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Fractional Maclaurin-type inequalities for multiplicatively convex functions and multiplicatively P-functions

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Abstract. In this paper, we present a fractional integral identity, and then based upon it we establish the Maclaurin's inequalities for multiplicatively convex functions and multiplicatively P-functions via multiplicative Riemann–Liouville fractional integrals.

1. Introduction

We shall recall the definitions of convex functions, multiplicatively convex functions and multiplicatively P-functions.

A function $f\colon I \subseteq \mathbb{R}$ is said to be convex on I, where I is a real-valued interval, if the following inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

A function $f\colon I \subseteq \mathbb{R} \to (0, \infty)$ is said to be multiplicatively convex or log-convex, if $\log f$ is convex or equivalently the following inequality

$$f(tx + (1-t)y) \leq [f(x)]^t [f(y)]^{1-t}$$

holds for all $x, y \in I$ and $t \in [0, 1]$ (see [21]).

In accordance with the content above, it follows that

$$f(tx + (1-t)y) \leq [f(x)]^t [f(y)]^{1-t} \leq tf(x) + (1-t)f(y),$$

which reveals that every multiplicatively convex function is a convex function, but the converse is not true.

A function $f\colon I \subseteq \mathbb{R} \to (0, \infty)$ is said to be multiplicatively P-functions or log-P-functions, if the following inequality

$$f(tx + (1-t)y) \leq f(x)f(y),$$

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holds for all $x, y \in I$ and $t \in [0,1]$ (see [16]).

One of the most classical inequalities in mathematics for convex functions is the so called Hermite–Hadamard's integral inequality, which can be stated as

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}, \tag{1}$$

where f is a real-valved convex function on the finite interval $[a,b]$.

The inequality (1) and its variants have attracted the attentions of many researchers, and various generalizations, refinements, extensions and variants have appeared in the literature, see for example the published articles [2, 3, 6, 8, 10, 11, 14, 24, 25] and the bibliographies quoted in them.

Niculescu [19] established the following Hermite–Hadamard's inequalities for multiplicatively convex functions

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \log f(x)\mathrm{d}x \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{2}$$

Dragomir [13] gave the analogue of the inequality (1) for multiplicatively convex functions

$$f\left(\frac{a+b}{2}\right) \leq \exp\left\{\frac{1}{b-a}\int_a^b \ln f(x)\mathrm{d}x\right\} \leq \sqrt{f(a)f(b)}. \tag{3}$$

Sarikaya et al. [23] proved the following Riemann–Liouville (RL) fractional Hermite–Hadamard-type inequalities for convex functions, where $\alpha \in (0,1]$

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[\mathcal{I}_{a+}^{\alpha} f(b) + \mathcal{I}_{b-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2}. \tag{4}$$

In 2016, Abdeljawad and Grossman introduced a type of fractional integrals, called the multiplicative RL-fractional integrals, in the following way:

Definition 1.1. *[1] The multiplicative left-sided RL-fractional integral ${}_a\mathcal{I}_*^{\alpha} f(x)$ of order $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$ is defined by*

$$ {}_a\mathcal{I}_*^{\alpha} f(x) = \exp\left\{\left(\mathcal{I}_{a+}^{\alpha}(\ln \circ f)\right)(x)\right\}$$

and the multiplicative right-sided one ${}_*\mathcal{I}_b^{\alpha} f(x)$ is defined by

$$ {}_*\mathcal{I}_b^{\alpha} f(x) = \exp\left\{\left(\mathcal{I}_{b-}^{\alpha}(\ln \circ f)\right)(x)\right\},$$

where the symbols $\mathcal{I}_{a+}^{\alpha} f(x)$ and $\mathcal{I}_{b-}^{\alpha} f(x)$ denote respectively the left- and right-sided RL-fractional integrals, which are defined, correspondingly, by the following expressions

$$\mathcal{I}_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, \quad x > a$$

and

$$\mathcal{I}_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\mathrm{d}t, \quad x < b.$$

Budak et al. [12] derived the following Hermite–Hadamard's inequality for multiplicative RL-fractional integrals

$$f\left(\frac{a+b}{2}\right) \leq \left[{}_a\mathcal{I}_*^{\alpha} f(b) \cdot_* \mathcal{I}_b^{\alpha} f(a)\right]^{\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}} \leq \sqrt{f(a)f(b)} \tag{5}$$

and

$$f\left(\frac{a+b}{2}\right) \le \left[{}_{\frac{a+b}{2}}\mathcal{I}_{*}^{\alpha} f(b) \cdot_{*} \mathcal{I}_{\frac{a+b}{2}}^{\alpha} f(a)\right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}} \le \sqrt{f(a)f(b)}. \tag{6}$$

For recent results in connection with multiplicative integral operators, see [4, 15, 17, 20, 22].

The following inequality is referred to as the Maclaurin's quadrature rule, which can be stated as follows:

$$\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{7(b-a)^4}{51840}\|f^{(4)}\|_\infty, \tag{7}$$

where f is a four-order continuously differentiable function on (a,b) and $\|f^{(4)}\|_\infty = \sup_{x\in(a,b)} |f^{(4)}(x)|$ (see [7]).

Motivated by all of the results above, in this study we present an identity for multiplicative differentiable functions. Based upon it, we establish the Maclaurin-type inequalities for multiplicatively convex functions and multiplicatively P-functions through multiplicative RL-fractional integrals. Some applications of the obtained results are given as well.

