Filomat 37:28 (2023), 9487–9496
https://doi.org/10.2298/FIL2328487B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Remarks on the sum of powers of normalized signless Laplacian eigenvalues of graphs

Şerife Burcu Bozkurt Altındağ[a], Igor Milovanović[b], Emina Milovanović[b], Marjan Matejić[b]

[a]*Faculty of Science, Selçuk University, Konya, Turkey*
[b]*Faculty of Electronic Engineering, University of Niš, Niš, Serbia*

Abstract. Let $G = (V, E)$, $V = \{v_1, v_2, \dots, v_n\}$, be a simple connected graph of order n and size m. Denote by $\gamma_1^+ \geq \gamma_2^+ \geq \cdots \geq \gamma_n^+ \geq 0$ the normalized signless Laplacian eigenvalues of G, and by $\sigma_\alpha(G)$ the sum of α-th powers of the normalized signless Laplacian eigenvalues of a connected graph. The paper deals with bounds of σ_α. Some special cases, when $\alpha = \frac{1}{2}$ and $\alpha = -1$, are also considered.

1. Introduction

Let $G = (V, E)$, $V = \{v_1, v_2, \dots, v_n\}$, be a simple connected graph with n vertices, m edges and a sequence of vertex degrees $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta > 0$, $d_i = d(v_i)$. With $i \sim j$ we denote the adjacency of vertices v_i and v_j in graph G.

Let $A = (a_{ij})_{n\times n}$ and $D = \mathrm{diag}(d_1, d_2, \dots, d_n)$ be the adjacency and the diagonal degree matrix of G, respectively. Then $L = D - A$ is the Laplacian matrix of G. Because graph G is assumed to be connected, it has no isolated vertices and therefore the matrix $D^{-1/2}$ is well-defined. The normalized Laplacian is defined as $\mathcal{L} = D^{-1/2}LD^{-1/2} = I - D^{-1/2}AD^{-1/2} = I - R$, signless Laplacian matrix as $L^+ = D + A$, and normalized signless Laplacian as $\mathcal{L}^+ = D^{-1/2}L^+D^{-1/2} = I + D^{-1/2}AD^{-1/2} = I + R$, where R is the Randić matrix [5]. For more information on these matrices one can refer to [11, 13, 18]. Each of these matrices completely represents the graph. However, for a graph with large number of nodes it requires a large amount of memory to store the matrix. As an alternative we might study the eigenvalues of the matrix. Eigenvalues of the corresponding graph matrix form the spectrum of G. These eigenvalues (spectra) give us some useful information about the matrix which can be translated into useful information about the graph [7].

Let $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_{n-1} > \gamma_n = 0$ be the normalized Laplacian eigenvalues of G. Some well known properties of these eigenvalues are [30]:

$$\sum_{i=1}^{n-1} \gamma_i = n \quad \text{and} \quad \sum_{i=1}^{n-1} \gamma_i^2 = n + 2M_2^*(G),$$

2020 *Mathematics Subject Classification*. Primary 15A18; Secondary 05C50
Keywords. normalized signless Laplacian eigenvalues, Laplacian incidence energy, Randić incidence energy
Received: 28 August 2020; Revised: 01 April 2023; Accepted: 06 June 2023
Communicated by Dragan S. Djordjević
Research supported by the Serbian Ministry of Education, Science and Technological Development, grant No. 451-03-68/2022-14/200102
Email addresses: srf_burcu_bozkurt@hotmail.com (Şerife Burcu Bozkurt Altındağ), igor@elfak.ni.ac.rs (Igor Milovanović), ema@elfak.ni.ac.rs (Emina Milovanović), marjan.matejic@elfak.ni.ac.rs (Marjan Matejić)

where

$$M_2^*(G) = \sum_{i\sim j} \frac{1}{d_i d_j},$$

is a graph invariant known as modified second Zagreb index [24]. It is also met under the name general Randić index R_{-1},(see [8, 26]).

