



(n_1, \dots, n_d) -quasi- (p, q) -isometric commuting tuple of operators

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Abstract. In this work we construct the concept based on the extension of n -quasi- p -isometric operators of a single operator studied in [11, 14] to the multi-dimensional operators. we are introducing some new interesting results of these family of tuples of operators that are expanding some results of recently published works based on a single operator.

1. Introduction

Let X (resp. \mathcal{Y}) be a complex Banach (resp. Hilbert) space and $\mathcal{B}[X]$ (resp. $\mathcal{B}[\mathcal{Y}]$) be the algebra of bounded operators on X (resp. \mathcal{Y}) and say I_X (resp. $I_{\mathcal{Y}}$) be the identity of $\mathcal{B}[X]$ (resp. $\mathcal{B}[\mathcal{Y}]$), then an operator $A \in \mathcal{B}[\mathcal{Y}]$, its range, kernel and adjoint of A represent by $\mathcal{R}(A)$, $\ker(A)$ and A^* respectively.

Any $A \in \mathcal{B}[\mathcal{Y}]$ is called p -isometry for $p > 0$ if

$$\sum_{l=0}^p (-1)^{p-l} \binom{p}{l} A^{*l} A^l = 0, \quad (1.1)$$

or equivalently,

$$\sum_{l=0}^p (-1)^{p-l} \binom{p}{l} \|A^l x\|^2 = 0 \quad \forall x \in \mathcal{Y}, \quad (1.2)$$

where $\binom{p}{l} = \frac{p!}{l!(p-l)!}$ [1–3].

Such p -isometry was represent by Authors [1–3]. Several properties have been studied as nilpotent perturbations, products and tensor products of such operators (see [2, 3]. The extension of these family of operators to (p, q) -isometry on general Banach spaces has been studied by Sid Ahmed [13], Bayart [4] and

2020 Mathematics Subject Classification. Primary 47B15; Secondary 47B20, 47A15 optionally.

Keywords. p -isometries, n -quasi- p -isometries, p -isometric tuple of operators.

Received: 27 December 2022; Accepted: 02 June 2023

Communicated by Dragan S. Djordjević

Research supported by the Deanship of Scientific Research at Jouf University under grant No (DSR-2021-03-0335).

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Hoffman et al. [9]. Given $p \in \mathbb{N}$ and $q \in (0, \infty)$, an operator $A \in \mathcal{B}[\mathcal{X}]$ is called an (p, q) -isometry if and only if

$$\sum_{l=0}^p (-1)^{p-l} \binom{p}{l} \|A^l x\|^q = 0 \quad (\forall x \in \mathcal{X} \quad \text{see [4, 15]}).$$

In [14], the authors has construct the concept of the family of n -quasi- p -isometry operators on Hilbert space which extended the family of p -isometry operators. An operator $A \in \mathcal{B}[\mathcal{Y}]$ is called n -quasi- p -isometry if it hold the following inequalities

$$A^{*n} \left(\sum_{l=0}^p (-1)^{p-l} \binom{p}{l} A^{*l} A^l \right) A^n = 0,$$

or if

$$\sum_{l=0}^p (-1)^{p-l} \binom{p}{l} \|A^{l+n} x\|^2 = 0, \quad \forall x \in \mathcal{Y}.$$

We find in [11] that the products of some n -quasi- p -isometry are p -isometry, the powers of an n -quasi- p -isometry are n -quasi- p -isometry and the perturbation of n -quasi- p -isometry by nilpotent operators are n -quasi- p -isometries.

Recently, we have seen the concept of n -quasi- (p, q) -isometries on a Banach spaces has been introduced in [10] which gives and extension of these family of (p, q) -isometry on Banach spaces as follows: An $A \in \mathcal{B}[\mathcal{X}]$ is called an n -quasi- (p, q) -isometry if it is hold the relation

$$\sum_{l=0}^p (-1)^{p-l} \binom{p}{l} \|A^{l+n} x\|^q = 0, \quad \forall x \in \mathcal{Y},$$

for some positive integers n, p and real $q \in (0, \infty)$.

For $d \in \mathbb{Z}_+$, let $\mathbf{A} = (A_1, \dots, A_d) \in \mathcal{B}[\mathcal{X}]^d$ be c.t.o. (abbreviated c.t.o.) and $\gamma = (\gamma_1, \dots, \gamma_d) \in \mathbb{Z}_+^d$ (multi-indices) and set $\gamma! := \gamma_1! \cdots \gamma_d!$. Further, we define $\mathbf{A}^\gamma := A_1^{\gamma_1} A_2^{\gamma_2} \cdots A_d^{\gamma_d}$ where $A_j^{\gamma_j}$ is represent the product of A_j times itself γ_j times.