2. Preliminaries

In 2008, Bashirov et al. [9] proposed a family of multiplicative integral operators, called the *integral operators, in the following way:

$$\int_a^b (f(x))^{\mathrm{d}x} = \exp\left\{\int_a^b \ln(f(x))\mathrm{d}x\right\}.$$

Proposition 2.1. *[9] Given that the positive function f is *integrable on the real-valued interval $[a,b]$, then we have the following properties:*

(i) $\int_a^b ((f(x))^p)^{\mathrm{d}x} = \left(\int_a^b (f(x))^{\mathrm{d}x}\right)^p, \ p \in \mathbb{R}$,

(ii) $\int_a^b (f(x)g(x))^{\mathrm{d}x} = \int_a^b (f(x))^{\mathrm{d}x} \cdot \int_a^b (g(x))^{\mathrm{d}x}$,

(iii) $\int_a^b \left(\frac{f(x)}{g(x)}\right)^{\mathrm{d}x} = \frac{\int_a^b (f(x))^{\mathrm{d}x}}{\int_a^b (g(x))^{\mathrm{d}x}}$,

(iv) $\int_a^b (f(x))^{\mathrm{d}x} = \int_a^c (f(x))^{\mathrm{d}x} \cdot \int_c^b (f(x))^{\mathrm{d}x}, \quad a \le c \le b$,

(v) $\int_a^a (f(x))^{\mathrm{d}x} = 1$ *and* $\int_b^a (f(x))^{\mathrm{d}x} = \left(\int_a^b (f(x))^{\mathrm{d}x}\right)^{-1}$.

Bashirov et al. also proposed the multiplicative derivative of the functions.

Definition 2.2. *[9] Given that the function $f : \mathbb{R} \to \mathbb{R}$ is positive. The multiplicative derivative of function f, denoted it by f^*, is given as*

$$\frac{\mathrm{d}^* f(x)}{\mathrm{d}x} = f^*(x) = \lim_{h\to 0}\left(\frac{f(x+h)}{f(x)}\right)^{\frac{1}{h}}.$$

The relation between the f^* and the ordinary derivative f' is the following:

$$f^*(x) = \exp\big\{[\ln f(x)]'\big\} = \exp\left\{\frac{f'(x)}{f(x)}\right\}.$$

Proposition 2.3. *[9] Given that the functions f and g are both multiplicative derivative, and h is differentiable. If c is a positive constant, then functions cf, $f+g$, fg, $\frac{f}{g}$, f^h and $f\circ h$ are all multiplicative derivative, and we have the following properties:*

(i) $(cf)^*(x) = f^*(x)$,

(ii) $(f+g)^*(x) = f^*(x)^{\frac{f(x)}{f(x)+g(x)}} \cdot g^*(x)^{\frac{g(x)}{f(x)+g(x)}}$,

(iii) $(fg)^*(x) = f^*(x)g^*(x)$,

(iv) $\left(\frac{f}{g}\right)^*(x) = \frac{f^*(x)}{g^*(x)}$,

(v) $\left(f^h\right)^*(x) = f^*(x)^{h(x)} \cdot f(x)^{h'(x)}$,

(vi) $(f\circ h)^*(x) = f^*(h(x))^{h'(x)}$.

The formulas of the multiplicative integration by parts are the following ones.

Theorem 2.4. *[9] Let $f:[a,b]\to\mathbb{R}$ be multiplicative differentiable, and let $g:[a,b]\to\mathbb{R}$ be differentiable. Then the function f^g is multiplicative integrable. And we have that*

$$\int_a^b \left(f^*(x)^{g(x)}\right)^{\mathrm{d}x} = \frac{f(b)^{g(b)}}{f(a)^{g(a)}} \cdot \frac{1}{\int_a^b \left(f(x)^{g'(x)}\right)^{\mathrm{d}x}}.$$

Lemma 2.5. *[5]. Let $f:[a,b]\to\mathbb{R}$ be multiplicative differentiable, and let $g:[a,b]\to\mathbb{R}$ and $h:J\subset\mathbb{R}\to[a,b]$ be two differentiable functions. Then we have that*

$$\int_a^b \left(f^*(h(x))^{g(x)h'(x)}\right)^{\mathrm{d}x} = \frac{f(h(b))^{g(b)}}{f(h(a))^{g(a)}} \cdot \frac{1}{\int_a^b \left(f(h(x))^{g'(x)}\right)^{\mathrm{d}x}}.$$

3. Main Results

Our main results depend on the following lemma.

Lemma 3.1. *Let $f:[a,b]\to\mathbb{R}^+$ be a multiplicative differentiable function on (a,b) with $a<b$. If f^* is multiplicative integrable on $[a,b]$, then we have the following multiplicative RL-fractional integral identity*

$$\begin{aligned}
{}_*\mathcal{J}_f(\alpha;a,b) = &\left(\int_0^1 \left(\left(f^*\left((1-t)a+t\frac{5a+b}{6}\right)\right)^{\frac{t^\alpha}{6}}\right)^{\mathrm{d}t}\right)^{\frac{b-a}{6}} \\
&\times \left(\int_0^1 \left(\left(f^*\left((1-t)\frac{5a+b}{6}+t\frac{a+b}{2}\right)\right)^{\frac{t^\alpha}{3}-\frac{5}{24}}\right)^{\mathrm{d}t}\right)^{\frac{b-a}{3}} \\
&\times \left(\int_0^1 \left(\left(f^*\left((1-t)\frac{a+b}{2}+t\frac{a+5b}{6}\right)\right)^{\frac{t^\alpha}{3}-\frac{1}{8}}\right)^{\mathrm{d}t}\right)^{\frac{b-a}{3}} \\
&\times \left(\int_0^1 \left(\left(f^*\left((1-t)\frac{a+5b}{6}+tb\right)\right)^{\frac{t^\alpha}{6}-\frac{1}{6}}\right)^{\mathrm{d}t}\right)^{\frac{b-a}{6}},
\end{aligned} \tag{8}$$

where

$$
{}_*\mathcal{J}_f(\alpha;a,b) := \frac{\left[\left(f\left(\frac{5a+b}{6}\right)\right)^3\left(f\left(\frac{a+b}{2}\right)\right)^2\left(f\left(\frac{a+5b}{6}\right)\right)^3\right]^{\frac{1}{8}}}{\left[{}_*\mathcal{I}^{\alpha}_{\left(\frac{5a+b}{6}\right)^-}f(a)\cdot{}_*\mathcal{I}^{\alpha}_{b^-}f\left(\frac{a+5b}{6}\right)\right]^{\frac{6^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}\cdot\left[{}_*\mathcal{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f\left(\frac{5a+b}{6}\right)\cdot{}_*\mathcal{I}^{\alpha}_{\left(\frac{a+5b}{6}\right)^-}f\left(\frac{a+b}{2}\right)\right]^{\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}}.
$$

