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Abelian theorems involving the fractional wavelet transforms

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Abstract. In this paper, the initial-value and the final-value Abelian theorems are presented for the continuous fractional wavelet transform of functions and distributions. An application of these Abelian theorems to the continuous fractional wavelet transforms is also investigated by using the Mexican hat wavelet function.

1. Introduction and Motivation

The fractional Fourier transform, which was studied by Luchko *et al.* [9] in the year 2008, plays a significant role for finding fractional derivatives (see, for details, [8]). Later, in the year 2010, Kilbas *et al.* [7] discussed the composition of the fractional Fourier transforms with some modified fractional integrals and fractional derivatives. More recently, the calculus of pseudo-differential operators, which are associated with the fractional Fourier transform on the Schwartz space, were considered in [10] and [31]. Motivated by these and other related developments, Srivastava *et al.* [19] investigated various potentially useful properties of the continuous fractional wavelet transform.

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The theory of the Hankel transform, which was presented by Haimo [3], was applied by Srivastava *et al.* [22] who introduced the fractional Bessel wavelet transform and studied the associated Parseval formula and the inversion formula, and also considered a discrete version of the aforesaid transform.

Abelian theorems are known to be useful for finding the initial value with the help of the final value and also for finding the final value by using the initial value by means of several different integral transform techniques. Many authors have used these integral transform techniques to investigate Abelian theorems in the classical sense as well as in the distributional sense. For example, in the year 1955, Griffith [2] proved a theorem concerning the asymptotic behavior of the Hankel transforms. Later, in the year 1966, Zemanian [36] considered some Abelian theorems for the distributional Hankel transformation and the K-transformations. Hayek and González [4] established Abelian theorems for the generalized index ${}_2F_1$-transform in 1992. Pathak (see [11] and [12]) investigated the Abelian theorems for the wavelet transform by applying the theory of the Fourier transforms in 2001. More recently, in the year 2020, Upadhyay *et al.* [33] established the Abelian theorems for the Bessel wavelet transform. Abelian theorems for the Laplace transform and the Mehler-Fock transform of general order over distributions of compact support and over certain spaces of generalized functions were proved by González and Negrín [1] in 2020. Subsequently, in the year 2022, Prasad *et al.* [13] studied the Abelian theorems for the quadratic-phase Fourier wavelet transform.

In this sequel to the above-mentioned developments, our main objective is to investigate the Abelian theorems for the fractional wavelet transform in the classical sense and in the distributional sense. In our investigation, we apply the techniques which are based upon the fractional Fourier transform. As an application of the Abelian theorems to the continuous fractional wavelet transform, we make use of the Mexican hat wavelet function (see, for example, [28]).

2. Definitions, Notations and Preliminaries

In this section, we first present the definitions and notations which we shall need in our present investigation (see, for details, [7, 9, 10, 12, 19, 34]).

Definition 1. Let $\alpha \in (0, 1]$. The fractional Fourier transform of order α is defined, for a given function ϕ, by

$$\widehat{\phi}_\alpha(w) = (\mathcal{F}_\alpha\phi)(w) = \int_{\mathbb{R}} e^{-i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}} x}\phi(x)\, \mathrm{d}x \qquad (\forall\ w \in \mathbb{R}), \tag{1}$$

provided the integral on the right-hand side of Eq. (1) is convergent.

Definition 2. The inverse fractional Fourier transform of $\mathcal{F}_\alpha\phi$ of order α is given by

$$\mathcal{F}_\alpha^{-1}(\mathcal{F}_\alpha\phi)(x) = \frac{1}{2\pi\alpha}\int_{\mathbb{R}} e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}} x}|w|^{\frac{1}{\alpha}-1}(\mathcal{F}_\alpha\phi)(w)\, \mathrm{d}w \qquad (\forall\ x \in \mathbb{R}), \tag{2}$$

provided the integral on the right-hand side of Eq. (2) is convergent.

Definition 3. The Schwartz space $S(\mathbb{R})$ is the vector space of all complex-valued infinitely differentiable functions ϕ on $\mathbb{R}$ such that, for all indices $\beta, \gamma \in \mathbb{N}_0$, we have

$$\gamma_{\beta,\gamma}(\phi) = \sup_{x\in\mathbb{R}} |x^\beta(D^\gamma\phi)(x)| < \infty. \tag{3}$$

Definition 4. A sequence (ϕ_j) of functions in the Schwartz space $S(\mathbb{R})$ is said to converge to zero in $S(\mathbb{R})$ $\big($denoted by $\phi_j \to 0$ in $S(\mathbb{R})\big)$ if, for all indices $\beta, \gamma \in \mathbb{N}_0$, we have

$$\sup_{x\in\mathbb{R}} |x^\beta(D^\gamma\phi_j)(x)| \to 0 \qquad (j \to \infty).$$

Definition 5. A linear functional T on $S(\mathbb{R})$ is said to be continuous if, for any sequence (ϕ_j) of functions in $S(\mathbb{R})$ converging to zero in $S(\mathbb{R})$, we have

$$T(\phi_j) \to 0 \qquad (j \to \infty).$$

Continuous linear functionals on $S(\mathbb{R})$ are called tempered distributions and are denoted by $S^{'}(\mathbb{R})$. If $T \in S^{'}(\mathbb{R})$, then $T : S(\mathbb{R}) \to \mathbf{C}$ is a continuous linear map such that

$$T(\phi) = \langle T, \phi \rangle \textit{ for all } \phi \in S(\mathbb{R}).$$

Definition 6. A sequence (T_j) of functions in $S^{'}(\mathbb{R})$ is said to converge to zero in $S^{'}(\mathbb{R})$ if, for any $\phi \in S(\mathbb{R})$, the sequence $\langle T_j, \phi\rangle$ converges to zero in $\mathbb{C}$ as $j \to \infty$, $\mathbb{C}$ being the set of complex numbers.

Definition 7. We denote by $\mathcal{D}(\mathbb{R})$ the set of all complex-valued infinitely differentiable functions on $\mathbb{R}$ having compact support. Moreover, $\mathcal{D}^{'}(\mathbb{R})$ is the dual of the space $\mathcal{D}(\mathbb{R})$ and its elements are called Schwartz distributions. The space of all those distributions in $\mathcal{D}^{'}(\mathbb{R})$ that have compact support is denoted by $\Xi^{'}(\mathbb{R})$.

Definition 8. Let $L^p(\mathbb{R}^n)(1 \leqq p \leqq \infty)$ be the space of measurable functions on $\mathbb{R}^n$ with the norm $\|\cdot\|_p$. Suppose also that $\psi \in L^2(\mathbb{R})$ and $0 < \alpha \leqq 1$. Then the fractional wavelet $\psi_{\alpha,a,b}(t)$ is defined by

$$\psi_{\alpha,a,b}(t) = \frac{1}{|a|^{\frac{1}{\alpha}}}\psi\left(\frac{t-b}{|a|^{\frac{1}{\alpha}}}\right) \qquad (a \neq 0;\ b \in \mathbb{R}). \tag{4}$$

Definition 9. Let $\psi \in L^2(\mathbb{R})$. Then the continuous fractional wavelet transform of a given signal $\phi \in L^2(\mathbb{R})$ for $0 < \alpha \leqq 1$ is defined by

$$\begin{aligned}\left(W_{\psi_\alpha}\phi\right)(b,a) &= \langle \phi, \psi_{\alpha,a,b}\rangle \\ &= \int_{-\infty}^{+\infty} \phi(t)\frac{1}{|a|^{\frac{1}{\alpha}}}\overline{\psi\left(\frac{t-b}{|a|^{\frac{1}{\alpha}}}\right)}dt.\end{aligned} \tag{5}$$