In [7], the authors has construct the family of p -isometry c.t.o. on a Hilbert space. An $\mathbf{S} = (S_1, \dots, S_d) \in \mathcal{B}[\mathcal{Y}]^d$ is called p -isometry c.t.o. if

$$\sum_{l=0}^p (-1)^{p-l} \binom{p}{l} \left(\sum_{|\gamma|=l} \frac{l!}{\gamma!} \mathbf{S}^{*\gamma} \mathbf{S}^\gamma \right) = 0 \tag{1.3}$$

or

$$\sum_{l=0}^p (-1)^{p-l} \binom{p}{l} \left(\sum_{|\gamma|=l} \frac{l!}{\gamma!} \|\mathbf{S}^\gamma x\|^2 \right) = 0 \quad \text{for all } x \in \mathcal{X}. \tag{1.4}$$

An extension of p -isometry c.t.o. to (n_1, \dots, n_d) -quasi- p -isometry c.t.o. was construct in [5] as follows. If $\mathbf{A} = (A_1, \dots, A_d) \in \mathcal{B}[\mathcal{Y}]^d$ be an (n_1, \dots, n_d) -quasi- p -isometry c.t.o. if

$$\prod_{j=1}^d A_j^{*n_j} \left(\sum_{l=0}^p (-1)^l \binom{p}{l} \sum_{|\gamma|=l} \frac{l!}{\gamma!} \mathbf{A}^{*\gamma} \mathbf{A}^\gamma \right) \prod_{j=1}^d A_j^{n_j} = 0,$$

or

$$\sum_{l=0}^p (-1)^l \binom{p}{l} \left(\sum_{|\gamma|=l} \frac{l!}{\gamma!} \|\mathbf{A}^{\gamma+\bar{n}} x\|^2 \right) = 0,$$

for all $x \in \mathcal{Y}$, where $\tilde{\mathbf{n}} = (n_1, \dots, n_d)$.

In this work, we see an extension of the n -quasi- (p, q) -isometry operators to the multi-dimensional operator theory on Banach spaces.

This work based on the family of multivariable operators on Banach space which shows a extended of n -quasi- (p, q) -isometry operators and the concept of (n_1, \dots, n_d) -quasi- (p, q) -isometry c.t.o.

2. (n_1, \dots, n_d) -quasi- (p, q) -isometric tuples

On Banach space, the idea of a (n_1, \dots, n_d) -quasi- (p, q) -isometry-(c.t.o.) has been developed

Definition 2.1. Let $\mathbf{A} = (A_1, \dots, A_d) \in \mathcal{B}[X]^d$, $\tilde{\mathbf{n}} = (n_1, \dots, n_d) \in \mathbb{Z}_+^d$ and p is a positive integer. \mathbf{A} is called (n_1, \dots, n_d) -quasi- (p, q) -isometry c.t.o. if

$$\sum_{l=0}^p (-1)^{p-l} \binom{p}{l} \left(\sum_{|\gamma|=l} \frac{l!}{\gamma!} \|\mathbf{A}^{\gamma+\tilde{\mathbf{n}}}x\|^q \right) = 0. \tag{2.1}$$

If $d = 1$, then (2.1) coincides with (1.3).

Remark 2.2. (1) If $\mathcal{R}(A_l^{n_l}) \subset \ker(A_k^{n_k})$ for $n_l n_k \neq 0$, for some l and k , then \mathbf{A} is a (n_1, \dots, n_d) -quasi- (p, q) -isometry c.t.o.

(2) Every (p, q) -isometric c.t.o. is an $\tilde{\mathbf{n}}$ -quasi- (p, q) -isometry c.t.o. for $(n_1, \dots, n_d) \in \mathbb{N}^d$.

Remark 2.3. Let $\mathbf{A} = (A_1, A_2) \in \mathcal{B}[X]^2$ be a c.t.o. Then we observe that

(i) \mathbf{A} is $(1, 1)$ -quasi- $(1, q)$ -isometry c.t.o. if

$$\|A_1 A_2 x\|^q - \|A_1^2 A_2 x\|^q - \|A_1 A_2^2 x\|^q = 0.$$

(ii) \mathbf{A} is an $(1, 1)$ -quasi- $(2, q)$ -isometry c.t.o. if

$$\|A_1 A_2 x\|^q - 2 \left(\|A_1^2 x\|^q + \|A_2^2 x\|^q - \|A_1^2 A_2^2 x\|^q \right) + \|A_1^3 x\|^q + \|A_2^3 x\|^q = 0.$$

Remark 2.4. Let $\mathbf{A} = (A_1, \dots, A_d) \in \mathcal{B}[X]^d$ be c.t.o. Then

(i) \mathbf{A} is an $(1, \dots, 1)$ -quasi- $(1, q)$ -isometry c.t.o. if and only if

$$\|A_1 \cdots A_d x\|^q - \sum_{j=1}^d \|A_1 \cdots A_{j-1} A_{j+1} \cdots A_d A_j^2 x\|^q = 0.$$

(ii) \mathbf{A} is an $(1, \dots, 1)$ -quasi- $(2, q)$ -isometry c.t.o.e if and only if

$$\begin{aligned} \|A_1 \cdots A_d x\|^q &- 2 \sum_{j=1}^d \|A_1 \cdots A_{j-1} A_{j+1} \cdots A_d A_j^2 x\|^q + \sum_{j=1}^d \|A_1 \cdots A_{j-1} A_{j+1} \cdots A_d A_j^3 x\|^q \\ &+ \sum_{1 \leq j \neq l \leq d} \|A_1 \cdots A_{j-1}^2 A_{j+1}^2 \cdots A_{l-1}^2 A_{l+1}^2 \cdots A_d A_j^2 A_l^2 x\|^q = 0. \end{aligned}$$

(iii) \mathbf{A} is an $\tilde{n} = (n_1, \dots, n_d)$ -quasi-(2, q)-isometry c.t.o. if and only if

$$\|A^{\tilde{n}}x\|^q - 2 \sum_{j=1}^d \|A^{\tilde{n}}A_jx\|^q + \sum_{j=1}^d \|A^{\tilde{n}}A_j^2x\|^q + 2 \sum_{1 \leq j < l \leq d} \|A^{\tilde{n}}A_lA_jx\|^q = 0. \tag{2.2}$$