Proof. For sake of brevity, we use the following notations:

$$
I_1 = \left(\int_0^1\left(\left(f^*\left((1-t)a+t\frac{5a+b}{6}\right)\right)^{\frac{t^{\alpha}}{6}}\right)^{\mathrm{d}t}\right)^{\frac{b-a}{6}},
$$

$$
I_2 = \left(\int_0^1\left(\left(f^*\left((1-t)\frac{5a+b}{6}+t\frac{a+b}{2}\right)\right)^{\frac{t^{\alpha}}{3}-\frac{5}{24}}\right)^{\mathrm{d}t}\right)^{\frac{b-a}{3}},
$$

$$
I_3 = \left(\int_0^1\left(\left(f^*\left((1-t)\frac{a+b}{2}+t\frac{a+5b}{6}\right)\right)^{\frac{t^{\alpha}}{3}-\frac{1}{8}}\right)^{\mathrm{d}t}\right)^{\frac{b-a}{3}}
$$

and

$$
I_4 = \left(\int_0^1\left(\left(f^*\left((1-t)\frac{a+5b}{6}+tb\right)\right)^{\frac{t^{\alpha}}{6}-\frac{1}{6}}\right)^{\mathrm{d}t}\right)^{\frac{b-a}{6}}.
$$

Using the integration by parts for multiplicative integrals, from I_1 we have that

$$
\begin{aligned}
I_1 &= \int_0^1\left(\left(f^*\left((1-t)a+t\frac{5a+b}{6}\right)\right)^{\frac{b-a}{6}\cdot\frac{t^{\alpha}}{6}}\right)^{\mathrm{d}t}\\
&= \frac{\left(f\left(\frac{5a+b}{6}\right)\right)^{\frac{1}{6}}}{\left(f\left(a\right)\right)^0}\cdot\frac{1}{\int_0^1\left(\left(f\left((1-t)a+t\frac{5a+b}{6}\right)\right)^{\frac{\alpha t^{\alpha-1}}{6}}\right)^{\mathrm{d}t}}\\
&= \frac{\left(f\left(\frac{5a+b}{6}\right)\right)^{\frac{1}{6}}}{\exp\left\{\int_0^1\frac{\alpha}{6}\cdot t^{\alpha-1}\ln f\left((1-t)a+t\frac{5a+b}{6}\right)\mathrm{d}t\right\}}\\
&= \frac{\left(f\left(\frac{5a+b}{6}\right)\right)^{\frac{1}{6}}}{\exp\left\{\int_a^{\frac{5a+b}{6}}\frac{\alpha}{6}\cdot\left(\frac{6(u-a)}{b-a}\right)^{\alpha-1}\cdot\frac{6}{b-a}\ln f\left(u\right)\mathrm{d}u\right\}}\\
&= \frac{\left(f\left(\frac{5a+b}{6}\right)\right)^{\frac{1}{6}}}{\exp\left\{\frac{6^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\mathcal{I}^{\alpha}_{\left(\frac{5a+b}{6}\right)^-}\ln f(a)\right\}}.
\end{aligned}
$$

We also derive that

$$\begin{aligned}
I_2 &= \int_0^1 \left(\left(f^* \left((1-t)\frac{5a+b}{6} + t\frac{a+b}{2} \right) \right)^{\frac{b-a}{3}\cdot\left(\frac{t^\alpha}{3}-\frac{5}{24}\right)} \right)^{\mathrm{d}t} \\
&= \frac{\left(f\left(\frac{a+b}{2}\right)\right)^{\frac{1}{8}}}{\left(f\left(\frac{5a+b}{6}\right)\right)^{-\frac{5}{24}}} \cdot \frac{1}{\int_0^1 \left(\left(f\left((1-t)\frac{5a+b}{6} + t\frac{a+b}{2} \right) \right)^{\frac{\alpha t^{\alpha-1}}{3}} \right)^{\mathrm{d}t}} \\
&= \frac{\left(f\left(\frac{a+b}{2}\right)\right)^{\frac{1}{8}} \left(f\left(\frac{5a+b}{6}\right)\right)^{\frac{5}{24}}}{\exp\left\{ \int_0^1 \frac{\alpha}{3} t^{\alpha-1} \ln f\left((1-t)\frac{5a+b}{6} + t\frac{a+b}{2} \right) \mathrm{d}t \right\}} \\
&= \frac{\left(f\left(\frac{a+b}{2}\right)\right)^{\frac{1}{8}} \left(f\left(\frac{5a+b}{6}\right)\right)^{\frac{5}{24}}}{\exp\left\{ \int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} \frac{\alpha}{3} \cdot \left(\frac{3(u-\frac{5a+b}{6})}{b-a} \right)^{\alpha-1} \cdot \frac{3}{b-a} \ln f(u)\, \mathrm{d}u \right\}} \\
&= \frac{\left(f\left(\frac{a+b}{2}\right)\right)^{\frac{1}{8}} \left(f\left(\frac{5a+b}{6}\right)\right)^{\frac{5}{24}}}{\exp\left\{ \frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \mathcal{I}^\alpha_{\left(\frac{a+b}{2}\right)^-} \ln f\left(\frac{5a+b}{6}\right) \right\}}.
\end{aligned}$$

Similarly,

$$I_3 = \frac{\left(f\left(\frac{a+5b}{6}\right)\right)^{\frac{5}{24}} \left(f\left(\frac{a+b}{2}\right)\right)^{\frac{1}{8}}}{\exp\left\{ \frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \mathcal{I}^\alpha_{\left(\frac{a+5b}{6}\right)^-} \ln f\left(\frac{a+b}{2}\right) \right\}}$$

and

$$I_4 = \frac{\left(f\left(\frac{a+5b}{6}\right)\right)^{\frac{1}{6}}}{\exp\left\{ \frac{6^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \mathcal{I}^\alpha_{b^-} \ln f\left(\frac{a+5b}{6}\right) \right\}}.$$