From Eq. (1) and Eq. (4), we have

$$F_\alpha(\psi_{\alpha,a,b}(t))(w) = e^{-i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}\widehat{\psi_\alpha}(aw). \tag{6}$$

Let $\phi, \psi \in L^2(\mathbb{R})$. Then the Parseval formula (see [19]) for the fractional Fourier transform is given by

$$\langle \phi, \psi\rangle = \frac{1}{2\pi\alpha}\left\langle |w|^{\frac{1}{\alpha}-1}\widehat{\phi}_\alpha(w), \widehat{\psi}_\alpha(w)\right\rangle. \tag{7}$$

In view of Eq. (5), Eq. (6) and Eq. (7), we get the following relation:

$$\left(W_{\psi_\alpha}\phi\right)(b,a) = \frac{1}{2\pi\alpha}\int_{-\infty}^{+\infty} e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}|w|^{\frac{1}{\alpha}-1}\widehat{\phi}_\alpha(w)\overline{\widehat{\psi}_\alpha(aw)}\,\mathrm{d}w. \tag{8}$$

3. Abelian Theorems for the Fractional Wavelet Transform of Functions

In this section, we present the initial-value and the final-value theorems for the fractional wavelet transform of functions.

Let us suppose that

$$\widehat{\psi}_\alpha(w) = O(|w|^\mu) \qquad (|w| \to 0) \tag{9}$$

and that

$$1+\frac{1}{\alpha}<\eta<\mu+1+\frac{1}{\alpha}.$$

Then the following integral:

$$\int_{-\infty}^{\infty}\overline{\widehat{\psi_\alpha}(w)}|w|^{\frac{1}{\alpha}-\eta}\,\mathrm{d}w$$

is convergent.

We now set

$$\int_{-\infty}^{\infty}\overline{\widehat{\psi_\alpha}(w)}|w|^{\frac{1}{\alpha}-\eta}\,\mathrm{d}w=F_1(\alpha,\eta). \tag{10}$$

Theorem 1. (**Initial-Value Theorem for the Fractional Wavelet Transform**)
Let

$$1+\frac{1}{\alpha}<\eta<\mu+1+\frac{1}{\alpha}\quad \textit{and}\quad \mu>0.$$

Assume also that

$$|w|^{\frac{1}{\alpha}-\eta}\widehat{\psi_\alpha}(w)\in L^1(\mathbb{R}),$$

$$|\widehat{\psi_\alpha}(w)|\leqq M\qquad (M>0)$$

and

$$|w|^{\frac{1}{\alpha}-1}\widehat{\phi_\alpha}(w)\in L^1(\delta,\infty)\qquad (\forall\ \ \delta>0).$$

If

$$\lim_{|w|\to 0}(2\pi\alpha)^{-1}|w|^{-1+\eta}\widehat{\phi_\alpha}(w)=F_2(\alpha,\eta), \tag{11}$$

then

$$\lim_{a\to\infty}a^{1-\eta+\frac{1}{\alpha}}\left(W_{\psi_\alpha}\phi\right)(b,a)=F_1(\alpha,\eta)F_2(\alpha,\eta). \tag{12}$$

Proof. From Eq. (8) and Eq. (10), we have

$$\begin{aligned}
&\left|a^{1-\eta+\frac{1}{\alpha}}\left(W_{\psi_\alpha}\phi\right)(b,a)-F_1(\alpha,\eta)F_2(\alpha,\eta)\right|\\
&\qquad=\left|a^{1-\eta+\frac{1}{\alpha}}\frac{1}{2\pi\alpha}\int_{-\infty}^{\infty}e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}|w|^{\frac{1}{\alpha}-1}\widehat{\phi_\alpha}(w)\overline{\widehat{\psi_\alpha}(aw)}\,\mathrm{d}w\right.\\
&\qquad\qquad\left.-F_2(\alpha,\eta)\int_{-\infty}^{\infty}\overline{\widehat{\psi_\alpha}(aw)}|aw|^{\frac{1}{\alpha}-\eta}a\,\mathrm{d}w\right|\\
&\qquad=a\left|\int_{-\infty}^{\infty}(2\pi\alpha)^{-1}e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}|aw|^{\frac{1}{\alpha}-1}|aw|^{1-\eta}|w|^{-1+\eta}\widehat{\phi_\alpha}(w)\overline{\widehat{\psi_\alpha}(aw)}\,\mathrm{d}w\right.\\
&\qquad\qquad\left.-F_2(\alpha,\eta)\int_{-\infty}^{\infty}\overline{\widehat{\psi_\alpha}(aw)}|aw|^{1-\eta}|aw|^{\frac{1}{\alpha}-1}\,\mathrm{d}w\right|\\
&\qquad=a\left|\int_{-\infty}^{\infty}\left[(2\pi\alpha)^{-1}e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}|w|^{-1+\eta}\widehat{\phi_\alpha}(w)-F_2(\alpha,\eta)\right]\right.\\
&\qquad\qquad\left.\cdot|aw|^{1-\eta}|aw|^{\frac{1}{\alpha}-1}\overline{\widehat{\psi_\alpha}(aw)}\,\mathrm{d}w\right|\\
&\qquad\leqq a\sup_{|w|<\delta}\left|(2\pi\alpha)^{-1}e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}|w|^{-1+\eta}\widehat{\phi_\alpha}(w)-F_2(\alpha,\eta)\right|
\end{aligned}$$

$$
\begin{aligned}
&\cdot \int_{|w|<\delta} |aw|^{1-\eta} |aw|^{\frac{1}{\alpha}-1} |\widehat{\psi}_\alpha(aw)| \,\mathrm{d}w \\
&+ a \int_{|w|>\delta} \left| (2\pi\alpha)^{-1} e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}} b} |w|^{-1+\eta} \widehat{\phi}_\alpha(w) - F_2(\alpha,\eta) \right| \\
&\cdot |aw|^{1-\eta} |aw|^{\frac{1}{\alpha}-1} |\widehat{\psi}_\alpha(aw)| \,\mathrm{d}w \\
\leqq\ & \sup_{|w|<\delta} \left| (2\pi\alpha)^{-1} e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}} b} |w|^{-1+\eta} \widehat{\phi}_\alpha(w) - F_2(\alpha,\eta) \right| \int_{-\infty}^{\infty} |w|^{\frac{1}{\alpha}-\eta} |\widehat{\psi}_\alpha(w)| \,\mathrm{d}w \\
&+ M a^{1+\frac{1}{\alpha}-\eta} \int_{|w|>\delta} \left| (2\pi\alpha)^{-1} e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}} b} |w|^{-1+\eta} \widehat{\phi}_\alpha(w) - F_2(\alpha,\eta) \right| |w|^{\frac{1}{\alpha}-\eta} \,\mathrm{d}w. \qquad (13)
\end{aligned}
$$

Since $|w|^{\frac{1}{\alpha}-\eta}\widehat{\psi}_\alpha(w) \in L^1(\mathbb{R})$, there exists a positive real number M_1 such that

$$\int_{-\infty}^{\infty} |w|^{\frac{1}{\alpha}-\eta} |\widehat{\psi}_\alpha(w)| \,\mathrm{d}w = M_1 < \infty.$$

Also, for any $\epsilon > 0$, we have

$$\sup_{|w|<\delta} \left| (2\pi\alpha)^{-1} e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}} b} |w|^{-1+\eta} \widehat{\phi}_\alpha(w) - F_2(\alpha,\eta) \right| < \frac{\epsilon}{2M_1},$$

by choosing δ small enough.