(iv) \mathbf{A} is an $\tilde{n} = (n_1, \dots, n_d)$ -quasi-(1, q)-isometry c.t.o. if and only if

$$\|A^{\tilde{n}}x\|^q - \sum_{j=1}^d \|A_jA^{\tilde{n}}x\|^q = 0. \tag{2.3}$$

Example 2.5. Let $\mathbf{A} = (A_1, A_2) \in \mathcal{B}[\mathbb{R}^3]^2$, where

$$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}.$$

by showing that $\|x\| = |x_1| + |x_2| + |x_3|$. Direct calculations show that

$$\|A_1x_0\| + \|A_2x_0\| - \|x_0\| \neq 0, \text{ for some } x_0 \in \mathbb{R}^3.$$

Moreover

$$\|A_1A_2x\| - \|A_1^2A_2x\| - \|A_1A_2^2x\| = 0, \forall x \in \mathbb{R}^3.$$

Therefore $\mathbf{A} = (A_1, A_2)$ is a (1, 1)-quasi-(1, 1)-isometry, but \mathbf{A} is not a (1, 1)-isometry.

Remark 2.6. From the above example, we observe that the family of \tilde{n} -quasi-(p, q)-isometry contain the family of (p, q)-isometry c.t.o. as a proper subfamily.

Example 2.7. Let $A \in \mathcal{B}[\mathcal{X}]$ be an n -quasi (p, q)-isometry and set $\mathbf{A} = \left(\frac{1}{\sqrt[n]{a}}A, \dots, \frac{1}{\sqrt[n]{a}}A\right)$, then \mathbf{A} is an (n, \dots, n) -quasi-

(p, q) -isometry.

$$\begin{aligned}
 & \sum_{l=0}^p (-1)^l \binom{p}{l} \sum_{|\gamma|=l} \frac{l!}{\gamma!} \left\| \left(\frac{1}{\sqrt[d]{d}} A \right)^{\gamma_1+n}, \dots, \left(\frac{1}{\sqrt[d]{d}} A \right)^{\gamma_d+n} x \right\|^q \\
 &= \sum_{l=0}^p (-1)^l \binom{p}{l} \sum_{|\gamma|=l} \frac{l!}{\gamma!} \left\| \left(\frac{1}{\sqrt[d]{d}} A \right)^{|\gamma|+nd} x \right\|^q \\
 &= \sum_{l=0}^p (-1)^l \binom{p}{l} \sum_{|\gamma|=l} \frac{l!}{\gamma!} \left\| \left(\frac{1}{\sqrt[d]{d}} A \right)^l \left(\frac{1}{\sqrt[d]{d}} A \right)^{nd} x \right\|^q \\
 &= \sum_{l=0}^p (-1)^l \binom{p}{l} d^l \left\| \left(\frac{1}{\sqrt[d]{d}} A \right)^l \left(\frac{1}{\sqrt[d]{d}} A \right)^{nd} x \right\|^q \\
 &= \sum_{l=0}^p (-1)^l \binom{p}{l} \left(\sqrt[d]{d} \right)^{ld} \left\| \left(\frac{1}{\sqrt[d]{d}} A \right)^l \left(\frac{1}{\sqrt[d]{d}} A \right)^{nd} x \right\|^q \\
 &= \sum_{l=0}^p (-1)^l \binom{p}{l} \left\| A^l \left(\frac{1}{\sqrt[d]{d}} A \right)^{nd} x \right\|^q \\
 &= \sum_{l=0}^p (-1)^l \binom{p}{l} \left\| A^l \left(\frac{1}{\sqrt[d]{d}} \right)^{nd} A^{nd} x \right\|^q \\
 &= \sum_{l=0}^p (-1)^l \binom{p}{l} \left\| A^{l+n} A^{n(d-1)} \left(\frac{1}{\sqrt[d]{d}} \right)^{nd} x \right\|^q \\
 &= \sum_{l=0}^p (-1)^l \binom{p}{l} \left\| A^{l+n} y \right\|^q = 0.
 \end{aligned}$$

where $y = A^{n(d-1)} \left(\frac{1}{\sqrt[d]{d}} \right)^{nd} x$. Therefore \mathbf{A} is an (n, \dots, n) -quasi- (p, q) -isometry tuple as required.

Proposition 2.8. Let $\mathbf{A} = (A_1, \dots, A_d) \in \mathcal{B}[\mathcal{X}]^d$, is an (n_1, \dots, n_d) -quasi- (p, q) -isometry c.t.o., if and only if \mathbf{A} is an (p, q) -isometry on $\overline{\mathcal{R}(\mathbf{A}^{\tilde{\mathbf{n}}})}$.