Multiplying four equalities above, we get that

$$\begin{aligned}
&I_1 \times I_2 \times I_3 \times I_4 \\
&= \frac{\left(f\left(\frac{5a+b}{6}\right)\right)^{\frac{3}{8}} \left(f\left(\frac{a+b}{2}\right)\right)^{\frac{2}{8}} \left(f\left(\frac{a+5b}{6}\right)\right)^{\frac{3}{8}}}{\exp\left\{ \varrho_1 \left[\mathcal{I}^\alpha_{\left(\frac{5a+b}{6}\right)^-} \ln f(a) + \mathcal{I}^\alpha_{b^-} \ln f\left(\frac{a+5b}{6}\right) \right] \right\} \cdot \exp\left\{ \varrho_2 \left[\mathcal{I}^\alpha_{\left(\frac{a+b}{2}\right)^-} \ln f\left(\frac{5a+b}{6}\right) + \mathcal{I}^\alpha_{\left(\frac{a+5b}{6}\right)^-} \ln f\left(\frac{a+b}{2}\right) \right] \right\}} \\
&= \frac{\left(f\left(\frac{5a+b}{6}\right)\right)^{\frac{3}{8}} \left(f\left(\frac{a+b}{2}\right)\right)^{\frac{2}{8}} \left(f\left(\frac{a+5b}{6}\right)\right)^{\frac{3}{8}}}{\left[{}_*\mathcal{I}^\alpha_{\left(\frac{5a+b}{6}\right)} f(a) \cdot_* \mathcal{I}^\alpha_{b} f\left(\frac{a+5b}{6}\right) \right]^{\varrho_1} \cdot \left[{}_*\mathcal{I}^\alpha_{\left(\frac{a+b}{2}\right)} f\left(\frac{5a+b}{6}\right) \cdot_* \mathcal{I}^\alpha_{f\left(\frac{a+5b}{6}\right)} f\left(\frac{a+b}{2}\right) \right]^{\varrho_2}},
\end{aligned}$$

where

$$\varrho_1 = \frac{6^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}$$

and

$$\varrho_1 = \frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}.$$

The proof is completed.

Remark 3.2. *If we consider taking $\alpha = 1$ in Lemma 3.1, then the result reduces to Lemma 3.1 established by Meftah in [18].*

Theorem 3.3. *Let $f : [a, b] \to \mathbb{R}^+$ be an increasing multiplicative differentiable function on (a, b) with $a < b$. If f^* is multiplicative convex on $[a, b]$, then for $\alpha > 0$ the following inequality in related to multiplicative RL-fractional integrals holds:*

$$\left|{}_*\mathcal{J}_f(\alpha; a, b)\right| \leq (f^*(a))^{\frac{b-a}{6}\Delta_1} \left(f^*\left(\frac{5a+b}{6}\right)\right)^{\frac{b-a}{3}\left(\frac{1}{2}\Delta_2+\Delta_3+\Delta_4\right)} \left(f^*\left(\frac{a+b}{2}\right)\right)^{\frac{b-a}{3}(\Delta_5+\Delta_6+\Delta_7+\Delta_8)} \times \left(f^*\left(\frac{a+5b}{6}\right)\right)^{\frac{b-a}{3}\left(\Delta_9+\Delta_{10}+\frac{1}{2}\Delta_{11}\right)} (f^*(b))^{\frac{b-a}{6}\Delta_{12}},$$

where

$$\Delta_1 = \frac{1}{6(\alpha+1)(\alpha+2)}, \ \Delta_2 = \frac{1}{6(\alpha+2)}, \ \Delta_3 = \frac{5\alpha}{24(\alpha+1)}\left(\frac{5}{8}\right)^{\frac{1}{\alpha}} - \frac{5\alpha}{48(\alpha+2)}\left(\frac{5}{8}\right)^{\frac{2}{\alpha}},$$

$$\Delta_4 = \frac{1}{3(\alpha+1)(\alpha+2)} - \frac{5}{48} + \Delta_3, \ \Delta_5 = \frac{5\alpha}{48(\alpha+2)}\left(\frac{5}{8}\right)^{\frac{2}{\alpha}}, \ \Delta_6 = \frac{1}{3(\alpha+2)} - \frac{5}{48} + \Delta_5,$$

$$\Delta_7 = \frac{\alpha}{8(\alpha+1)}\left(\frac{3}{8}\right)^{\frac{1}{\alpha}} - \frac{\alpha}{16(\alpha+2)}\left(\frac{3}{8}\right)^{\frac{2}{\alpha}}, \ \Delta_8 = \frac{1}{3(\alpha+1)(\alpha+2)} - \frac{1}{16} + \Delta_7, \ \Delta_9 = \frac{\alpha}{16(\alpha+2)}\left(\frac{3}{8}\right)^{\frac{2}{\alpha}},$$

$$\Delta_{10} = \frac{1}{3(\alpha+2)} - \frac{1}{16} + \Delta_9, \ \Delta_{11} = \frac{1}{12} - \frac{1}{6(\alpha+1)(\alpha+2)}, \ \Delta_{12} = \frac{1}{12} - \frac{1}{6(\alpha+2)}.$$

Proof. Taking advantage of Lemma 3.1, the property of multiplicative integrals, the multiplicative convexity of f^* together with the increased monotonicity of f, it follows that

$$\begin{aligned}\left|{}_*\mathcal{J}_f(\alpha; a, b)\right| \leq & \left(\exp\left\{\frac{b-a}{6}\int_0^1 \left|\ln\left(f^*\left((1-t)a + t\frac{5a+b}{6}\right)\right)^{\frac{t^\alpha}{6}}\right| dt\right\}\right) \\ & \times \left(\exp\left\{\frac{b-a}{3}\int_0^1 \left|\ln\left(f^*\left((1-t)\frac{5a+b}{6} + t\frac{a+b}{2}\right)\right)^{\frac{t^\alpha}{3}-\frac{5}{24}}\right| dt\right\}\right) \\ & \times \left(\exp\left\{\frac{b-a}{3}\int_0^1 \left|\ln\left(f^*\left((1-t)\frac{a+b}{2} + t\frac{a+5b}{6}\right)\right)^{\frac{t^\alpha}{3}-\frac{1}{8}}\right| dt\right\}\right) \\ & \times \left(\exp\left\{\frac{b-a}{6}\int_0^1 \left|\ln\left(f^*\left((1-t)\frac{a+5b}{6} + tb\right)\right)^{\frac{t^\alpha}{6}-\frac{1}{6}}\right| dt\right\}\right)\end{aligned}$$