For a real number α, the sum of α-th powers of normalized Laplacian eigenvalues of a connected graph was defined by [2]

$$S_\alpha(G) = \sum_{i=1}^{n-1} \gamma_i^\alpha .$$

More details about this subject can be found in [1, 12, 20]. For $\alpha = \frac{1}{2}$, $S_{1/2}(G) = LIE\,(G)$ which is known as Laplacian incidence energy (see [21, 28]) is obtained. For $\alpha = -1$, the Kemeny's constant,

$$K(G) = S_{-1}(G) = \sum_{i=1}^{n-1} \frac{1}{\gamma_i},$$

defined in [17] (see also [6, 19, 21]) is obtained. Let us note that a graph invariant

$$K_f^*(G) = 2mK(G),$$

defined in [9] is known as the degree Kirchhoff index.

Let $\gamma_1^+ \geq \gamma_2^+ \geq \cdots \geq \gamma_n^+ \geq 0$ be the normalized signless Laplacian eigenvalues of G. Denote by N_k the following auxiliary quantity

$$N_k = \sum_{i=2}^{k+1} \gamma_i^+,$$

where $1 \leq k \leq n-2$.

By analogy with Kemeny's constant, for the connected non–bipartite graphs, we introduce "signless Kemeny's" constant

$$K^+(G) = \sum_{i=1}^{n} \frac{1}{\gamma_i^+}.$$

For a real number α, the sum of α-th powers of the normalized signless Laplacian eigenvalues of a connected graph was defined in [3] as

$$\sigma_\alpha(G) = \sum_{i=1}^{n} \left(\gamma_i^+\right)^\alpha .$$

For $\alpha = \frac{1}{2}$, $\sigma_{1/2}(G) = I_R E(G)$, which is known as Randić (normalized) incidence energy (see [3, 4]), and for $\alpha = -1$, $\sigma_{-1}(G) = K^+(G)$. Notice that the normalized Laplacian and normalized signless Laplacian eigenvalues coincide in the case of bipartite graphs [3]. Therefore, for connected bipartite graphs, $S_\alpha(G)$ is equal to $\sigma_\alpha(G)$, $LIE\,(G)$ is equal to $I_R E(G)$ and $K(G)$ is equal to $K^+(G)$.

This paper deals with bounds of σ_α and special cases $\alpha = \frac{1}{2}$ and $\alpha = -1$.

2. Preliminaries

In this section we recall some results from the literature that will be used hereafter.

Lemma 2.1. [10] *Let G be a graph of order n with no isolated vertices. Then*

$$\sum_{i=1}^{n} \gamma_i^+ = n \quad \text{and} \quad \sum_{i=1}^{n} (\gamma_i^+)^2 = n + 2M_2^*(G)\,.$$

The basic result for γ_1^+ was obtained in [15].

Lemma 2.2. [15] *For any connected graph G, the largest normalized signless Laplacian eigenvalue is*

$$\gamma_1^+ = 2\,.$$

Lemma 2.3. [15] *Let G be a graph of order $n \geq 2$ with no isolated vertices. Then*

$$\gamma_2^+ = \gamma_3^+ = \cdots = \gamma_n^+ = \frac{n-2}{n-1}\,,$$

if and only if $G \cong K_n$.

Lemma 2.4. [14] *Let G be a connected graph with $n > 2$ vertices. Then $\gamma_2 = \gamma_3 = \cdots = \gamma_{n-1}$ if and only if $G \cong K_n$ or $G \cong K_{p,q}$.*

The following was proved in [27] for an arbitrary square matrix A of order $n \times n$ with only real valued eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

Lemma 2.5. [27] *Let A be an $n \times n$ matrix with only real eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Suppose that λ_1 is known. Let $1 \leq k \leq n-2$. Then*

$$\sum_{i=2}^{k+1} \lambda_i \leq \frac{k(\mathrm{tr}A - \lambda_1)}{n-1} + \sqrt{\frac{k(n-k-1)g(A)}{n-1}}\,,$$

where

$$g(A) = \mathrm{tr}\left(A - \frac{\mathrm{tr}A}{n} I\right)^2 - \frac{n}{n-1}\left(\lambda_1 - \frac{\mathrm{tr}A}{n}\right)^2\,.$$

3. Main results

Lemma 3.1. *Let G be a connected non–bipartite graph with $n \geq 3$ vertices. Then, for $1 \leq k \leq n-2$,*

$$N_k \geq \frac{(n-2)k}{n-1}\,. \tag{1}$$

The equality in (1) is achieved for $G \cong K_n$.