Next, since $\eta > 1 + \frac{1}{\alpha}$, we find that

$$a^{1+\frac{1}{\alpha}-\eta} \to 0 \ \text{ as } \ a \to \infty$$

and that the integral in the second term of Eq. (13) is convergent. So, for any $\epsilon > 0$, we can make the second term in Eq. (13) less than $\frac{\epsilon}{2}$. Hence, for any $\epsilon > 0$, we have the following consequence:

$$\left| a^{1-\eta+\frac{1}{2\alpha}} \left(W_{\psi_\alpha}\phi \right)(b,a) - F_1(\alpha,\eta) F_2(\alpha,\eta) \right| < \epsilon$$

for sufficiently large a. □

Theorem 2. (**Final-Value Theorem for the Fractional Wavelet Transform**)
Let

$$1 + \frac{1}{\alpha} < \eta < \mu + 1 + \frac{1}{\alpha} \quad \textit{and} \quad \mu > 0.$$

Suppose also that

$$|w|^{\frac{1}{\alpha}-\eta}\widehat{\psi}_\alpha(w) \in L^1(\mathbb{R})$$

and that

$$|w|^{\mu+\frac{1}{\alpha}-1}\widehat{\phi}_\alpha(w) \in L^1(-X,X) \qquad (\forall\, X > 0).$$

If

$$\lim_{|w|\to\infty} (2\pi\alpha)^{-1} e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}} b} |w|^{-1+\eta} \widehat{\phi}_\alpha(w) = F_3(\alpha,b,\eta), \tag{14}$$

then

$$\lim_{a\to 0} a^{1-\eta+\frac{1}{\alpha}} \left(W_{\psi_\alpha}\phi \right)(b,a) = F_1(\alpha,\eta) F_3(\alpha,b,\eta). \tag{15}$$

Proof. In view of the proof of Theorem 1, we find $\forall\ X > 0$ that

$$\begin{aligned}
&\left|a^{1-\eta+\frac{1}{\alpha}}\Big(W_{\psi_\alpha}\phi\Big)(b,a) - F_1(\alpha,\eta)F_3(\alpha,b,\eta)\right| \\
&\quad \leqq a\int_{|w|<X}\left|(2\pi\alpha)^{-1}e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}|w|^{-1+\eta}\widehat{\phi}_\alpha(w) - F_3(\alpha,b,\eta)\right| \\
&\qquad \cdot |aw|^{1-\eta}|aw|^{\frac{1}{\alpha}-1}|\widehat{\psi}_\alpha(aw)|\ \mathrm{d}w \\
&\qquad + a\int_{|w|>X}\left|(2\pi\alpha)^{-1}e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}|w|^{-1+\eta}\widehat{\phi}_\alpha(w) - F_3(\alpha,b,\eta)\right| \\
&\qquad \cdot |aw|^{1-\eta}|aw|^{\frac{1}{\alpha}-1}|\widehat{\psi}_\alpha(aw)|\ \mathrm{d}w \\
&\quad \leqq a^{1-\eta+\frac{1}{\alpha}}\int_{-X}^{X}\left|(2\pi\alpha)^{-1}e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}|w|^{-1+\eta}\widehat{\phi}_\alpha(w) - F_3(\alpha,b,\eta)\right| \\
&\qquad \cdot |w|^{1-\eta}|w|^{\frac{1}{\alpha}-1}|\widehat{\psi}_\alpha(aw)|\ \mathrm{d}w \\
&\qquad + \sup_{|w|>X}\left|(2\pi\alpha)^{-1}e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}|w|^{-1+\eta}\widehat{\phi}_\alpha(w) - F_3(\alpha,b,\eta)\right| \\
&\qquad \cdot \int_{-\infty}^{\infty}|w|^{1-\eta}|w|^{\frac{1}{\alpha}-1}|\widehat{\psi}_\alpha(w)|\ \mathrm{d}w.
\end{aligned}$$

Thus, by using Eq. (9), there exists a positive constant $M_2 > 0$ such that

$$|\widehat{\psi}_\alpha(aw)| \leqq M_2(a|w|)^\mu.$$

Hence we have

$$\begin{aligned}
&\left|a^{1-\eta+\frac{1}{\alpha}}\Big(W_{\psi_\alpha}\phi\Big)(b,a) - F_1(\alpha,\eta)F_3(\alpha,b,\eta)\right| \\
&\quad \leqq M_2 a^{\mu+1-\eta+\frac{1}{\alpha}}\int_{-X}^{X}\left|(2\pi\alpha)^{-1}e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}\widehat{\phi}_\alpha(w) - F_3(\alpha,b,\eta)|w|^{1-\eta}\right||w|^{\frac{1}{\alpha}-1+\mu}\ \mathrm{d}w \\
&\qquad + \sup_{|w|>X}\left|(2\pi\alpha)^{-1}e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}|w|^{-1+\eta}\widehat{\phi}_\alpha(w) - F_3(\alpha,b,\eta)\right| \\
&\qquad \cdot \int_{-\infty}^{\infty}|w|^{\frac{1}{\alpha}-\eta}|\widehat{\psi}_\alpha(w)|\ \mathrm{d}w.
\end{aligned} \tag{16}$$

Now, since both of the integrals on the right-hand side of Eq. (16) are convergent and

$$\eta < 1 + \frac{1}{\alpha} + \mu,$$

therefore, as $a \to 0$, the first term can be made less than $\frac{\epsilon}{2}$. Also, for sufficiently large X, the second term, which is independent of a, can be made less than $\frac{\epsilon}{2}$. Hence we obtain

$$\left|a^{1-\eta+\frac{1}{\alpha}}\Big(W_{\psi_\alpha}\phi\Big)(b,a) - F_1(\alpha,\eta)F_3(\alpha,b,\eta)\right| < \epsilon$$

for sufficiently small a. □

4. Abelian Theorems for the Fractional Wavelet Transform of Distributions

In this section, Abelian theorems for the fractional wavelet transform of distributions are investigated and their properties are obtained by exploiting the theory of the fractional Fourier transform.

Definition 10. Let $\phi \in S'(\mathbb{R})$. Then the fractional wavelet transform of the distribution:

$$|w|^{\frac{1}{\alpha}-1}\widehat{\phi}_\alpha(w) \in S'(\mathbb{R})$$

is defined by

$$\left(W_{\psi_\alpha}\phi\right)(b,a) = \frac{1}{2\pi\alpha}\left\langle |w|^{\frac{1}{\alpha}-1}\widehat{\phi}_\alpha(w), e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}\overline{\widehat{\psi}_\alpha(aw)}\right\rangle. \tag{17}$$

Theorem 3. *If $\phi \in S'(\mathbb{R})$, then the differentiability of the fractional wavelet transform:*

$$\left(W_{\psi_\alpha}\phi\right)(b,a)$$

is exhibited by

$$\begin{aligned}&\left(\frac{\partial}{\partial a}\right)^m\left(\frac{\partial}{\partial b}\right)^n\left(W_{\psi_\alpha}\phi\right)(b,a)\\ &\qquad = \frac{1}{2\pi\alpha}\left\langle |w|^{\frac{1}{\alpha}-1}\widehat{\phi}_\alpha(w),\ \left(i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}\right)^n e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}\left(\frac{\partial}{\partial a}\right)^m\overline{\widehat{\psi}_\alpha(aw)}\right\rangle\end{aligned} \tag{18}$$

for $a > 0$ and for all $m, n \in \mathbb{N}_0$.