$$
\begin{aligned}
&= \left(\exp\left\{\frac{b-a}{6}\int_0^1 \frac{t^\alpha}{6}\cdot\left|\ln\left(f^*\left((1-t)a+t\frac{5a+b}{6}\right)\right)\right|\mathrm{d}t\right\}\right)\\
&\quad\times\left(\exp\left\{\frac{b-a}{3}\int_0^1 \left|\frac{t^\alpha}{3}-\frac{5}{24}\right|\cdot\left|\ln\left(f^*\left((1-t)\frac{5a+b}{6}+t\frac{a+b}{2}\right)\right)\right|\mathrm{d}t\right\}\right)\\
&\quad\times\left(\exp\left\{\frac{b-a}{3}\int_0^1 \left|\frac{t^\alpha}{3}-\frac{1}{8}\right|\cdot\left|\ln\left(f^*\left((1-t)\frac{a+b}{2}+t\frac{a+5b}{6}\right)\right)\right|\mathrm{d}t\right\}\right)\\
&\quad\times\left(\exp\left\{\frac{b-a}{6}\int_0^1 \left|\frac{t^\alpha}{6}-\frac{1}{6}\right|\cdot\left|\ln\left(f^*\left((1-t)\frac{a+5b}{6}+tb\right)\right)\right|\mathrm{d}t\right\}\right)\\
&= \left(\exp\left\{\frac{b-a}{6}\int_0^1 \frac{t^\alpha}{6}\cdot\left|\ln\left(f^*\left((1-t)a+t\frac{5a+b}{6}\right)\right)\right|\mathrm{d}t\right\}\right)\\
&\quad\times\left(\exp\left\{\frac{b-a}{3}\int_0^{\left(\frac{5}{8}\right)^{\frac{1}{\alpha}}} \left(\frac{5}{24}-\frac{t^\alpha}{3}\right)\cdot\left|\ln\left(f^*\left((1-t)\frac{5a+b}{6}+t\frac{a+b}{2}\right)\right)\right|\mathrm{d}t\right\}\right)\\
&\quad\times\left(\exp\left\{\frac{b-a}{3}\int_{\left(\frac{5}{8}\right)^{\frac{1}{\alpha}}}^1 \left(\frac{t^\alpha}{3}-\frac{5}{24}\right)\cdot\left|\ln\left(f^*\left((1-t)\frac{5a+b}{6}+t\frac{a+b}{2}\right)\right)\right|\mathrm{d}t\right\}\right)\\
&\quad\times\left(\exp\left\{\frac{b-a}{3}\int_0^{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}} \left(\frac{1}{8}-\frac{t^\alpha}{3}\right)\cdot\left|\ln\left(f^*\left((1-t)\frac{a+b}{2}+t\frac{a+5b}{6}\right)\right)\right|\mathrm{d}t\right\}\right)\\
&\quad\times\left(\exp\left\{\frac{b-a}{3}\int_{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}^1 \left(\frac{t^\alpha}{3}-\frac{1}{8}\right)\cdot\left|\ln\left(f^*\left((1-t)\frac{a+b}{2}+t\frac{a+5b}{6}\right)\right)\right|\mathrm{d}t\right\}\right)\\
&\quad\times\left(\exp\left\{\frac{b-a}{6}\int_0^1 \left(\frac{1}{6}-\frac{t^\alpha}{6}\right)\cdot\left|\ln\left(f^*\left((1-t)\frac{a+5b}{6}+tb\right)\right)\right|\mathrm{d}t\right\}\right)\\
&\le \left(\exp\left\{\frac{b-a}{6}\int_0^1 \frac{t^\alpha}{6}\cdot\ln\left((f^*(a))^{1-t}\left(f^*\left(\frac{5a+b}{6}\right)\right)^t\right)\mathrm{d}t\right\}\right)\\
&\quad\times\left(\exp\left\{\frac{b-a}{3}\int_0^{\left(\frac{5}{8}\right)^{\frac{1}{\alpha}}} \left(\frac{5}{24}-\frac{t^\alpha}{3}\right)\cdot\ln\left(\left(f^*\left(\frac{5a+b}{6}\right)\right)^{1-t}\left(f^*\left(\frac{a+b}{2}\right)\right)^t\right)\mathrm{d}t\right\}\right)\\
&\quad\times\left(\exp\left\{\frac{b-a}{3}\int_{\left(\frac{5}{8}\right)^{\frac{1}{\alpha}}}^1 \left(\frac{t^\alpha}{3}-\frac{5}{24}\right)\cdot\ln\left(\left(f^*\left(\frac{5a+b}{6}\right)\right)^{1-t}\left(f^*\left(\frac{a+b}{2}\right)\right)^t\right)\mathrm{d}t\right\}\right)\\
&\quad\times\left(\exp\left\{\frac{b-a}{3}\int_0^{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}} \left(\frac{1}{8}-\frac{t^\alpha}{3}\right)\cdot\ln\left(\left(f^*\left(\frac{a+b}{2}\right)\right)^{1-t}\left(f^*\left(\frac{a+5b}{6}\right)\right)^t\right)\mathrm{d}t\right\}\right)\\
&\quad\times\left(\exp\left\{\frac{b-a}{3}\int_{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}^1 \left(\frac{t^\alpha}{3}-\frac{1}{8}\right)\cdot\ln\left(\left(f^*\left(\frac{a+b}{2}\right)\right)^{1-t}\left(f^*\left(\frac{a+5b}{6}\right)\right)^t\right)\mathrm{d}t\right\}\right)\\
&\quad\times\left(\exp\left\{\frac{b-a}{6}\int_0^1 \left(\frac{1}{6}-\frac{t^\alpha}{6}\right)\cdot\ln\left(\left(f^*\left(\frac{a+5b}{6}\right)\right)^{1-t}(f^*(b))^t\right)\mathrm{d}t\right\}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\exp\left\{\frac{b-a}{6}\left(\ln\left(f^{*}(a)\right)\cdot\int_{0}^{1}\frac{1}{6}t^{\alpha}(1-t)\mathrm{d}t\right)\right\}\right)\\
&\quad\times\left(\exp\left\{\frac{b-a}{6}\left(\ln\left(f^{*}\left(\frac{5a+b}{6}\right)\right)\cdot\int_{0}^{1}\frac{1}{6}t^{\alpha}t\mathrm{d}t\right)\right\}\right)\\
&\quad\times\left(\exp\left\{\frac{b-a}{3}\left(\ln\left(f^{*}\left(\frac{5a+b}{6}\right)\right)\cdot\int_{0}^{\left(\frac{5}{8}\right)^{\frac{1}{\alpha}}}\left(\frac{5}{24}-\frac{1}{3}t^{\alpha}\right)(1-t)\mathrm{d}t\right)\right\}\right)\\
&\quad\times\left(\exp\left\{\frac{b-a}{3}\left(\ln\left(f^{*}\left(\frac{a+b}{2}\right)\right)\cdot\int_{0}^{\left(\frac{5}{8}\right)^{\frac{1}{\alpha}}}\left(\frac{5}{24}-\frac{1}{3}t^{\alpha}\right)t\mathrm{d}t\right)\right\}\right)\\
&\quad\times\left(\exp\left\{\frac{b-a}{3}\left(\ln\left(f^{*}\left(\frac{5a+b}{6}\right)\right)\cdot\int_{\left(\frac{5}{8}\right)^{\frac{1}{\alpha}}}^{1}\left(\frac{1}{3}t^{\alpha}-\frac{5}{24}\right)(1-t)\mathrm{d}t\right)\right\}\right)\\
&\quad\times\left(\exp\left\{\frac{b-a}{3}\left(\ln\left(f^{*}\left(\frac{a+b}{2}\right)\right)\cdot\int_{\left(\frac{5}{8}\right)^{\frac{1}{\alpha}}}^{1}\left(\frac{1}{3}t^{\alpha}-\frac{5}{24}\right)t\mathrm{d}t\right)\right\}\right)\\
&\quad\times\left(\exp\left\{\frac{b-a}{3}\left(\ln\left(f^{*}\left(\frac{a+b}{2}\right)\right)\cdot\int_{0}^{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}\left(\frac{1}{8}-\frac{1}{3}t^{\alpha}\right)(1-t)\mathrm{d}t\right)\right\}\right)\\
&\quad\times\left(\exp\left\{\frac{b-a}{3}\left(\ln\left(f^{*}\left(\frac{a+5b}{6}\right)\right)\cdot\int_{0}^{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}\left(\frac{1}{8}-\frac{1}{3}t^{\alpha}\right)t\mathrm{d}t\right)\right\}\right)\\
&\quad\times\left(\exp\left\{\frac{b-a}{3}\left(\ln\left(f^{*}\left(\frac{a+b}{2}\right)\right)\cdot\int_{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}^{1}\left(\frac{1}{3}t^{\alpha}-\frac{1}{8}\right)(1-t)\mathrm{d}t\right)\right\}\right)\\
&\quad\times\left(\exp\left\{\frac{b-a}{3}\left(\ln\left(f^{*}\left(\frac{a+5b}{6}\right)\right)\cdot\int_{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}^{1}\left(\frac{1}{3}t^{\alpha}-\frac{1}{8}\right)t\mathrm{d}t\right)\right\}\right)\\
&\quad\times\left(\exp\left\{\frac{b-a}{6}\left(\ln\left(f^{*}\left(\frac{a+5b}{6}\right)\right)\cdot\int_{0}^{1}\left(\frac{1}{6}-\frac{1}{6}t^{\alpha}\right)(1-t)\mathrm{d}t\right)\right\}\right)\\
&\quad\times\left(\exp\left\{\frac{b-a}{6}\left(\ln\left(f^{*}(b)\right)\cdot\int_{0}^{1}\left(\frac{1}{6}-\frac{1}{6}t^{\alpha}\right)t\mathrm{d}t\right)\right\}\right).
\end{aligned}
$$