Proof. Let $\mathbf{A} = (A_1, \dots, A_d) \in \mathcal{B}[\mathcal{X}]^d$, \mathbf{A} is an (n_1, \dots, n_d) -quasi- (p, q) -isometry c.t.o. if and only if, $\forall x \in \mathcal{X}$.

$$\begin{aligned}
 0 &= \sum_{l=0}^p (-1)^l \binom{p}{l} \sum_{|\gamma|=l} \frac{l!}{\gamma!} \|\mathbf{A}^{\gamma+\tilde{\mathbf{n}}} x\|^q \\
 &= \sum_{l=0}^p (-1)^l \binom{p}{l} \sum_{|\gamma|=l} \frac{l!}{\gamma!} \|\mathbf{A}^\gamma (\mathbf{A}^{\tilde{\mathbf{n}}} x)\|^q \\
 &= \sum_{l=0}^p (-1)^l \binom{p}{l} \sum_{|\gamma|=l} \frac{l!}{\gamma!} \|\mathbf{A}^\gamma x\|^q, \forall x \in \overline{\mathcal{R}(\mathbf{A}^{\tilde{\mathbf{n}}})}.
 \end{aligned}$$

Equivalently, \mathbf{A} is an (p, q) -isometry on $\overline{\mathcal{R}(\mathbf{A}^{\tilde{\mathbf{n}}})}$. \square

For $\tilde{\mathbf{n}} = (n_1, \dots, n_d)$, $\tilde{\mathbf{n}}' = (n'_1, \dots, n'_d) \in \mathbb{Z}_+^d$, we will say that $\tilde{\mathbf{n}} \leq \tilde{\mathbf{n}}'$, if $n_i \leq n'_i$, $i = 1, \dots, d$.

Proposition 2.9. Let $\mathbf{A} = (A_1, \dots, A_d) \in \mathcal{B}[X]^d$ be an $\tilde{\mathbf{n}}$ -quasi- (p, q) -isometry c.t.o., then \mathbf{A} is an $\tilde{\mathbf{n}}$ -quasi- (p, q) -isometry c.t.o., $\forall \tilde{\mathbf{n}} \geq \mathbf{\bar{n}}$.

Proof. Consider \mathbf{A} is an $\tilde{\mathbf{n}}$ -quasi- (p, q) -isometry c.t.o. By Proposition 2.8, we shows that \mathbf{A} is an (p, q) -isometry c.t.o. on $\overline{\mathcal{R}(\mathbf{A}^{\tilde{\mathbf{n}}})}$. It is obvious that

$$\overline{\mathcal{R}(\mathbf{A}^{\tilde{\mathbf{n}}})} \supset \overline{\mathcal{R}(\mathbf{A}^{\mathbf{\bar{n}}})} \quad \forall \quad \tilde{\mathbf{n}} \geq \mathbf{\bar{n}}.$$

This shows that \mathbf{A} is an (p, q) -isometry c.t.o. on $\overline{\mathcal{R}(\mathbf{A}^{\mathbf{\bar{n}}})}$. Hence, \mathbf{A} is an $\tilde{\mathbf{n}}$ -quasi- (p, q) -isometry c.t.o. $\forall \tilde{\mathbf{n}} \geq \mathbf{\bar{n}}$. \square

The following theorem extended [10, Theorem 2.2].

Theorem 2.10. Let $\mathbf{A} = (A_1, \dots, A_d) \in \mathcal{B}[X]^d$ be an $\tilde{\mathbf{n}}$ -quasi- (p, q) -isometry c.t.o., then \mathbf{A} is an $\tilde{\mathbf{n}}$ -quasi- $(p + l, q)$ -isometry c.t.o., $\forall l$.

Proof. we need to show that that \mathbf{A} is an $\tilde{\mathbf{n}}$ -quasi- $(p + 1, q)$ -isometry c.t.o. .