Proof. By Lemmas 2.1 and 2.2, it is elementary to see that

$$\frac{N_k}{k} = \frac{\sum_{i=2}^{k+1} \gamma_i^+}{k} \geq \frac{\sum_{i=k+2}^{n} \gamma_i^+}{n-k-1} = \frac{n-2-N_k}{n-k-1}\,,$$

that is (1).

By Lemma 2.3 one can easily check that the equality in (1) is achieved for $G \cong K_n$. □

From Lemmas 2.1, 2.2, 2.3 and 2.5 the following result can be proved.

Lemma 3.2. *Let G be a connected non–bipartite graph with $n \geq 3$ vertices. Then, for $1 \leq k \leq n-2$,*

$$N_k \leq \frac{(n-2)k + \sqrt{k(n-k-1)(2(n-1)M_2^*(G)-n)}}{n-1} \,. \tag{2}$$

The equality in (2) *is achieved for $G \cong K_n$.*

Theorem 3.3. *Let G be a connected non–bipartite graph with $n \geq 3$ vertices and k, $1 \leq k \leq n-2$, be a positive integer.*

(i) If $0 \leq \alpha \leq 1$, then

$$\sigma_\alpha \leq 2^\alpha + \frac{(n-2)^\alpha}{(n-1)^{\alpha-1}} \,, \tag{3}$$

with equality if and only if either $\alpha = 0$, or $\alpha = 1$, or $G \cong K_n$.

(ii) If $\alpha \geq 1$, then

$$\sigma_\alpha \geq 2^\alpha + \frac{(n-2)^\alpha}{(n-1)^{\alpha-1}} \,, \tag{4}$$

with equality if and only if $\alpha = 1$ or $G \cong K_n$.

(iii) If $\alpha \leq 0$, then

$$\sigma_\alpha(G) \leq 2^\alpha + k^{1-\alpha}\left(\frac{(n-2)\,k + \sqrt{k\,(n-k-1)\,(2\,(n-1)\,M_2*(G)-n)}}{n-1}\right)^\alpha \tag{5}$$
$$+\ (n-k-1)^{1-\alpha}\left(\frac{(n-2)\,(n-k-1) - \sqrt{(n-k-1)k\Big(2(n-1)M_2^*(G)-n\Big)}}{n-1}\right)^\alpha .$$

with equality achieved for $\alpha = 0$ or $G \cong K_n$.

Proof. (i) We start with the case $0 \leq \alpha \leq 1$. From the power mean inequality, see for example [22], we have

$$\left(\frac{\sum_{i=2}^{k+1}(\gamma_i^+)^\alpha}{k}\right)^{1/\alpha} \leq \frac{N_k}{k} \,,$$

that is

$$\sum_{i=2}^{k+1}(\gamma_i^+)^\alpha \leq k^{1-\alpha}N_k^\alpha \,, \tag{6}$$

where the equality holds if and only if $\gamma_2^+ = \gamma_3^+ = \cdots = \gamma_{k+1}^+$.

Considering Lemmas 2.1 and 2.2 with the same idea as in the above

$$\sum_{i=k+2}^{n}(\gamma_i^+)^\alpha \leq (n-k-1)^{1-\alpha}(n-2-N_k)^\alpha \,, \tag{7}$$

where the the equality holds if and only if $\gamma_{k+2}^+ = \gamma_{k+3}^+ = \cdots = \gamma_n^+$.