The desired inequality yields from the above by noting that

$$\Delta_1 := \int_0^1 \frac{1}{6}t^{\alpha}(1-t)\mathrm{d}t = \frac{1}{6(\alpha+1)(\alpha+2)},$$

$$\Delta_2 := \int_0^1 \frac{1}{6}t^{\alpha}t\mathrm{d}t = \frac{1}{6(\alpha+2)},$$

$$\Delta_3 := \int_0^{\left(\frac{5}{8}\right)^{\frac{1}{\alpha}}}\left(\frac{5}{24}-\frac{1}{3}t^{\alpha}\right)(1-t)\mathrm{d}t = \frac{5\alpha}{24(\alpha+1)}\left(\frac{5}{8}\right)^{\frac{1}{\alpha}} - \frac{5\alpha}{48(\alpha+2)}\left(\frac{5}{8}\right)^{\frac{2}{\alpha}},$$

$$\Delta_4 := \int_{\left(\frac{5}{8}\right)^{\frac{1}{\alpha}}}^1\left(\frac{1}{3}t^{\alpha}-\frac{5}{24}\right)(1-t)\mathrm{d}t = \frac{1}{3(\alpha+1)(\alpha+2)} - \frac{5}{48} + \Delta_3,$$

$$\Delta_5 := \int_0^{\left(\frac{5}{8}\right)^{\frac{1}{\alpha}}} \left(\frac{5}{24} - \frac{1}{3}t^\alpha\right) t\mathrm{d}t = \frac{5\alpha}{48(\alpha+2)}\left(\frac{5}{8}\right)^{\frac{2}{\alpha}},$$

$$\Delta_6 := \int_{\left(\frac{5}{8}\right)^{\frac{1}{\alpha}}}^1 \left(\frac{1}{3}t^\alpha - \frac{5}{24}\right) t\mathrm{d}t = \frac{1}{3(\alpha+2)} - \frac{5}{48} + \Delta_5,$$