Proof. For $h > 0$, we have

$$\begin{aligned}&\frac{1}{h}\Big[\left(W_{\psi_\alpha}\phi\right)(b,a+h) - \left(W_{\psi_\alpha}\phi\right)(b,a)\Big] - \frac{1}{2\pi\alpha}\left\langle |w|^{\frac{1}{\alpha}-1}\widehat{\phi}_\alpha(w),\ e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}\frac{\partial}{\partial a}\overline{\widehat{\psi}_\alpha(aw)}\right\rangle\\ &\qquad = \frac{1}{2\pi\alpha}\left\langle |w|^{\frac{1}{\alpha}-1}\widehat{\phi}_\alpha(w),\ e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}\left[\frac{1}{h}\left(\overline{\widehat{\psi}_\alpha\big((a+h)w\big)} - \overline{\widehat{\psi}_\alpha(aw)}\right) - \frac{\partial}{\partial a}\overline{\widehat{\psi}_\alpha(aw)}\right]\right\rangle.\end{aligned}$$

Thus, clearly, we have to show that

$$e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}\left[\frac{1}{h}\left(\overline{\widehat{\psi}_\alpha((a+h)w)} - \overline{\widehat{\psi}_\alpha(aw)}\right) - \frac{\partial}{\partial a}\overline{\widehat{\psi}_\alpha(aw)}\right] \to 0 \text{ in } S(\mathbb{R}) \text{ as } h \to 0,$$

since

$$\begin{aligned}&\left|w^k\left(\frac{\partial}{\partial w}\right)^m\left[e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}\left\{\frac{1}{h}\left(\overline{\widehat{\psi}_\alpha((a+h)w)} - \overline{\widehat{\psi}_\alpha(aw)}\right) - \frac{\partial}{\partial a}\overline{\widehat{\psi}_\alpha(aw)}\right\}\right]\right|\\ &\qquad = \left|w^k\sum_{r=0}^{m}\binom{m}{r}\left[\left(\frac{\partial}{\partial w}\right)^{m-r}e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}\right]\right.\\ &\qquad\qquad \left.\cdot\left[\left(\frac{\partial}{\partial w}\right)^r\left\{\frac{1}{h}\left(\overline{\widehat{\psi}_\alpha((a+h)w)} - \overline{\widehat{\psi}_\alpha(aw)}\right) - \frac{\partial}{\partial a}\overline{\widehat{\psi}_\alpha(aw)}\right\}\right]\right|\\ &\qquad = \left|w^k\sum_{r=0}^{m}\binom{m}{r}\left[\left(\frac{\partial}{\partial w}\right)^{m-r}e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}\right]\right.\\ &\qquad\qquad \left.\cdot\left\{\frac{1}{h}\left(\left(\frac{\partial}{\partial w}\right)^r\overline{\widehat{\psi}_\alpha((a+h)w)} - \left(\frac{\partial}{\partial w}\right)^r\overline{\widehat{\psi}_\alpha(aw)}\right) - \frac{\partial}{\partial a}\left(\frac{\partial}{\partial w}\right)^r\overline{\widehat{\psi}_\alpha(aw)}\right\}\right|\\ &\qquad = \left|w^k\sum_{r=0}^{m}\binom{m}{r}\left[\left(\frac{\partial}{\partial w}\right)^{m-r}e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}\right]\right.\\ &\qquad\qquad \left.\cdot\frac{1}{h}\left[\int_a^{a+h}\left\{\left(\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial w}\right)^r\overline{\widehat{\psi}_\alpha(tw)} - \left(\frac{\partial}{\partial a}\right)\left(\frac{\partial}{\partial w}\right)^r\overline{\widehat{\psi}_\alpha(aw)}\right\}\mathrm{d}t\right]\right|\end{aligned}$$

$$
\begin{aligned}
&= \Bigg| w^k \sum_{r=0}^{m} \binom{m}{r} \Bigg[\left(\frac{\partial}{\partial w}\right)^{m-r} e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}} b} \Bigg] \\
&\quad \cdot \frac{1}{h} \int_a^{a+h} \left(\int_a^t \left(\frac{\partial}{\partial u}\right)^2 \left(\frac{\partial}{\partial w}\right)^r \overline{\widehat{\psi}_\alpha(uw)} du \right) \mathrm{d}t \Bigg| \\
&\leqq \Bigg| w^k \sum_{r=0}^{m} \binom{m}{r} \Bigg[\left(\frac{\partial}{\partial w}\right)^{m-r} e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}} b} \Bigg] \Bigg| \\
&\quad \cdot \frac{h}{2} \sup_{a \leqq u \leqq a+h} \left| \left(\frac{\partial}{\partial u}\right)^2 \left(\frac{\partial}{\partial w}\right)^r \overline{\widehat{\psi}_\alpha(uw)} \right|. \qquad (19)
\end{aligned}
$$

Now, from [31, p. 535], we can write

$$
\left(\frac{\partial}{\partial w}\right)^{m-r} e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}} b} \qquad (20)
$$

$$
\begin{aligned}
&= (m-r)! \Bigg[\frac{(ib) e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}} b} \left(\frac{1}{\alpha}\right)\left(\frac{1}{\alpha}-1\right)\left(\frac{1}{\alpha}-2\right)\cdots\left(\frac{1}{\alpha}-(m-r)+1\right)}{1!(m-r)!} \\
&\quad \cdot |w|^{\frac{1}{\alpha}-(m-r)} (\operatorname{sgn} w)^{m-r+1} + \cdots \\
&\quad + \frac{(ib)^{m-r} e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}} b} \frac{1}{\alpha^{m-r}}}{(1!)^{m-r}(m-r)!} |w|^{\frac{m-r}{\alpha}-(m-r)} (\operatorname{sgn} w)^{2(m-r)} \Bigg] \\
&= (m-r)! \Bigg[A_1 (ib) e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}} b} (\operatorname{sgn} w)^{m-r+1} |w|^{\frac{1}{\alpha}-(m-r)} + \cdots \\
&\quad + A_{m-r} (ib)^{m-r} e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}} b} (\operatorname{sgn} w)^{2(m-r)} |w|^{\frac{m-r}{\alpha}-(m-r)} \Bigg], \qquad (21)
\end{aligned}
$$

where $A_1, A_2, \cdots, A_{m-r}$ are constants.