Indeed, we have

$$\begin{aligned} & \sum_{l=0}^{p+1} (-1)^l \binom{p+1}{l} \sum_{|\gamma|=k} \frac{l!}{\gamma!} \|\mathbf{A}^\gamma \mathbf{A}^{\tilde{\mathbf{n}}} x\|^q \\ = & \|\mathbf{A}^{\tilde{\mathbf{n}}} x\|^q + \sum_{l=0}^p (-1)^l \left[\binom{p}{l} + \binom{p}{l-1} \right] \sum_{|\gamma|=l} \frac{l!}{\gamma!} \|\mathbf{A}^\gamma \mathbf{A}^{\tilde{\mathbf{n}}} x\|^q - (-1)^p \sum_{|\gamma|=p+1} \frac{(p+1)!}{\gamma!} \|\mathbf{A}^\gamma \mathbf{A}^{\tilde{\mathbf{n}}} x\|^q \\ = & \sum_{l=0}^p (-1)^l \binom{p}{l} \sum_{|\gamma|=l} \frac{l!}{\gamma!} \|\mathbf{A}^\gamma \mathbf{A}^{\tilde{\mathbf{n}}} x\|^q - \sum_{l=0}^{p-1} (-1)^l \binom{p}{l} \sum_{|\gamma|=l+1} \frac{(l+1)!}{\gamma!} \|\mathbf{A}^\gamma \mathbf{A}^{\tilde{\mathbf{n}}} x\|^q \\ & - (-1)^p \sum_{|\gamma|=p+1} \frac{(p+1)!}{\gamma!} \|\mathbf{A}^\gamma \mathbf{A}^{\tilde{\mathbf{n}}} x\|^q \\ = & - \sum_{l=0}^{p-1} (-1)^l \binom{p}{l} \sum_{|\gamma|=l+1} \frac{l!(\gamma_1 + \dots + \gamma_d)}{\gamma_1! \cdot \gamma_2! \cdot \dots \cdot \gamma_d!} \|\mathbf{A}^\gamma \mathbf{A}^{\tilde{\mathbf{n}}} x\|^q \\ & - (-1)^p \sum_{|\gamma|=p+1} \frac{p!(\gamma_1 + \dots + \gamma_d)}{\gamma_1! \cdot \gamma_2! \cdot \dots \cdot \gamma_d!} \|\mathbf{A}^\gamma \mathbf{A}^{\tilde{\mathbf{n}}} x\|^q \\ = & - \sum_{j=1}^d \sum_{l=0}^{p-1} (-1)^l \binom{p}{l} \sum_{|\gamma|=l+1} (-1)^l \binom{p}{l} \frac{l! \gamma_j}{\gamma_1! \cdot \gamma_2! \cdot \dots \cdot \gamma_d!} \|A^{\gamma_1} \dots A_j^{\gamma_j-1} A_j^{\gamma_j+1} \dots A_d^{\gamma_d} A_j \mathbf{A}^{\tilde{\mathbf{n}}} x\|^q \\ & - (-1)^p \sum_{j=1}^d \sum_{|\gamma|=p+1} \frac{p! \gamma_j}{\gamma_1! \cdot \gamma_2! \cdot \dots \cdot \gamma_d!} \|A^{\gamma_1} \dots A_j^{\gamma_j-1} A_j^{\gamma_j+1} \dots A_d^{\gamma_d} A_j \mathbf{A}^{\tilde{\mathbf{n}}} x\|^q \\ = & - \sum_{j=1}^d \sum_{l=0}^{p-1} (-1)^l \binom{p}{l} \sum_{|\delta|=l} (-1)^l \binom{p}{l} \frac{l!}{\delta!} \|\mathbf{A}^\delta A_j \mathbf{A}^{\tilde{\mathbf{n}}} x\|^q \\ & - (-1)^p \sum_{j=1}^d \sum_{|\gamma|=p} \frac{p!}{\delta!} \|\mathbf{A}^\delta A_j \mathbf{A}^{\tilde{\mathbf{n}}} x\|^q \\ = & - \sum_{j=1}^d \sum_{l=0}^p (-1)^l \binom{p}{l} \sum_{|\delta|=l} \frac{l!}{\delta!} \|\mathbf{A}^\delta A_j \mathbf{A}^{\tilde{\mathbf{n}}} x\|^q \\ = & 0. \end{aligned}$$

□

Proposition 2.11. Let \mathfrak{S}_d be a symmetry group on d symbols $\{1, 2, \dots, d\}$ and let $\mathbf{A} = (A_1, \dots, A_d) \in \mathcal{B}[\mathcal{X}]^d$. If \mathbf{A} is an $\tilde{\mathbf{n}} = (n_1, \dots, n_d)$ -quasi- (p, q) -isometry c.t.o., then for every $\sigma \in \mathfrak{S}_d$, $\mathbf{A}_\sigma := (A_{\sigma(1)}, \dots, A_{\sigma(d)})$ is an $\tilde{\mathbf{n}}_\sigma = (n_{\sigma(1)}, \dots, n_{\sigma(d)})$ -quasi- (p, q) -isometry c.t.o.

Proof. It follows from the condition that $\prod_{j=1}^d A_j = \prod_{j=1}^d A_{\sigma(j)}$ and the identity

$$\begin{aligned} & \sum_{l=0}^p (-1)^l \binom{p}{l} \sum_{|\alpha|=l} \frac{l!}{\gamma!} \|\mathbf{A}_\sigma^{\gamma + \tilde{\mathbf{n}}_\sigma} x\|^q \\ & \sum_{l=0}^p (-1)^l \binom{p}{l} \sum_{|\alpha|=l} \frac{l!}{\gamma!} \|\mathbf{A}^{\gamma + \tilde{\mathbf{n}}} x\|^q \\ & = 0. \end{aligned}$$

□

The following theorem describe the condition under which an (n_1, \dots, n_d) -quasi- (p, q) -isometry c.t.o. to be any $(1, \dots, 1)$ -quasi- (p, q) -isometry c.t.o.

Theorem 2.12. If $\mathbf{A} = (A_1, \dots, A_d) \in \mathcal{B}[\mathcal{X}]^d$ is an $\tilde{\mathbf{n}}$ -quasi- (p, q) -isometric c.t.o. such that $\mathcal{R}(A_j) = \mathcal{R}(A_j^{n_j})$ for each $j, 1 \leq j \leq d$, then \mathbf{A} is an quasi- (p, q) -isometric c.t.o.

Proof. Since \mathbf{A} is an $\tilde{\mathbf{n}}$ -quasi- (p, q) -isometric c.t.o. for $j = 1, \dots, d$, then we have

$$\begin{aligned} & \sum_{l=0}^p (-1)^l \binom{p}{l} \sum_{|\gamma|=l} \frac{l!}{\gamma!} \|\mathbf{A}^{\gamma + \tilde{\mathbf{n}}} x\|^q = 0 \\ \Rightarrow & \sum_{l=0}^p (-1)^l \binom{p}{l} \sum_{|\gamma|=l} \frac{l!}{\gamma!} \|\mathbf{A}^\gamma A_1^{n_1} \dots A_d^{n_d} x\|^q = 0 \\ \Rightarrow & \sum_{l=0}^p (-1)^l \binom{p}{l} \sum_{|\gamma|=l} \frac{l!}{\gamma!} \|\mathbf{A}^\gamma A_1^{n_1} \dots A_{d-1}^{n_{d-1}} A_d^{\gamma_d} y\|^q = 0 \\ \Rightarrow & \sum_{l=0}^p (-1)^l \binom{p}{l} \sum_{|\gamma|=l} \frac{l!}{\gamma!} \|\mathbf{A}^\gamma A_d A_1^{n_1} \dots A_{d-1}^{n_{d-1}} y\|^q = 0 \\ \Rightarrow & \vdots \\ \Rightarrow & \sum_{l=0}^p (-1)^l \binom{p}{l} \sum_{|\gamma|=l} \frac{l!}{\gamma!} \|\mathbf{A}^\gamma A_1 \dots A_d x\|^q = 0. \end{aligned}$$