Then by Eqs. (6) and (7), we obtain

$$\begin{aligned}\sigma_\alpha(G) &= 2^\alpha + \sum_{i=2}^{k+1}(\gamma_i^+)^\alpha + \sum_{i=k+2}^{n}(\gamma_i^+)^\alpha \leq \\ &\leq 2^\alpha + k^{1-\alpha}N_k^\alpha + (n-k-1)^{1-\alpha}(n-2-N_k)^\alpha .\end{aligned}$$

For $x \geq \frac{k(n-2)}{n-1}$, let

$$f(x) = 2^\alpha + k^{1-\alpha}x^\alpha + (n-k-1)^{1-\alpha}(n-2-x)^\alpha .$$

It is easy to see that f is decreasing for $x \geq \frac{k(n-2)}{n-1}$, since $0 \leq \alpha \leq 1$. Therefore, by Lemma 3.1

$$\sigma_\alpha \leq 2^\alpha + k^{1-\alpha}\left(\frac{(n-2)k}{n-1}\right)^\alpha + (n-k-1)\left(\frac{n-2}{n-1}\right)^\alpha = 2^\alpha + \frac{(n-2)^\alpha}{(n-1)^{\alpha-1}} .$$

Hence, we get the upper bound in (3). If the equality holds in (3), then $\gamma_2^+ = \gamma_3^+ = \cdots = \gamma_{k+1}^+$, $\gamma_{k+2}^+ = \gamma_{k+3}^+ = \cdots = \gamma_n^+$ and $N_k = \frac{(n-2)k}{n-1}$. This implies that $\gamma_2^+ = \gamma_3^+ = \cdots = \gamma_n^+ = \frac{n-2}{n-1}$. Thus, by Lemma 2.3, we arrive at $G \cong K_n$. Conversely, if $G \cong K_n$, it can be easily seen that the equality holds in (3).

(ii) Note that f is increasing for $x \geq \frac{(n-2)k}{n-1}$, since $\alpha \geq 1$. Then, for $\alpha \geq 1$, by power mean inequality and Lemmas 2.1, 2.2 and 3.1, we have

$$\sigma_\alpha(G) \geq 2^\alpha + k^{1-\alpha}\left(\frac{(n-2)k}{n-1}\right)^\alpha + (n-k-1)\left(\frac{n-2}{n-1}\right)^\alpha = 2^\alpha + \frac{(n-2)^\alpha}{(n-1)^{\alpha-1}} .$$

Hence, the lower bound in (4) holds. Similarly to the above, one can show that the equality in (4) holds if and only $G \cong K_n$.

(iii) Note that f is increasing for $x \geq \frac{(n-2)k}{n-1}$, since $\alpha \leq 0$. By Lemmas 3.1 and 3.2

$$\frac{(n-2)k}{n-1} \leq N_k \leq \frac{(n-2)k + \sqrt{k(n-k-1)(2(n-1)M_2^*(G)-n)}}{n-1} .$$

Therefore, we get

$$\sigma_\alpha(G) \leq f\left(\frac{(n-2)k + \sqrt{k(n-k-1)(2(n-1)M_2^*(G)-n}}{n-1}\right) .$$

This leads to the upper bound in (5). By Lemma 2.3, one can easily check that equality in (5) is achieved for $G \cong K_n$.
□

From Theorem 3.3, we have:

Corollary 3.4. *Let G be a connected non–bipartite graph with $n \geq 3$ vertices. Then*

$$I_R E(G) \leq \sqrt{2} + \sqrt{(n-1)(n-2)} . \tag{8}$$

Equality holds if and only if $G \cong K_n$.

The inequality (8) was proven in [10, 15].

Corollary 3.5. *Let G be a connected non-bipartite graph with $n \geq 3$ vertices and k, $1 \leq k \leq n-2$, be a positive integer. Then*

$$K^+(G) \leq \frac{1}{2} + \frac{k^2(n-1)}{(n-2)k + \sqrt{k(n-k-1)(2(n-1)M_2^*(G)-n)}} + \frac{(n-k-1)^2(n-1)}{(n-2)(n-k-1) - \sqrt{k(n-k-1)(2(n-1)M_2^*(G)-n)}} .$$

Equality is achieved for $G \cong K_n$.

Remark 3.6. *The bipartite graph case of Theorem* 3.3 *can be found in Theorem* 3.7 *of* [20].

In the next theorem we establish a relationship between $\sigma_\alpha(G)$ and $\sigma_{\alpha-1}(G)$.