Next, from Eq. (20), we have

$$
\left| \left(\frac{\partial}{\partial w}\right)^{m-r} e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}} b} \right| \leqq (m-r)! \left(|A_1||b||w|^{\frac{1}{\alpha}-(m-r)} + \cdots + |A_{m-r}||b|^{m-r}|w|^{(m-r)(\frac{1}{\alpha}-1)} \right). \qquad (22)
$$

Hence, by using Eq. (19) and Eq. (22), we obtain

$$
\begin{aligned}
&\left| w^k \left(\frac{\partial}{\partial w}\right)^m \left(e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}} b} \left[\frac{1}{h}\left(\overline{\widehat{\psi}_\alpha\big((a+h)w\big)} - \overline{\widehat{\psi}_\alpha(aw)} \right) - \frac{\partial}{\partial a} \overline{\widehat{\psi}_\alpha(aw)} \right] \right) \right| \\
&\quad \leqq \sum_{r=0}^{m} \binom{m}{r} (m-r)! \left(|A_1| \cdot |b| \cdot |w|^{\frac{1}{\alpha}-(m-r)} + \cdots + |A_{m-r}||b|^{m-r}|w|^{(m-r)(\frac{1}{\alpha}-1)} \right) \\
&\quad\quad \cdot \frac{h}{2} \sup_{a \leqq u \leqq a+h} \left| w^k \left(\frac{\partial}{\partial u}\right)^2 \left(\frac{\partial}{\partial w}\right)^r \overline{\widehat{\psi}_\alpha(uw)} \right|. \qquad (23)
\end{aligned}
$$

By substituting $uw = z$ into Eq. (23), we get

$$
\left| w^k \left(\frac{\partial}{\partial w}\right)^m e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}} b} \left[\frac{1}{h}\left(\overline{\widehat{\psi}_\alpha\big((a+h)w\big)} - \overline{\widehat{\psi}_\alpha(aw)} \right) - \frac{\partial}{\partial a} \overline{\widehat{\psi}_\alpha(aw)} \right] \right|
$$

$$\leqq \sum_{r=0}^{m}\binom{m}{r}(m-r)!\left(|A_1||b|\left|\frac{z}{u}\right|^{\frac{1}{\alpha}-(m-r)}+\cdots+|A_{m-r}||b|^{m-r}\left|\frac{z}{u}\right|^{(m-r)(\frac{1}{\alpha}-1)}\right)$$
$$\cdot\frac{h}{2}\sup_{a\leqq u\leqq a+h}\left|z^{k+2}u^{r-2-k}\left(\frac{\partial}{\partial z}\right)^{r+2}\overline{\widehat{\psi_\alpha}(z)}\right|$$
$$\leqq \sum_{r=0}^{m}\binom{m}{r}(m-r)!\ \frac{h}{2}\left[|A_1||b|\sup_{z\in\mathbb{R}}\left|z^{k+2+\frac{1}{\alpha}-(m-r)}\left(\frac{\partial}{\partial z}\right)^{r+2}\overline{\widehat{\psi_\alpha}(z)}\right|\right.$$
$$\cdot\sup_{a\leqq u\leqq a+h}|u|^{r-2-k-\frac{1}{\alpha}+(m-r)}+\cdots+|A_{m-r}||b|^{m-r}$$
$$\cdot\sup_{z\in\mathbb{R}}\left|z^{k+2+(m-r)(\frac{1}{\alpha}-1)}\left(\frac{\partial}{\partial z}\right)^{r+2}\overline{\widehat{\psi_\alpha}(z)}\right|$$
$$\left.\cdot\sup_{a\leqq u\leqq a+h}|u|^{r-2-k-(m-r)(\frac{1}{\alpha}-1)}\right]$$
$$\leqq \frac{h}{2}\sum_{r=0}^{m}\binom{m}{r}(m-r)!\left[|A_1||b|\sup_{a\leqq u\leqq a+h}|u|^{m-2-k-\frac{1}{\alpha}}\right.$$
$$\cdot\gamma_{k+2+\frac{1}{\alpha}-(m-r),r+2}(\widehat{\psi_\alpha})+\cdots+|A_{m-r}||b|^{m-r}$$
$$\cdot\sup_{a\leqq u\leqq a+h}|u|^{r-2-k-(m-r)(\frac{1}{\alpha}-1)}$$
$$\left.\cdot\gamma_{k+2+(m-r)(\frac{1}{\alpha}-1),r+2}(\widehat{\psi_\alpha})\right]\to 0\ \text{ as }\ h\to 0.$$

Hence, finally, we find that

$$\lim_{h\to 0}\frac{\left(W_{\psi_\alpha}\phi\right)(b,a+h)-\left(W_{\psi_\alpha}\phi\right)(b,a)}{h}$$
$$=\frac{1}{2\pi\alpha}\left\langle |w|^{\frac{1}{\alpha}-1}\widehat{\phi}_\alpha(w),\ e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}\frac{\partial}{\partial a}\overline{\widehat{\psi_\alpha}(aw)}\right\rangle.$$

Similarly, we can prove the differentiability with respect to the variable b and, in general, we can find Eq. (18). □

Theorem 4. *Let $\left(W_{\psi_\alpha}\phi\right)(b,a)$ be the fractional wavelet transform of the following distribution*:

$$|w|^{\frac{1}{\alpha}-1}\widehat{\phi}_\alpha(w)\in S'(\mathbb{R}).$$

Then, for large k and $a>0$, it is asserted that

$$\left(W_{\psi_\alpha}\phi\right)(b,a)=O\left(a^{-k-\frac{k}{\alpha}}|b|^k\right)\qquad (a\to 0); \tag{24}$$
$$=O\left(a^{2k-\frac{k}{\alpha}}\right)\qquad (a\to\infty); \tag{25}$$
$$=O\left(a^{-\frac{k}{\alpha}}(1+a^2)^k\right)\qquad (|b|\to 0); \tag{26}$$
$$=O\left(a^{-k-\frac{k}{\alpha}}(1+a^2)^k|b|^k\right)\qquad (|b|\to\infty). \tag{27}$$

Proof. In view of the boundedness property of generalized functions (see [35, p. 111]), there exists a constant $C>0$ and a non-negative integer k depending on $|w|^{\frac{1}{\alpha}-1}\widehat{\phi}_\alpha(w)$ such that

$$\left|\left(W_{\psi_\alpha}\phi\right)(b,a)\right|\leqq C\sup_{w}\left|(1+w^2)^k\left(\frac{\partial}{\partial w}\right)^k\left[e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}\widehat{\psi_\alpha}(aw)\right]\right|$$

$$= C\sup_{w}\left|(1+w^2)^k\sum_{s=0}^{k}\binom{k}{s}\Big[\Big(\frac{\partial}{\partial w}\Big)^{s}e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}\Big]\cdot\Big[\Big(\frac{\partial}{\partial w}\Big)^{k-s}\widehat{\psi_\alpha}(aw)\Big]\right|$$

$$= C\sup_{w}\left|\sum_{s=0}^{k}\sum_{r=0}^{k}\binom{k}{s}\binom{k}{r}w^{2r}\Big[\Big(\frac{\partial}{\partial w}\Big)^{s}e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}\Big]\cdot\Big[\Big(\frac{\partial}{\partial w}\Big)^{k-s}\widehat{\psi_\alpha}(aw)\Big]\right|.$$

On the other hand, in view of Eq. (22), there exist positive constants $A_1, A_2, \cdots, A_s$ such that

$$|\big(W_{\psi_\alpha}\phi\big)(b,a)| \leqq C\sup_{w}\Bigg|\sum_{s=0}^{k}\sum_{r=0}^{k}\binom{k}{s}\binom{k}{r}w^{2r}\ s!\ \Big[A_1|b|\cdot|w|^{\frac{1}{\alpha}-s}+\cdots$$
$$+A_s|b|^{s}\cdot|w|^{\frac{s}{\alpha}-s}\Big]\cdot\Big[\Big(\frac{\partial}{\partial w}\Big)^{k-s}\widehat{\psi_\alpha}(aw)\Big]\Bigg|.$$