$$\Delta_7 := \int_0^{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}} \left(\frac{1}{8} - \frac{1}{3}t^\alpha\right)(1-t)\mathrm{d}t = \frac{\alpha}{8(\alpha+1)}\left(\frac{3}{8}\right)^{\frac{1}{\alpha}} - \frac{\alpha}{16(\alpha+2)}\left(\frac{3}{8}\right)^{\frac{2}{\alpha}},$$

$$\Delta_8 := \int_{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}^1 \left(\frac{1}{3}t^\alpha - \frac{1}{8}\right)(1-t)\mathrm{d}t = \frac{1}{3(\alpha+1)(\alpha+2)} - \frac{1}{16} + \Delta_7,$$

$$\Delta_9 := \int_0^{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}} \left(\frac{1}{8} - \frac{1}{3}t^\alpha\right) t\mathrm{d}t = \frac{\alpha}{16(\alpha+2)}\left(\frac{3}{8}\right)^{\frac{2}{\alpha}},$$

$$\Delta_{10} := \int_{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}^1 \left(\frac{1}{3}t^\alpha - \frac{1}{8}\right) t\mathrm{d}t = \frac{1}{3(\alpha+2)} - \frac{1}{16} + \Delta_9,$$

$$\Delta_{11} := \int_0^1 \left(\frac{1}{6} - \frac{1}{6}t^\alpha\right)(1-t)\mathrm{d}t = \frac{1}{12} - \frac{1}{6(\alpha+1)(\alpha+2)}$$

and

$$\Delta_{12} := \int_0^1 \left(\frac{1}{6} - \frac{1}{6}t^\alpha\right) t\mathrm{d}t = \frac{1}{12} - \frac{1}{6(\alpha+2)}.$$

The proof is completed.

Corollary 3.4. *In Theorem 3.3, using the multiplicative convexity of f^*, i.e., $f^*\left(\frac{a+b}{2}\right) \leq \sqrt{f^*(a)\, f^*(b)}$, we deduce that*

$$\left|{}_*\mathcal{J}_f(\alpha; a, b)\right| \leq \left[(f^*(a))^{(\Delta_1+\Delta_5+\Delta_6+\Delta_7+\Delta_8)} \left(f^*\left(\frac{5a+b}{6}\right)\right)^{(\Delta_2+2\Delta_3+2\Delta_4)} \times \left(f^*\left(\frac{a+5b}{6}\right)\right)^{(2\Delta_9+2\Delta_{10}+\Delta_{11})} (f^*(b))^{(\Delta_5+\Delta_6+\Delta_7+\Delta_8+\Delta_{12})}\right]^{\frac{b-a}{6}}.$$

Corollary 3.5. *In Theorem 3.3, using the multiplicative convexity of f^*, i.e., $f^*\left(\frac{a+b}{2}\right) \leq \sqrt{f^*\left(\frac{5a+b}{6}\right) f^*\left(\frac{a+5b}{6}\right)}$, we deduce that*

$$\left|{}_*\mathcal{J}_f(\alpha; a, b)\right| \leq \left[(f^*(a))^{\Delta_1} \left(f^*\left(\frac{5a+b}{6}\right)\right)^{(\Delta_2+2\Delta_3+2\Delta_4+\Delta_5+\Delta_6+\Delta_7+\Delta_8)} \times \left(f^*\left(\frac{a+5b}{6}\right)\right)^{(\Delta_5+\Delta_6+\Delta_7+\Delta_8+2\Delta_9+2\Delta_{10}+\Delta_{11})} (f^*(b))^{\Delta_{12}}\right]^{\frac{b-a}{6}}.$$

Corollary 3.6. *In Theorem 3.3, if we assume that $f^* \leq M,\ M > 0$, then we deduce that*

$$\left|{}_*\mathcal{J}_f(\alpha; a, b)\right| \leq M^{\frac{b-a}{6}\left[\frac{5\alpha}{6(\alpha+1)}\left(\frac{5}{8}\right)^{\frac{1}{\alpha}}+\frac{\alpha}{2(\alpha+1)}\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}+\frac{8+4\alpha}{3(\alpha+1)(\alpha+2)}-\frac{1}{2}\right]}.$$

Theorem 3.7. *Let $f : [a, b] \to \mathbb{R}^+$ be an increasing multiplicative differentiable function on (a, b) with $a < b$. If f^* is a multiplicatively P-function on $[a, b]$, then for $\alpha > 0$ the following inequality in related to multiplicative RL-fractional integrals holds:*

$$\left|{}_*\mathcal{J}_f(\alpha; a, b)\right| \leq (f^*(a))^{\eta_1}\left(f^*\left(\frac{5a+b}{6}\right)\right)^{\eta_2}\left(f^*\left(\frac{a+b}{2}\right)\right)^{\eta_3}\left(f^*\left(\frac{a+5b}{6}\right)\right)^{\eta_4}(f^*(b))^{\eta_5},$$

where

$$\eta_1 = \frac{b-a}{6}\cdot\frac{1}{6(\alpha+1)},$$

$$\eta_2 = \frac{b-a}{6}\left[\frac{5}{6(\alpha+1)}+\frac{5\alpha}{6(\alpha+1)}\left(\frac{5}{8}\right)^{\frac{1}{\alpha}}-\frac{5}{12}\right],$$

$$\eta_3 = \frac{b-a}{3}\left[\frac{5\alpha}{12(\alpha+1)}\left(\frac{5}{8}\right)^{\frac{1}{\alpha}}+\frac{\alpha}{4(\alpha+1)}\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}+\frac{2}{3(\alpha+1)}-\frac{1}{3}\right],$$

$$\eta_4 = \frac{b-a}{6}\left[\frac{\alpha}{2(\alpha+1)}\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}+\frac{1}{2(\alpha+1)}-\frac{1}{12}\right],$$

and

$$\eta_5 = \frac{b-a}{6}\cdot\frac{\alpha}{6(\alpha+1)}.$$

Proof. By the similar procedure are used in the proof of Theorem 3.3, with Lemma 3.1, the property of the multiplicatively P-functions of f^*, and some integral computations, we get the desired result of Theorem 3.7.