Theorem 3.7. *Let G be a connected non-bipartite graph with $n \geq 3$ vertices. Then, for any real α, $\alpha \leq 1$ or $\alpha \geq 2$, holds*

$$\sigma_\alpha(G) \leq 2\sigma_{\alpha-1}(G) - \frac{(n-2M_2^*(G))^{\alpha-1}}{n^{\alpha-2}} . \tag{9}$$

When $1 \leq \alpha \leq 2$, the sense of inequality reverses. Equality holds if and only if either $\alpha = 1$, or $\alpha = 2$, or $G \cong K_n$.

Proof. For any non–bipartite graph with $n \geq 3$ vertices holds

$$2\sigma_{\alpha-1}(G) - \sigma_\alpha(G) = \sum_{i=1}^{n} (2-\gamma_i^+)(\gamma_i^+)^{\alpha-1} . \tag{10}$$

Let $p = (p_i)$, $i = 1, 2, \ldots, n$, be a non negative real number sequence and $a = (a_i)$, $i = 1, 2, \ldots, n$ positive real number sequence. In [16] (see also [23]) it was proven that for any real r, $r \leq 0$ or $r \geq 1$, holds

$$\left(\sum_{i=1}^{n} p_i\right)^{r-1} \sum_{i=1}^{n} p_i a_i^r \geq \left(\sum_{i=1}^{n} p_i a_i\right)^r . \tag{11}$$

When $0 \leq r \leq 1$ the opposite inequality is valid.

For $r = \alpha - 1$, $\alpha \leq 1$ or $\alpha \geq 2$, $p_i = 2 - \gamma_i^+$, $a_i = \gamma_i^+$, $i = 1, 2, \ldots, n$, the inequality (11) becomes

$$\left(\sum_{i=1}^{n} (2-\gamma_i^+)\right)^{\alpha-2} \sum_{i=1}^{n} (2-\gamma_i^+)(\gamma_i^+)^{\alpha-1} \geq \left(\sum_{i=1}^{n} (2-\gamma_i^+)\gamma_i^+\right)^{\alpha-1} ,$$

Then, by Lemma 2.1

$$n^{\alpha-2} \sum_{i=1}^{n} (2-\gamma_i^+)(\gamma_i^+)^{\alpha-1} \geq \left(n - 2M_2^*(G)\right)^{\alpha-1} . \tag{12}$$

From the above inequality and identity (10) we obtain (9). The case when $1 \leq \alpha \leq 2$ can be proved analogously.

Equality in (12) holds if and only if either $\alpha = 1$, or $\alpha = 2$, or $2 = \gamma_1^+ = \cdots = \gamma_t^+ > \gamma_{t+1}^+ = \cdots = \gamma_n^+$, for some t, $1 \leq t \leq n-1$, or $\gamma_2^+ = \cdots = \gamma_n^+$. By Lemma 2.3, this implies that equality in (9) holds if and only if either $\alpha = 1$, or $\alpha = 2$, or $G \cong K_n$. □

From Theorem 3.7, we have:

Corollary 3.8. *Let G be a connected non-bipartite graph with $n \geq 3$ vertices. Then*

$$K^+(G) \geq \frac{n(n - M_2^*(G))}{n - 2M_2^*(G)} .$$

Equality holds if and only if $G \cong K_n$.

Considering the similar proof techniques in Theorem 3.7 together with Lemma 2.4, we get:

Theorem 3.9. *Let G be a connected bipartite graph with $n \geq 3$ vertices. Then, for any real α, $\alpha \leq 1$ or $\alpha \geq 2$, holds*

$$S_\alpha(G) = \sigma_\alpha(G) \leq 2\sigma_{\alpha-1}(G) - \frac{(n - 2M_2^*(G))^{\alpha-1}}{(n-2)^{\alpha-2}} . \tag{13}$$

When $1 \leq \alpha \leq 2$, the sense of inequality reverses. Equality holds if and only if either $\alpha = 1$, or $\alpha = 2$, or $G \cong K_{p,q}$.

From Theorem 3.9, we obtain:

Corollary 3.10. *Let G be a connected bipartite graph with $n \geq 3$ vertices. Then*

$$K(G) \geq \frac{(n-1)\,(n - 2M_2^*(G)) + (n-2)^2}{2\left(n - 2M_2^*(G)\right)} .$$

Equality holds if and only if $G \cong K_{p,q}$.