Hence, \mathbf{A} is an quasi- (p, q) -isometric c.t.o. □

Corollary 2.13. If $\mathbf{A} = (A_1, \dots, A_d) \in \mathcal{B}[\mathcal{X}]^d$ is an $\tilde{\mathbf{n}}$ -quasi- (p, q) -isometric c.t.o. such that each $A_j^2 = A_j$ for $j = 1, \dots, d$, then \mathbf{A} is a quasi- (p, q) -isometric c.t.o.

Proof. Each $A_j^2 = A_j$, then $\mathcal{R}(A_j) = \mathcal{R}(A_j^{n_j}) \forall j = 1, \dots, d$. Hence the prove is consequence Theorem 2.12. □

Proposition 2.14. Let $\mathbf{A} = (A_1, \dots, A_d) \in \mathcal{B}[\mathcal{X}]^d$ be an $\tilde{\mathbf{n}}$ -quasi-(2, q)-isometry c.t.o., then the following axioms are true.

$$\sum_{|l|=l} \frac{l!}{\gamma^l} \|\mathbf{A}^{\gamma+\tilde{\mathbf{n}}}x\|^q = (1-l)\|\mathbf{A}^{\tilde{\mathbf{n}}}x\|^q + l \left(\sum_{j=1}^d \|\mathbf{A}^{\tilde{\mathbf{n}}}A_jx\|^q \right), \quad \forall x \in \mathcal{X}, \forall l \in \mathbb{N}. \tag{2.4}$$

$$\sum_{j=1}^d \|\mathbf{A}^{\tilde{\mathbf{n}}}A_jx\|^q \geq \frac{l}{l-1} \|\mathbf{A}^{\tilde{\mathbf{n}}}x\|^q \quad \forall x \in \mathcal{X}, l \in \mathbb{N}, l \neq 1. \tag{2.5}$$

$$\sum_{j=1}^d \|\mathbf{A}^{\tilde{\mathbf{n}}}A^njx\|^q \geq \|\mathbf{A}^{\tilde{\mathbf{n}}}x\|^q \quad \forall x \in \mathcal{X}. \tag{2.6}$$

$$\lim_{l \rightarrow \infty} \left(\sum_{|l|=l} \frac{l!}{\gamma^l} \|\mathbf{A}^{\gamma+\tilde{\mathbf{n}}}x\|^q \right)^{\frac{1}{l}} = 1. \tag{2.7}$$

Proof. We use induction on l to prove inequality (2.4). For $l \in \{0, 1\}$ it is obvious. presume that (2.4) hold l and prove it for $l + 1$. In view of [9, Lemma 2.1], it follows that

Therefore by the induction hypothesis, we get

$$\begin{aligned} & \sum_{|l|=l+1} \frac{(l+1)!}{\gamma^l} \|\mathbf{A}^{\alpha+\tilde{\mathbf{n}}}x\|^q \\ &= (1-l) \sum_{i=1}^d \|A_i\mathbf{A}^{\tilde{\mathbf{n}}}x\|^q + l \sum_{i=1}^d \left(\sum_{j=1}^d \|A_jA_i\mathbf{A}^{\tilde{\mathbf{n}}}x\|^q \right) \\ &= (1-l) \sum_{i=1}^d \|A_i\mathbf{A}^{\tilde{\mathbf{n}}}x\|^q + k \sum_{j=1}^d \|A_j^2\mathbf{A}^{\tilde{\mathbf{n}}}x\|^q + 2l \left(\sum_{1 \leq j \neq i \leq d} \|A_jA_i\mathbf{A}^{\tilde{\mathbf{n}}}x\|^q \right). \end{aligned}$$

Since \mathbf{A} is an $\tilde{\mathbf{n}}$ -quasi-(2, q)-isometry c.t.o., it follows from (2.2)

$$\begin{aligned} & \sum_{|l|=l+1} \frac{(l+1)!}{\gamma^l} \|\mathbf{A}^{\alpha+\tilde{\mathbf{n}}}x\|^q \\ &= (1-l) \sum_{i=1}^d \|A_i\mathbf{A}^{\tilde{\mathbf{n}}}x\|^q + l \left(-\|\mathbf{A}^{\tilde{\mathbf{n}}}x\|^q + 2 \sum_{i=1}^d \|\mathbf{A}^{\tilde{\mathbf{n}}}A_ix\|^q \right) \\ &= -l\|\mathbf{A}^{\tilde{\mathbf{n}}}x\|^q + (l+1) \left(\sum_{1 \leq i \leq d} \|A_i\mathbf{A}^{\tilde{\mathbf{n}}}x\|^q \right), \end{aligned}$$

so that (2.4) holds for $l + 1$.