Thus, upon setting $z = aw$, we can find that

$$\Big|\big(W_{\psi_\alpha}\phi\big)(b,a)\Big| \leqq C\sup_{z}\Bigg|\sum_{s=0}^{k}\sum_{r=0}^{k}\binom{k}{s}\binom{k}{r}a^{k-s-2r}z^{2r}\ s!\ \Big[A_1|b|\Big|\frac{z}{a}\Big|^{\frac{1}{\alpha}-s}+\cdots$$
$$+A_s|b|^{s}\Big|\frac{z}{a}\Big|^{\frac{s}{\alpha}-s}\Big]\cdot\Big[\Big(\frac{\partial}{\partial w}\Big)^{k-s}\widehat{\psi_\alpha}(z)\Big]\Bigg|$$
$$\leqq C\sup_{z}\Bigg|\sum_{s=0}^{k}\sum_{r=0}^{k}\binom{k}{s}\binom{k}{r}\ s!\ \Big[A_1|b|a^{k-\frac{1}{\alpha}-2r}|z|^{2r+\frac{1}{\alpha}-s}+\cdots$$
$$+A_s|b|^{s}\ a^{k-\frac{s}{\alpha}-2r}\ |z|^{2r+\frac{s}{\alpha}-s}\Big]\cdot\Big[\Big(\frac{d}{dz}\Big)^{k-s}\widehat{\psi_\alpha}(z)\Big]\Bigg|$$
$$\leqq C\sum_{s=0}^{k}\sum_{r=0}^{k}\binom{k}{s}\binom{k}{r}\ s!\ \Big[A_1|b|a^{k-\frac{1}{\alpha}-2r}\sup_{z}\Big|z^{2r+\frac{1}{\alpha}-s}\Big(\frac{d}{dz}\Big)^{k-s}$$
$$\cdot\widehat{\psi_\alpha}(z)\Big|+\cdots+A_s|b|^{s}a^{k-\frac{s}{\alpha}-2r}\sup_{z}\Big|z^{2r+\frac{s}{\alpha}-s}\Big(\frac{d}{dz}\Big)^{k-s}\widehat{\psi_\alpha}(z)\Big|\Big]$$
$$\leqq C\sum_{s=0}^{k}\sum_{r=0}^{k}\binom{k}{s}\binom{k}{r}\ s!\ \Big[A_1|b|a^{k-\frac{1}{\alpha}-2r}\gamma_{2r+\frac{1}{\alpha}-s,k-s}\Big(\widehat{\psi_\alpha}(z)\Big)+$$
$$\cdots+A_s|b|^{s}a^{k-\frac{s}{\alpha}-2r}\gamma_{2r+\frac{s}{\alpha}-s,k-s}\Big(\widehat{\psi_\alpha}(z)\Big)\Big],$$

that is, that

$$\Big|\big(W_{\psi_\alpha}\phi\big)(b,a)\Big| \leqq C\sum_{s=0}^{k}\sum_{r=0}^{k}\binom{k}{s}\binom{k}{r}\ s!\ \Big[A_1|b|a^{k-\frac{1}{\alpha}-2r}\gamma_{2r+\frac{1}{\alpha}-s,k-s}\Big(\widehat{\psi_\alpha}(z)\Big)$$
$$\cdots+A_s|b|^{s}a^{k-\frac{s}{\alpha}-2r}\gamma_{2r+\frac{s}{\alpha}-s,k-s}\Big(\widehat{\psi_\alpha}(z)\Big)\Big]$$
$$= C\sum_{s=0}^{k}\sum_{r=0}^{k}\sum_{l=1}^{s}\binom{k}{s}\binom{k}{r}\ s!\ A_l|b|^{l}a^{k-\frac{l}{\alpha}-2r}\gamma_{2r+\frac{l}{\alpha}-s,k-s}\Big(\widehat{\psi_\alpha}(z)\Big)$$
$$\leqq C'\sum_{s=0}^{k}\sum_{r=0}^{k}\binom{k}{s}\binom{k}{r}\ s!\ A_s|b|^{s}a^{k-\frac{s}{\alpha}-2r}\gamma_{2r+\frac{s}{\alpha}-s,k-s}\Big(\widehat{\psi_\alpha}(z)\Big)$$
$$\leqq C''\sum_{r=0}^{k}\binom{k}{r}a^{-2r}a^{k-\frac{k}{\alpha}}(a+|b|)^{k}$$

$$= C''(1+a^{-2})^k a^{k-\frac{k}{\alpha}}(a+|b|)^k. \tag{28}$$

Thus, from Eq. (28), we are led to the assertions given by Eq. (24), Eq. (25), Eq. (26) and Eq. (27). □

We now state and prove our next result (Theorem 5 below) which is useful to obtain Abelian theorems for the distributional fractional wavelet transform. For this purpose, we assume that

$$D^s\widehat{\psi}_\alpha(w) = O(|w|^\mu), \quad |w| \to 0 \qquad (\forall\ s \in \mathbb{N}_0) \tag{29}$$

for some real number μ.

Theorem 5. *Let $\psi \in S(\mathbb{R})$ and $\phi \in S'(\mathbb{R})$ be distributions of compact support in $\mathbb{R}$. Then*

$$\left(W_{\psi_\alpha}\phi\right)(b,a) = \frac{1}{2\pi\alpha}\left\langle |w|^{\frac{1}{\alpha}-1}\widehat{\phi}_\alpha(w), e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}\overline{\widehat{\psi}_\alpha(aw)}\right\rangle$$

is a smooth function on $\mathbb{R}\times\mathbb{R}_+$ and satisfies the following condition:

$$\left(W_{\psi_\alpha}\phi\right)(b,a) = O\left(a^\mu(1+a+|b|)^k\right) \qquad \left(|a| \to 0;\ k \in \mathbb{N}\right). \tag{30}$$

Proof. Let $\phi \in S'(\mathbb{R})$. Then, from Theorem 2.3 of [31], we have $\widehat{\phi}_\alpha \in S'(\mathbb{R})$. We assume that $\widehat{\phi}_\alpha$ is of compact support $K \subset \mathbb{R}$. We also let $\lambda(w) \in \mathcal{D}(\mathbb{R})$, the space of all C^∞-functions of compact support such that $\lambda(w) = 1$ in a neighborhood of K. Therefore, we get

$$\begin{aligned}\left(W_{\psi_\alpha}\phi\right)(b,a) &= \frac{1}{2\pi\alpha}\left\langle |w|^{\frac{1}{\alpha}-1}\widehat{\phi}_\alpha(w),\ e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}\overline{\widehat{\psi}_\alpha(aw)}\right\rangle\\ &= \frac{1}{2\pi\alpha}\left\langle |w|^{\frac{1}{\alpha}-1}\widehat{\phi}_\alpha(w), \lambda(w)e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}\overline{\widehat{\psi}_\alpha(aw)}\right\rangle.\end{aligned}$$

So, by Theorem 3, $\left(W_{\psi_\alpha}\phi\right)(b,a)$ is infinitely differentiable with respect to the variables b and a. Thus, by the boundedness property of generalized functions as used in Theorem 4, we have

$$\begin{aligned}\left|\left(W_{\psi_\alpha}\phi\right)(b,a)\right| &= \frac{1}{2\pi\alpha}\left|\left\langle |w|^{\frac{1}{\alpha}-1}\widehat{\phi}_\alpha(w),\ e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}\overline{\widehat{\psi}_\alpha(aw)}\right\rangle\right|\\ &\leqq C\max_r\sup_{w\in K}\left|D_w^r\left[\lambda(w)e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}\overline{\widehat{\psi}_\alpha(aw)}\right]\right|\\ &\leqq C\max_r\sup_{w\in K}\sum_{n=0}^{r}\binom{r}{n}\left|\left(D_w^{r-n}\lambda(w)\right)D_w^n\left(e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}\overline{\widehat{\psi}_\alpha(aw)}\right)\right|\\ &\leqq C\max_r\sup_{w\in K}\sum_{n=0}^{r}\binom{r}{n}\left|\left(D_w^{r-n}\lambda(w)\right)\sum_{s=0}^{n}\binom{n}{s}\left(D_w^{n-s}e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}\right)\left(D_w^s\overline{\widehat{\psi}_\alpha(aw)}\right)\right|.\end{aligned}$$