Theorem 3.11. *Let G be a connected non–bipartite graph, with $n \geq 3$ vertices. Then for any real α holds*

$$\sigma_\alpha(G) \leq 2^\alpha + \sqrt{(n-2)(\sigma_{2\alpha-1}(G) - 2^{2\alpha-1})} . \tag{14}$$

Equality holds if and only if $\alpha = 1$ or $G \cong K_n$.

Proof. The following identities are valid for any real α

$$\sigma_{2\alpha-1}(G) - 2^{2\alpha-1} = \sum_{i=2}^{n} (\gamma_i^+)^{2\alpha-1} = \sum_{i=2}^{n} \frac{\left((\gamma_i^+)^\alpha\right)^2}{\gamma_i^+} . \tag{15}$$

On the other hand, for positive real number sequences $x = (x_i)$ and $a = (a_i)$, $i = 1, 2, \ldots, n$, and arbitrary real $r \geq 0$, in [25] the following inequality was proved

$$\sum_{i=1}^{n} \frac{x_i^{r+1}}{a_i^r} \geq \frac{\left(\sum_{i=1}^{n} x_i\right)^{r+1}}{\left(\sum_{i=1}^{n} a_i\right)^r} . \tag{16}$$

For $r = 1$, $x_i = (\gamma_i^+)^\alpha$, $a_i = \gamma_i^+$, $i = 2, \ldots, n$, the above inequality transforms into

$$\sum_{i=2}^{n} \frac{\left((\gamma_i^+)^\alpha\right)^2}{\gamma_i^+} \geq \frac{\left(\sum_{i=2}^{n} (\gamma_i^+)^\alpha\right)^2}{\sum_{i=2}^{n} \gamma_i^+} ,$$

Then, by Lemmas 2.1 and 2.2

$$\sum_{i=2}^{n} \frac{\left((\gamma_i^+)^\alpha\right)^2}{\gamma_i^+} \geq \frac{(\sigma_\alpha(G) - 2^\alpha)^2}{n-2} . \tag{17}$$

Combining (15) and (17) we obtain (14).

Equality in (17) holds if and only if $\alpha = 1$ or $\frac{(\gamma_2^+)^\alpha}{\gamma_2^+} = \frac{(\gamma_3^+)^\alpha}{\gamma_3^+} = \cdots = \frac{(\gamma_n^+)^\alpha}{\gamma_n^+}$. By Lemma 2.3, this implies that equality in (14) holds if and only if $\alpha = 1$ or $G \cong K_n$. $\square$

Remark 3.12. *It can be easily observed that for $\alpha = \frac{1}{2}$, from* (14) *the inequality* (8) *is obtained.*

By taking $\alpha = 0$ in Eq. (14), we also have:

Corollary 3.13. *Let G be a connected non–bipartite graph, with $n \geq 3$ vertices. Then*

$$K^{+}(G) \geq \frac{n(2n-3)}{2(n-2)}\,.$$

Equality holds if and only if $G \cong K_n$.

Using the similar proof techniques in Theorem 3.11 together with Lemma 2.4, we obtain:

Theorem 3.14. *Let G be a connected bipartite graph, with $n \geq 3$ vertices. Then for any real α holds*

$$S_{\alpha}(G) = \sigma_{\alpha}(G) \leq 2^{\alpha} + \sqrt{(n-2)(\sigma_{2\alpha-1}(G) - 2^{2\alpha-1})}\,. \tag{18}$$

Equality holds if and only if $\alpha = 1$ or $G \cong K_{p,q}$.

From Theorem 3.14, we get:

Corollary 3.15. *[15] Let G be a connected bipartite graph with $n \geq 3$ vertices. Then*

$$LIE(G) = I_R E(G) \leq \sqrt{2} + n - 2\,.$$

Equality holds if and only if $G \cong K_{p,q}$.

Corollary 3.16. *[29] Let G be a connected bipartite graph, with $n \geq 3$ vertices. Then*

$$K_f^{*}(G) \geq (2n-3)\,m.$$

Equality holds if and only if $G \cong K_{p,q}$.

