big|_{l^q_k} &\le \frac{1}{|t|^n} \Big| \sum_{l\in E} \|\Box_{k+l}f\|_{L^1_x L^\infty_v} \Big|_{l^q_k} \\ &\le \frac{1}{|t|^n} \cdot 5^n \cdot (5 + 2t)^n \Big| \|\Box_k f\|_{L^1_x L^\infty_v} \Big|_{l^q_k}, \end{aligned} \tag{26}$$

which is equivalently to say that

$$\|U(t)f\|_{M_{(\infty,1),q}} \le \frac{1}{|t|^n} \|f\|_{M_{(1,\infty),q}}, \text{ if } 0 < t < 1; \tag{27}$$

and

$$\|U(t)f\|_{M_{(\infty,1),q}} \le \|f\|_{M_{(1,\infty),q}}, \text{ if } t \ge 1. \tag{28}$$

Thus, we end the proof of Theorem 2.1. □

Remark 2.5. *The dispersive estimate of kinetic transport equation in Besov spaces can refer to [2] , and its application in other partial differential equations, such as shallow water waves equation, see [12, 13].*

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