In view of Eq. (22), there exist positive constants $A_1,\cdots,A_{n-s}$ such that

$$\begin{aligned}\left|\left(W_{\psi_\alpha}\phi\right)(b,a)\right| &\leqq C\max_r\sup_{w\in K}\sum_{n=0}^{r}\sum_{s=0}^{n}\binom{r}{n}\binom{n}{s}\left|\left(D_w^{r-n}\lambda(w)\right)\right|\\ &\quad\cdot(n-s)!\left(A_1|b||w|^{\frac{1}{\alpha}-(n-s)}+\cdots+A_{n-s}|b|^{n-s}|w|^{(n-s)\left(\frac{1}{\alpha}-1\right)}\right)\left|\left(D_w^s\overline{\widehat{\psi}_\alpha(aw)}\right)\right|\\ &\leqq C'\max_r\sup_{w\in K}\sum_{n=0}^{r}\sum_{s=0}^{n}\binom{r}{n}\binom{n}{s}\left|\left(D_w^{r-n}\lambda(w)\right)\right|\left[\sum_{l=1}^{n-s}A_l|b|^l|w|^{\frac{l}{\alpha}-(n-s)}\right]\left|\left(D_w^s\overline{\widehat{\psi}_\alpha(aw)}\right)\right|\end{aligned}$$

$$\leqq C'' \max_r \sup_{w\in K} \sum_{n=0}^{r} \sum_{s=0}^{n} \binom{r}{n}\binom{n}{s} |b|^{n-s} |w|^{(n-s)\left(\frac{1}{\alpha}-1\right)} a^{s+\mu} |w|^{\mu}$$
$$\leqq C'' \max_r \sum_{n=0}^{r} \binom{r}{n} \left(\sum_{s=0}^{n} \binom{n}{s} |b|^{n-s} \right) a^{s+\mu}$$
$$\leqq C'' \max_r \sum_{n=0}^{r} \binom{r}{n} \left(a + |b|\right)^n a^{\mu}$$
$$= C'' \max_r \left(1 + a + |b|\right)^r a^{\mu},$$

where C'' is a positive constant. Hence we have

$$\left|\left(W_{\psi_\alpha}\phi\right)(b,a)\right| \leqq C'' \max_r \left(1 + a + |b|\right)^r a^{\mu}.$$

□

For the distributional fractional wavelet transform given by Eq. (17), we have the following initial-value theorem.

Theorem 6. *Let* $\widehat{\phi}_\alpha \in S'(\mathbb{R})$ *which can be decomposed into*

$$\widehat{\phi}_\alpha = \phi_1 + \phi_2,$$

where ϕ_1 *is an ordinary function and* $\phi_2 \in \Xi'(\mathbb{R} \setminus \{0\})$ *is of order* k*. Also let the real numbers* μ *and* η *be such that*

$$1 + \frac{1}{\alpha} + 2k - \frac{k}{\alpha} < \eta < \mu + 1 + \frac{1}{\alpha}.$$

Suppose also that

$$|w|^{\frac{1}{\alpha}-\eta} \widehat{\psi_\alpha}(w) \in L^1(\mathbb{R})$$

and

$$|w|^{\frac{1}{\alpha}-1} \phi_1(w) \in L^1(\delta, \infty) \qquad (\forall\ \delta > 0),$$

and assume that

$$\left(W_{\psi_\alpha}\phi\right)(b,a)$$

is the distributional wavelet transform of

$$|w|^{\frac{1}{\alpha}-1} \widehat{\phi}_\alpha,$$

which is defined by Eq. (17)*. Then*

$$\lim_{a\to\infty} a^{1-\eta+\frac{1}{\alpha}} \left(W_{\psi_\alpha}\phi\right)(b,a) = F_1(\alpha,\eta) \lim_{|w|\to 0} (2\pi\alpha)^{-1} |w|^{-1+\eta} \widehat{\phi}_\alpha(w). \tag{31}$$

Proof. By Theorem 3, we see that

$$\left(W_{\psi_\alpha}\phi_2\right)(b,a) = \frac{1}{2\pi\alpha} \left\langle |w|^{\frac{1}{\alpha}-1} \phi_2(w),\ e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}} b} \overline{\widehat{\psi_\alpha}(aw)} \right\rangle$$

is an infinitely differentiable function on $\mathbb{R} \times \mathbb{R}_+$. Furthermore, by Theorem 4,

$$\left(W_{\psi_\alpha}\phi_2\right)(b,a) = O(a^{2k-\frac{k}{\alpha}}) \qquad (a \to \infty).$$

Hence there exists a constant $C > 0$ such that

$$\left| a^{1-\eta+\frac{1}{\alpha}} \left(W_{\psi_\alpha}\phi_2\right)(b,a) \right| \leqq C a^{1-\eta+\frac{1}{\alpha}+2k-\frac{k}{\alpha}}. \tag{32}$$

Since

$$1-\eta+\frac{1}{\alpha}+2k-\frac{k}{\alpha}<0,$$

the right-hand side of Eq. (32) tends to 0 as $a\to\infty$. Also, since the support of $\phi_2\in\Xi'(\mathbb{R}\setminus\{0\})$ is a compact subset of $\mathbb{R}-\{0\}$, we get

$$\lim_{w\to 0} e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}|w|^{-1+\eta}\phi_2(w)=0.$$

The result asserted by Theorem 6 follows by an application of Theorem 1 with $\widehat{\phi}_\alpha(w)$ replaced by $\phi_1(w)$. □

The following result is the final-value theorem for the distributional fractional wavelet transform given by Eq. (17).

Theorem 7. *Let*

$$1+\frac{1}{\alpha}<\eta<\mu+1+\frac{1}{\alpha}\qquad(\mu>0).$$

Assume thet $\widehat{\phi}_\alpha\in S'(\mathbb{R})$ can be decomposed into $\widehat{\phi}_\alpha=\phi_1+\phi_2$, where ϕ_1 is an ordinary function satisfying the following condition:

$$|w|^{\mu+\frac{1}{\alpha}-1}\phi_1(w)\in L^1(-X,X)\qquad(\forall\ X>0)$$

and $\phi_2\in\Xi'(\mathbb{R}\setminus\{0\})$. If $\left(W_{\psi_\alpha}\phi\right)(b,a)$ is the distributional wavelet transform of $|w|^{\frac{1}{\alpha}-1}\widehat{\phi}_\alpha$ defined by Eq. (17), *then*

$$\lim_{a\to 0} a^{1-\eta+\frac{1}{\alpha}}\left(W_{\psi_\alpha}\phi\right)(b,a)=F_1(\alpha,\eta)(2\pi\alpha)^{-1}\lim_{w\to\infty} e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}|w|^{-1+\eta}\widehat{\phi}_\alpha(w). \tag{33}$$

Proof. By Theorem 3 and Theorem 5, we observe that

$$\left(W_{\psi_\alpha}\phi_2\right)(b,a)=\frac{1}{2\pi\alpha}\left\langle |w|^{\frac{1}{\alpha}-1}\phi_2(w),\ e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}\overline{\widehat{\psi}_\alpha(aw)}\right\rangle,$$

is an infinitely differentiable function on $\mathbb{R}\times\mathbb{R}_+$ and

$$\left(W_{\psi_\alpha}\phi_2\right)(b,a)=Ca^{\mu}(1+|b|)^k\ \text{ as }\ a\to 0,$$

C being a large constant. Since

$$1-\eta+\frac{1}{\alpha}+\mu>0,$$

we have

$$a^{1-\eta+\frac{1}{\alpha}}\left|\left(W_{\psi_\alpha}\phi_2\right)(b,a)\right|\leqq Ca^{1-\eta+\frac{1}{\alpha}+\mu}\ (1+|b|)^k\to 0\ \text{ as }\ a\to 0.$$

By taking $\widehat{\phi}_\alpha(w)$ to be $\phi_2(w)$, the final result follows from Theorem 2. □

5. Application

As an application of the theory presented in this article, we consider the fractional wavelet transform defined by the Mexican hat wavelet function (see, for details, [28]).