Similarly as in previous theorems, the following results can be proved.

Theorem 3.17. *Let G be a connected non–bipartite graph with $n \geq 3$ vertices. Then, for any real α, $\alpha \leq 0$ or $\alpha \geq 1$, holds*

$$\sigma_{\alpha}(G) \leq 2\sigma_{\alpha-1}(G) - \frac{n^{\alpha}}{(2K^{+}(G) - n)^{\alpha-1}}\,.$$

When $0 \leq \alpha \leq 1$, the opposite inequality is valid. Equality holds if and only if either $\alpha = 0$, or $\alpha = 1$, or $G \cong K_n$.

Theorem 3.18. *Let G be a connected bipartite graph with $n \geq 3$ vertices. Then, for any real α, $\alpha \leq 0$ or $\alpha \geq 1$, holds*

$$S_{\alpha}(G) = \sigma_{\alpha}(G) \leq 2\sigma_{\alpha-1}(G) - \frac{(n-2)^{\alpha}}{(2K(G) - n + 1)^{\alpha-1}}\,.$$

When $0 \leq \alpha \leq 1$, the opposite inequality is valid. Equality holds if and only if either $\alpha = 0$, or $\alpha = 1$, or $G \cong K_{p,q}$.

Theorem 3.19. *Let G be a connected non–bipartite graph with $n \geq 3$ vertices. Then, for any real α holds*

$$\sigma_{\alpha}(G) \leq 2^{\alpha} + \sqrt{(\sigma_{2\alpha+1}(G) - 2^{2\alpha+1})\left(K^{+}(G) - \frac{1}{2}\right)}.$$

Equality holds if and only if $\alpha = -1$, or $G \cong K_n$.

From Theorem 3.19, we get the following relation between $K^+(G)$, $M_2^*(G)$ and $I_RE(G)$.

Corollary 3.20. *Let G be a connected non–bipartite graph with $n \geq 3$ vertices. Then*

$$\left(K^+(G) - \frac{1}{2}\right)(n + 2M_2^*(G) - 4) \geq \left(I_RE(G) - \sqrt{2}\right)^2 .$$

Equality holds if and only if $G \cong K_n$.

Theorem 3.21. *Let G be a connected bipartite graph with $n \geq 3$ vertices. Then, for any real α holds*

$$S_\alpha(G) = \sigma_\alpha(G) \leq 2^\alpha + \sqrt{(\sigma_{2\alpha+1}(G) - 2^{2\alpha+1})\left(K(G) - \frac{1}{2}\right)}.$$

Equality holds if and only if $\alpha = -1$, or $G \cong K_{p,q}$.

From Theorem 3.21, we have:

Corollary 3.22. *Let G be a connected bipartite graph with $n \geq 3$ vertices. Then*

$$\left(K(G) - \frac{1}{2}\right)(n + 2M_2^*(G) - 4) \geq \left(LIE(G) - \sqrt{2}\right)^2 .$$

Equality holds if and only if $G \cong K_{p,q}$.

References

[1] M. Bianchi, A. Cornaro, J. L. Palacios, A. Torriero, Bounding the sum of powers of normalized Laplacian eigenvalues of graph through majorization method, MATCH Commun. Math. Comput. Chem. 70 (2013) 707–716.

[2] Ş. B. Bozkurt, D. Bozkurt, On the sum of powers of normalized Laplacian eigenvalues of graphs, MATCH Commun. Math. Comput. Chem. 68 (2012) 917–930.

[3] Ş. B. Bozkurt Altındağ, Note on the sum of powers of normalized signless Laplacian eigenvalues of graphs, Math. Interdisc. Res. 4 (2) (2019) 171–182.

[4] Ş. B. Bozkurt Altındağ, Sum of powers of normalized signless Laplacian eigenvalues and Randić (normalized) incidence energy of graphs, Bull. Inter. Math. Virtual Inst. 11 (1) (2021) 135–146.

[5] Ş. B. Bozkurt, A. D. Gungor, I. Gutman, A. S. Cevik, Randić matrix and Randić energy, MATCH Commun. Math. Comput. Chem. 64 (2010), 239–250.