The inequality (2.5) follows from (2.4) and the inequality (2.6) follows from (2.5) by taking $l \rightarrow \infty$. To proof (2.7), it follows from (2.4)

$$\limsup_{l \rightarrow \infty} \left(\sum_{|l|=l} \frac{l!}{\gamma^l} \|\mathbf{A}^{\gamma+\tilde{\mathbf{n}}}x\|^q \right)^{\frac{1}{l}} \leq 1.$$

However according to (2.6), $\left(\sum_{|\gamma|=l} \frac{l!}{\gamma!} \|\mathbf{A}^{\gamma+\tilde{\mathbf{n}}}x\|^q\right)^{\frac{1}{q}}$ is monotonically increasing, so that

$$\liminf_{l \rightarrow \infty} \left(\sum_{|\gamma|=l} \frac{l!}{\gamma!} \|\mathbf{A}^{\gamma+\tilde{\mathbf{n}}}x\|^q\right)^{\frac{1}{q}} \geq \lim_{l \rightarrow \infty} \left(\|\mathbf{A}^{\gamma+\tilde{\mathbf{n}}}x\|^q\right) = 1.$$

□

Definition 2.15. Let $\mathbf{A} = (A_1, \dots, A_d) \in \mathcal{B}[\mathcal{X}]^d$ be a c.t.o. Then \mathbf{A} is called bounded if

$$\sup \left\{ \sum_{|\gamma|=l} \frac{l!}{\gamma!} \|A^\gamma x\|^q, \forall l \in \mathbb{N} \right\} < \infty$$

for all $x \in \mathcal{X}$.

Theorem 2.16. Let $\mathbf{A} = (A_1, \dots, A_d)$ be an $\tilde{\mathbf{n}}$ -quasi- (p, q) -isometry c.t.o. and bounded. Then

$$\sum_{i=1}^d \|A_i \mathbf{A}^{\tilde{\mathbf{n}}}x\|^q = \|\mathbf{A}^{\tilde{\mathbf{n}}}x\|^q,$$

i.e., \mathbf{A} is an $\tilde{\mathbf{n}}$ -quasi- $(1, q)$ -isometry c.t.o.

Proof. Since \mathbf{A} is an $\tilde{\mathbf{n}}$ -quasi- (p, q) -isometry c.t.o., then \mathbf{A} is an (p, q) -isometric on $\overline{\mathcal{R}(\mathbf{A}^{\tilde{\mathbf{n}}})}$ (Proposition 2.8). Therefore

$$\sum_{|\gamma|=l} \frac{l!}{\gamma!} \|A^\gamma \mathbf{A}^{\tilde{\mathbf{n}}}x\|^q = \sum_{j=1}^p \binom{l}{j} \Psi_j(A, \mathbf{A}^{\tilde{\mathbf{n}}}x)$$

Hence there exist real numbers $\delta_0(x), \delta_1(x), \dots, \delta_{p-1}(x)$ such that

$$\sum_{i=1}^d \|A_i^l \mathbf{A}^{\tilde{\mathbf{n}}}x\|^q = \sum_{j=0}^{p-1} \delta_j(x) l^j. \tag{2.8}$$

Since \mathbf{A} is power bounded, for $x \in \mathcal{X}$,

$$M = \sup \left\{ \sum_{|\gamma|=l} \frac{l!}{\alpha!} \|A^\gamma x\|^l, l = 0, 1, 2, \dots \right\} < \infty.$$

Then we have

$$0 \leq \sup \left\{ \sum_{j=0}^{p-1} \delta_j(x) l^j : l = 0, 1, 2, \dots \right\} \leq M^q.$$

From the fact that l is arbitrary, we obtain $\delta_1(x) = \delta_2(x) = \dots = \delta_{p-1}(x) = 0$. Hence

$$\sum_{i=0}^d \|A_i^l \mathbf{A}^{\tilde{\mathbf{n}}}x\|^q = \|\mathbf{A}^{\tilde{\mathbf{n}}}x\|^q.$$

Since l is arbitrary, by showing $l = 1$ we get equality. Hence \mathbf{A} is an $\tilde{\mathbf{n}}$ -quasi- $(1, q)$ -isometry c.t.o. by (2.4). □

3. Spectral properties of \tilde{n} -quasi- (p, q) -isometry

In this section we study some spectral properties of c.t.o. Our inspiration can be from referred to [6–8, 12, 15].

Definition 3.1. ([6]). Let $\mathbf{A} = (A_1, \dots, A_d) \in \mathcal{B}[\mathcal{X}]^d$ be any c.t.o. on complex Banach space \mathcal{X} .

Let $\mu = (\mu_1, \dots, \mu_d) \in \mathbb{C}^d$, then μ is said to be point eigenvalue of \mathbf{A} if

$$\bigcap_{j=1}^d \ker(A_j - \mu_j) \neq 0,$$

the set of all joint spectrum of \mathbf{A} by $\sigma_p(\mathbf{A})$.