The Mexican hat wavelet function $\psi(x)$ is given by

$$\psi(x)=(1-x^2)e^{-\frac{1}{2}x^2}. \tag{34}$$

Also, from Example 1.6.4 of [12], the Fourier transform $\widehat{\psi}(w)$ of the function $\psi(x)$ in Eq. (34) is given by

$$\widehat{\psi}(w) = \sqrt{2\pi}|w|^2 e^{-\frac{|w|^2}{2}}.$$

Now, by Remark 5 of [7], the fractional Fourier transform of Eq. (34) is given by

$$\begin{aligned}\widehat{\psi}_\alpha(w) &= \widehat{\psi}(\text{sgn }(w)|w|^{\frac{1}{\alpha}}) \\ &= \sqrt{2\pi}|w|^{\frac{2}{\alpha}} e^{-\frac{1}{2}|w|^{\frac{2}{\alpha}}}.\end{aligned} \tag{35}$$

Furthermore, the following asymptotic order of $\widehat{\psi}_\alpha(w)$ holds true:

$$\widehat{\psi}_\alpha(w) = O\left(w^{\frac{2}{\alpha}}\right) \qquad (|w| \to 0). \tag{36}$$

Hence, in view of Eq. (8) and Eq. (35), we have the following fractional wavelet transform:

$$\left(W_{\psi_\alpha}\phi\right)(b,a) = \frac{1}{\sqrt{2\pi\alpha}} \int_{-\infty}^{\infty} e^{i(\text{sgn } w)|w|^{\frac{1}{\alpha}} b} |w|^{\frac{1}{\alpha}-1} \widehat{\phi}_\alpha(w) |aw|^{\frac{2}{\alpha}} e^{-\frac{1}{2}|aw|^{\frac{2}{\alpha}}} \, \mathrm{d}w. \tag{37}$$

Thus, from Eq. (10) and Eq. (35), we find the following expression of $F_1(\alpha, \eta)$:

$$\begin{aligned}F_1(\alpha,\eta) &= \int_{-\infty}^{\infty} \sqrt{2\pi}|w|^{\frac{2}{\alpha}} e^{-\frac{1}{2}|w|^{\frac{2}{\alpha}}} |w|^{\frac{1}{\alpha}-\eta} \, \mathrm{d}w \\ &= \int_{-\infty}^{\infty} \sqrt{2\pi}|w|^{\frac{3}{\alpha}-\eta} e^{-\frac{1}{2}|w|^{\frac{2}{\alpha}}} \, \mathrm{d}w,\end{aligned}$$

which, in view of the following familiar Gamma-function result:

$$\int_0^{\infty} x^{\mu} e^{-\lambda x^{\nu}} \, \mathrm{d}x = \frac{1}{\nu \lambda^{\left(\frac{\mu+1}{\nu}\right)}} \Gamma\left(\frac{\mu+1}{\nu}\right) \qquad \left(\Re(\mu) > -1;\ \min\{\Re(\nu), \Re(\lambda)\} > 0\right),$$

can be rewritten as follows:

$$F_1(\alpha,\eta) = \alpha \pi^{\frac{1}{2}} 2^{\frac{1}{2}(4+\alpha-\alpha\eta)} \Gamma\left(\frac{3-\alpha\eta+\alpha}{2}\right) \qquad \left(\eta < \frac{3}{\alpha}+1\right). \tag{38}$$

Therefore, by a modification of the proof of Theorem 1, for

$$\eta < \frac{3}{\alpha} + 1$$

and

$$e^{-\frac{1}{2}|w|^{\frac{2}{\alpha}}} |w|^{\frac{1}{\alpha}-1} \widehat{\phi}_\alpha(w) \in L^1(\delta, \infty) \qquad (\forall\ \delta > 0),$$

and, by using Eq. (38), we find that

$$\lim_{a\to\infty} a^{1-\eta+\frac{1}{\alpha}} \left(W_{\psi_\alpha}\phi\right)(b,a) = \pi^{-\frac{1}{2}} 2^{\frac{1}{2}(2+\alpha-\alpha\eta)} \Gamma\left(\frac{3-\alpha\eta+\alpha}{2}\right) \lim_{|w|\to 0} |w|^{-1+\eta} \widehat{\phi}_\alpha(w). \tag{39}$$

Furthermore, by applying Theorem 2, for

$$\eta < \frac{3}{\alpha} + 1$$

and

$$|w|^{\frac{3}{\alpha}-1} \widehat{\phi}_\alpha(w) \in L^1(-X, X) \qquad (\forall\ X > 0),$$

and, by using Eq. (38), we get

$$\lim_{a\to 0} a^{1-\eta+\frac{1}{\alpha}}\Big(W_{\psi_\alpha}\phi\Big)(b,a) = \pi^{-\frac{1}{2}}2^{\frac{1}{2}(2+\alpha-\alpha\eta)}\Gamma\left(\frac{3-\alpha\eta+\alpha}{2}\right) \cdot \lim_{|w|\to\infty} e^{i(\operatorname{sgn} w)|w|^{\frac{1}{\alpha}}b}|w|^{-1+\eta}\widehat{\phi}_\alpha(w). \tag{40}$$

Finally, upon taking into account the fact that the kernel $\widehat{\psi}_\alpha(w)$ is exponentially decreasing, the conditions of validity of the initial-value and final-value results are relaxed in this example. By using Eq. (39) and Eq. (40), we can obtain the corresponding results derivable from Theorem 6 and Theorem 7, respectively.

6. Conclusion

In several earlier developments (see, for example, [7, 9, 10, 19, 31]), one can find that the fractional wavelet transform has a rich theory and extensive mathematical background. This theory presents a study of Abelian theorems in the classical sense as well as in the distributional sense. In our investigation herein, we have established several Abelian theorems of the fractional wavelet transform. Moreover, with the help of an example, we have shown that the Mexican hat function is a fractional wavelet function, which contains adquate time and frequency localizations and justifies Abelian theorems involving the fractional wavelet transform. Upon a systematic survey of the existing literature on Abelian theorems for many different integral transforms, we conclude that the Abelian theorems associated with the fractional wavelet transform provide potentially useful information about the initial and final values of the fractional wavelet transform which we have investigated herein.

We choose to conclude our investigation by referring to several recent developments on the Fourier, Hankel and several other integral transforms, function spaces, distributional analysis, wavelet analysis, *et cetera* (see, for example, [5], [6], [14], [16], [17], [18], [20], [21], [23], [24], [25], [26], [27], [29], [30] and [32]), in each of which the reader can find some other presumably novel directions of further researches along the lines which we have developed in the present article.

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