[6] S. Butler, Algebraic aspects of the normalized Laplacian, in: Recent trends in combinatorics, IMA Vol. Math. Appl. 159, Springer, 2016, pp. 295–315.

[7] S. Butler, Eigenvalues and Structures of Graphs, PhD. thesis, University of California, San Diego, 2008.

[8] M. Cavers, S. Fallat, S. Kirkland, On the normalized Laplacian energy and general Randić index R_{-1} of graphs, Lin. Algebra Appl. 433 (2010) 172–190.

[9] H. Chen, F. Zhang, Resistance distance and the normalized Laplacian spectrum, Discr. Appl. Math. 155 (5) (2007) 654–661.

[10] B. Cheng, B. Liu, The normalized incidence energy of a graph, Lin. Algebra Appl. 438 (2013), 4510–4519.

[11] F. R. K. Chung, Spectral Graph Theory, Am. Math. Soc. Providence, 1997.

[12] G. P. Clemente, A. Cornaro, New bounds for the sum of powers of normalized Laplacian eigenvalues of graphs, Ars Math. Contemp. 11 (2016) 403–413.

[13] D. Cvetković, M. Doob, H. Sachs, Spectra of graphs, Academic press, New York, 1980.

[14] K. Ch. Das, A. D. Gungor, Ş. B. Bozkurt, On the normalized Laplacian eigenvalues of graphs, Ars Combin. 118 (2015) 143–154.

[15] R. Gu, F. Huang, X. Li, Randić incidence energy of graphs, Trans. Comb. 3 (2014), 1–9.

[16] J. L. W. Jensen, Sur les functions convexes of les integralities entre les values moyennes, Acta Math. 30 (1906) 175–193.

[17] J. G. Kemeny, J. L. Snell, Finite Markov chains, Van Nostrand, Princeton, NJ, 1960.

[18] S. Khan, S. Pirzada, Distance Signless Laplacian Eigenvalues, Diameter, and Clique Number, Discrete Math. Lett. 10 (2022) 28–31.

[19] M. Levene, G. Loizou, Kemeny's constant and the random surfer, Amer. Math. Monthly 109 (2002) 741–745.

[20] J. Li, J. M. Guo, W. C. Shiu, Ş. B. Bozkurt Altındağ, D. Bozkurt, Bounding the sum of powers of normalized Laplacian eigenvalues of a graph, Appl. Math. Comput. 324 (2018), 82–92.

[21] E. I. Milovanović, M. M. Matejić, I. Ž. Milovanović, On the normalized Laplacian spectral radius, Laplacian incidence energy and Kemeny's constant, Lin. Algebra Appl. 582 (2019) 181–196.

[22] D. S. Mitrinović, P. M. Vasić, Analytic inequalities, Springer Verlag, Berlin-Heidelberg-New York, 1970.

[23] D. S. Mitrinović, J. E. Pečarić, A. M. Fink, Classical and new inequalities in analysis, Kluwer Academic Publishers, Dordrecht, 1993.

[24] S. Nikolić, G. Kovačević, A. Milićević, N. Trinajstić, The Zagreb indices 30 years after, Croat. Chem. Acta 76 (2003) 113-124.
[25] J. Radon, Theorie und Anwendungen der absolut additiven Mengenfunktionen, Sitzungsber Acad. Wissen. Wien 122 (1913) 1295–1438.
[26] M. Randić, Characterization of molecular branching, J. Amer. Chem. Soc. 97 (1975) 6609–6615.
[27] O. Rojo, R. Soto, H. Rojo, Bounds for sums of eigenvalues and applications, Comput. Math. Appl. 39 (2000), 1–15.
[28] L. Shi, H. Wang, The Laplacian incidence energy of graphs, Linear Algebra Appl. 439 (2013), 4056–4062.
[29] B. Zhou, N. Trinajstić, On resistance-distance and Kirchhoff index, J. Math. Chem. 46 (2009) 283–289.
[30] P. Zumstein, Comparison of spectral methods through the adjacency matrix and the Laplacian of a graph, Th. Diploma, ETH Zürich, 2005.