Definition 3.2. ([6]). Let $\mathbf{A} = (A_1, \dots, A_d) \in \mathcal{B}[\mathcal{X}]^d$ be c.t.o, A number $\mu = (\mu_1, \dots, \mu_d) \in \mathbb{C}^d$ is in the joint approximate point spectrum $\sigma_{ap}(\mathbf{A})$ if

$$\exists (x_n)_n \in \mathcal{X}, \|x_n\| = 1$$

such that

$$(A_j - \mu_j)x_n \longrightarrow 0 \text{ as } n \longrightarrow \infty \text{ for every } j = 1, \dots, d.$$

We denote by

$$\mathbb{B}_q(\mathbb{C}^d) := \{ \mu = (\mu_1, \dots, \mu_d) \in \mathbb{C}^d / \|\mu\|_q = \left(\sum_{j=1}^d |\mu_j|^q \right)^{\frac{1}{q}} < 1 \}$$

and

$$\partial\mathbb{B}_q(\mathbb{C}^d) := \{ \mu = (\mu_1, \dots, \mu_d) \in \mathbb{C}^d / \|\mu\|_q = \left(\sum_{j=1}^d |\mu_j|^q \right)^{\frac{1}{q}} = 1 \}$$

Theorem 3.3. Let $\mathbf{A} = (A_1, \dots, A_d) \in \mathcal{B}[\mathcal{X}]^d$ be an \tilde{n} -quasi- (p, q) -isometry c.t.o. Then $\sigma_{ap}(\mathbf{A}) \subset [0] \cup \{ (\mu_1, \dots, \mu_d) \in \mathbb{C}^d / \left(\sum_{l=1}^d |\mu_l|^q \right)^{\frac{1}{q}} = 1 \}$, where

$$[0] := \{ (\mu_1, \dots, \mu_d) \in \mathbb{C}^d : \prod_{l=1}^d \mu_l = 0 \}.$$

Proof. Let $\mu = (\mu_1, \dots, \mu_d) \in \sigma_{ap}(\mathbf{A})$, then $\exists \{x_j\}_{j \geq 1} \subset \mathcal{X}$, with $\|x_j\| = 1$ such that $(A_j - \mu_j I)x_j \longrightarrow 0$ for all $j = 1, 2, \dots, d$. Since for $\gamma_j > 1$,

$$A_j^{\gamma_j} - \mu_j^{\gamma_j} = (A_j - \mu_j) \sum_{l=1}^{\gamma_j} \mu_j^{l-1} A_j^{\gamma_j-l}$$

By induction, for $\gamma \in \mathbb{Z}_+^d$, we have

$$(A^\gamma - \mu^\gamma I) = \sum_{l=0}^{\gamma} \left(\prod_{i \leq l} \mu_i^{\gamma_i} \right) (A_j^{\gamma_j} - \mu_j^{\gamma_j}) \prod_{i > l} A_i^{\gamma_i}.$$

$$\begin{aligned}
 0 &= \sum_{l=0}^p (-1)^l \binom{p}{l} \sum_{|\alpha|=l} \frac{l!}{\alpha!} \|\mathbf{A}^{\gamma+\bar{\mathbf{n}}} x_j\|^q \\
 &= \sum_{l=0}^p (-1)^l \binom{p}{l} \sum_{|\alpha|=l} \frac{l!}{\alpha!} \|(\mathbf{A}^{\gamma+\bar{\mathbf{n}}} - \mu^{\gamma+\bar{\mathbf{n}}})x_j + \mu^{\gamma+\bar{\mathbf{n}}}x_j\|^q \quad \forall j \\
 &= \sum_{l=0}^p (-1)^l \binom{p}{l} \sum_{|\alpha|=l} \frac{l!}{\alpha!} \|\mu^{\gamma+\bar{\mathbf{n}}}\|^q \\
 &= |\mu|^{\bar{q}\bar{\mathbf{n}}} \sum_{l=0}^p (-1)^l \binom{p}{l} \sum_{|\alpha|=l} \frac{l!}{\alpha!} |\mu|^{\alpha\gamma} \\
 0 &= |\mu|^{\bar{q}\bar{\mathbf{n}}} \left(1 - \left(\sum_{l=1}^d |\mu_l|^q\right)^{\frac{1}{q}}\right)^\gamma.
 \end{aligned}$$

Therefore $\mu = (\mu_1, \dots, \mu_d) \in [0]$ or $(\mu_1, \dots, \mu_d) \in \mathbb{C}^d / \left(\sum_{l=1}^d |\mu_l|^q\right)^{\frac{1}{q}} = 1$. \square

Theorem 3.4. *If $\mathbf{A} = (A_1 \dots, A_d) \in \mathcal{B}[X]^d$ is an $\bar{\mathbf{n}}$ -quasi- (p, q) -isometry c.t.o. Then $r(\mathbf{A}) = 1$. In particular $\sigma(\mathbf{A}) \subset [0]$ or $\sigma(\mathbf{A}) \subset [0] \cup \{(\mu_1, \dots, \mu_d) \in \mathbb{C}^d / \left(\sum_{l=1}^d |\mu_l|^q\right)^{\frac{1}{q}} = 1\}$.*

Proof. From Theorem 3.3, we obtain $\sigma_a(\mathbf{A}) \subset \partial\mathbb{B}_q(\mathbb{C}^d)$. From which it follows that $r(\mathbf{A}) = 1$. On the other hand, $\rho(\mathbf{A}) \cap \mathbb{B}_q(\mathbb{C}^d)$ is both open and closed subset of the domain $\mathbb{B}_q(\mathbb{C}^d)$. Consequently we find $\sigma(\mathbf{A}) \subset \partial\mathbb{B}_q(\mathbb{C}^d)$ or $\sigma(\mathbf{A}) = \bar{\mathbb{B}}_q(\mathbb{C}^d)$. \square

Acknowledgment. This work was funded by the Deanship of Scientific Research at Jouf University under grant No (DSR-2021-03-0335).

The authors would like to extend their thanks to the Editor and the Reviewer for the devoted time and efforts on our manuscript.

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