Corollary 3.8. *In Theorem 3.7, using the property of the multiplicatively P-functions of f^*, i.e., $f^*\left(\frac{a+b}{2}\right) \leq f^*(a)\, f^*(b)$, we deduce that*

$$\left|{}_*\mathcal{J}_f(\alpha; a, b)\right| \leq (f^*(a))^{\eta_1+\eta_3}\left(f^*\left(\frac{5a+b}{6}\right)\right)^{\eta_2}\left(f^*\left(\frac{a+5b}{6}\right)\right)^{\eta_4}(f^*(b))^{\eta_3+\eta_5}.$$

Corollary 3.9. *In Theorem 3.7, using the property of the multiplicatively P-functions of f^*, i.e., $f^*\left(\frac{a+b}{2}\right) \leq f^*\left(\frac{5a+b}{6}\right) f^*\left(\frac{a+5b}{6}\right)$, we deduce that*

$$\left|{}_*\mathcal{J}_f(\alpha; a, b)\right| \leq (f^*(a))^{\eta_1}\left(f^*\left(\frac{5a+b}{6}\right)\right)^{\eta_2+\eta_3}\left(f^*\left(\frac{a+5b}{6}\right)\right)^{\eta_3+\eta_4}(f^*(b))^{\eta_5}.$$

Corollary 3.10. *In Theorem 3.7, if we assume that f^* i.e., $f^* \leq M,\ M > 0$, then we deduce that*

$$\left|{}_*\mathcal{J}_f(\alpha; a, b)\right| \leq M^{\frac{b-a}{6}\left[\frac{5\alpha}{3(\alpha+1)}\left(\frac{5}{8}\right)^{\frac{1}{\alpha}}+\frac{\alpha}{\alpha+1}\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}+\frac{17+\alpha}{6(\alpha+1)}-\frac{7}{6}\right]}.$$

4. Applications to special means

We shall consider the means for finite arbitrary real numbers.

The arithmetic mean: $A(a_1, a_2, \cdots, a_n) = \frac{a_1+a_2+\cdots+a_n}{n}$.

The harmonic mean: $H_n(a_1, a_2, \cdots, a_n) = \frac{1}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{a_i}} = \frac{n}{\sum_{i=1}^{n}\frac{1}{a_i}}$, where $a_i \neq 0, i = 1, 2, \cdots, n$.

The logarithmic mean: $L(a, b) = \frac{b-a}{\ln b - \ln a}$, $a, b > 0$ and $a \neq b$.

The p-Logarithmic mean: $L_p(a, b) = \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}$, $a, b > 0, a \neq b$ and $p \in \mathbb{R}\setminus\{-1, 0\}$.

The identric mean of two positive numbers:

$$I(a,b) = \begin{cases} a & if\ a = b, \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{\frac{1}{b-a}} & if\ a \neq b. \end{cases}$$

We observe that

$$\ln I(a,b) = \frac{1}{b-a}\int_a^b \ln x \mathrm{d}x,$$

for $a, b > 0, a \neq b$.

Proposition 4.1. *Let $a, b \in \mathbb{R}$ with $0 < a < b$ together with $\alpha = 1$. Then we have that*

$$e^{\frac{3}{16}A^2(a,a,a,a,a,b)+\frac{1}{8}A^2(a,b)+\frac{3}{16}A^2(a,b,b,b,b,b)-\frac{1}{2}L_2^2(a,b)} \leq \left[e^{84A(a,b)+25A(a,a,a,a,a,b)+25A(a,b,b,b,b,b)}\right]^{\frac{b-a}{576}}.$$

Proof. Taking $\alpha = 1$ in Corollary 3.8, we have that

$$\begin{aligned} &\left(\left(f\left(\frac{5a+b}{6}\right)\right)^3\left(f\left(\frac{a+b}{2}\right)\right)^2\left(f\left(\frac{a+5b}{6}\right)\right)^3\right)^{\frac{1}{8}}\left(\int_a^b f(x)^{\mathrm{d}x}\right)^{\frac{1}{a-b}} \\ &\leq \left[(f^*(a))^{42}\left(f^*\left(\frac{5a+b}{6}\right)\right)^{25}\left(f^*\left(\frac{a+5b}{6}\right)\right)^{25}(f^*(b))^{42}\right]^{\frac{b-a}{576}}. \end{aligned} \tag{9}$$

The desired result can be obtained by applying the function $f(x) = e^{\frac{x^2}{2}}, x \in (0, \infty)$ to the inequality (9), where $f^*(x) = e^x$ and $\left(\int_a^b f(x)^{\mathrm{d}x}\right)^{\frac{1}{a-b}} = \exp\left\{-\frac{1}{2}L_2^2(a,b)\right\}$. This ends the proof.

Proposition 4.2. *Under the same assumptions of Proposition 4.1, we have that*

$$A^{\frac{3}{8}}(a,a,a,a,a,b)A^{\frac{1}{4}}(a,b)A^{\frac{3}{8}}(a,b,b,b,b,b)I^{-1}(a,b) \leq \left[e^{16H_2^{-1}(a,b)+59A^{-1}(a,a,a,a,a,b)+59A^{-1}(a,b,b,b,b,b)}\right]^{\frac{b-a}{576}}.$$

Proof. Taking $\alpha = 1$ in Corollary 3.9, we have that

$$\begin{aligned} &\left(\left(f\left(\frac{5a+b}{6}\right)\right)^3\left(f\left(\frac{a+b}{2}\right)\right)^2\left(f\left(\frac{a+5b}{6}\right)\right)^3\right)^{\frac{1}{8}}\left(\int_a^b f(x)^{\mathrm{d}x}\right)^{\frac{1}{a-b}} \\ &\leq \left[(f^*(a))^{8}\left(f^*\left(\frac{5a+b}{6}\right)\right)^{59}\left(f^*\left(\frac{a+5b}{6}\right)\right)^{59}(f^*(b))^{8}\right]^{\frac{b-a}{576}}. \end{aligned} \tag{10}$$

The desired result can be obtained by applying the function $f(x) = x, x \in (0, \infty)$ to the inequality (10), where $f^*(x) = e^{\frac{1}{x}}$ and $\left(\int_a^b f(x)^{dx}\right)^{\frac{1}{a-b}} = I^{-1}(a, b)$. This ends the proof.

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