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# Variants on digital covering maps

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**Abstract.** S-E Han's paper [11] discusses several variants of digital covering maps. We show several equivalences among these variants and discuss shortcomings in Han's paper.

## 1. Introduction

The notion of a covering map has been adapted from classical algebraic topology to digital topology, where it is an important tool for computing digital versions of fundamental groups for binary digital images. With varying success, attempts have been made to modify the notion of a digital covering map to obtain related results under less restrictive conditions. Among these attempts are Han's paper [11], which contains a proof that is murky (see section 4 for clarification) and citations that are inappropriate. We also discuss a strangely presented example in Han's related paper [9] (see Remark 6.6). Further, it turns out that some of Han's variants on covering maps don't really vary from covering maps (see Theorem 6.5). Also, some of material of [11] is superseded by other papers including [3, 12, 13]. We justify these claims in the current paper.

## 2. Preliminaries

We use $\mathbb{N}$ for the set of natural numbers, $\mathbb{Z}$ for the set of integers, and $\#X$ for the number of distinct members of $X$.

We typically denote a (binary) digital image as $(X, \kappa)$, where $X \subset \mathbb{Z}^n$ for some $n \in \mathbb{N}$ and $\kappa$ represents an adjacency relation of pairs of points in $X$. Thus, $(X, \kappa)$ is a graph, in which members of $X$ may be thought of as black points, and members of $\mathbb{Z}^n \setminus X$ as white points, of a picture of some "real world" object or scene.

### 2.1. Adjacencies

Let $u, n \in \mathbb{N}$, $1 \leq u \leq n$. Han's papers use "$k$-adjacency" sometimes to mean an arbitrary adjacency, sometimes as an abbreviation for what he calls "$k(u, n)$-adjacency," where the digital image $(X, k)$ satisfies $X \subset \mathbb{Z}^n$ and $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in X$ are $k(u, n)$-adjacent if and only if

- $x \neq y$, and

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- for at most $u$ indices $i$, $|\ x_i - y_i\ | = 1$, and
- for all indices $j$ such that $|\ x_j - y_j\ | \neq 1$, we have $x_j = y_j$.

Other authors refer to this adjacency as $c_u$-adjacency. We will prefer the latter notation in the current paper. The $c_u$ adjacencies are the adjacencies most used in digital topology, especially $c_1$ and $c_n$.

In low dimensions, it is also common to denote a $c_u$ adjacency by the number of points that can have this adjacency with a given point in $\mathbb{Z}^n$. E.g.,

- For subsets of $\mathbb{Z}^1$, $c_1$-adjacency is 2-adjacency.
- For subsets of $\mathbb{Z}^2$, $c_1$-adjacency is 4-adjacency and $c_2$-adjacency is 8-adjacency.
- For subsets of $\mathbb{Z}^3$, $c_1$-adjacency is 6-adjacency, $c_2$-adjacency is 18-adjacency, and $c_3$-adjacency is 26-adjacency.

We use the notations $y \leftrightarrow_\kappa x$, or, when the adjacency $\kappa$ can be assumed, $y \leftrightarrow x$, to mean $x$ and $y$ are $\kappa$-adjacent. The notations $y \leftrightarroweq_\kappa x$, or, when $\kappa$ can be assumed, $y \leftrightarroweq x$, mean either $y = x$ or $y \leftrightarrow_\kappa x$. For $x \in X$, let

$$N(X, x, \kappa) = \{\, y \in X \mid x \leftrightarrow_\kappa y \,\}.$$

When the image $(X, \kappa)$ under discussion is clear, we will use the notations $N(x)$ or $N_\kappa(x)$ as follows.

$$N(x) = \{\, y \in X \mid y \leftrightarroweq_\kappa x \,\} = N(X, x, \kappa) \cup \{x\}.$$

A sequence $P = \{y_i\}_{i=0}^m$ in a digital image $(X, \kappa)$ is a *$\kappa$-path from $a \in X$ to $b \in X$* if $a = y_0$, $b = y_m$, and $y_i \leftrightarroweq_\kappa y_{i+1}$ for $0 \le i < m$.

$X$ is *$\kappa$-connected* [14], or *connected* when $\kappa$ is understood, if for every pair of points $a, b \in X$ there exists a $\kappa$-path in $X$ from $a$ to $b$.

A *(digital) $\kappa$-closed curve* is a path $S = \{s_i\}_{i=0}^m$ such that $s_0 = s_m$, and $0 < |i - j| < m$ implies $s_i \neq s_j$. If, also, $0 \le i < m$ implies

$$N(S, x_i, \kappa) = \{x_{(i-1) \mod n}, x_{(i+1) \mod m}\}$$

then $S$ is a *(digital) $\kappa$-simple closed curve*.

*2.2. Digitally continuous functions*

Digital continuity is defined to preserve connectedness, as at Definition 2.1 below. By using adjacency as our standard of "closeness," we get Theorem 2.2 below.

**Definition 2.1.** [2] (generalizing a definition of [14]) *Let $(X, \kappa)$ and $(Y, \lambda)$ be digital images. A function $f : X \to Y$ is* $(\kappa, \lambda)$-continuous *if for every $\kappa$-connected $A \subset X$ we have that $f(A)$ is a $\lambda$-connected subset of $Y$.*

If either of $X$ or $Y$ is a subset of the other, we use the abbreviation $\kappa$-continuous for $(\kappa, \kappa)$-continuous.

When the adjacency relations are understood, we will simply say that $f$ is *continuous*. Continuity can be expressed in terms of adjacency of points:

**Theorem 2.2.** [2, 14] *A function $f : X \to Y$ is continuous if and only if $x \leftrightarrow x'$ in $X$ implies $f(x) \leftrightarroweq f(x')$.*

Han's papers generally use the equivalent formulation that $f$ is continuous if and only for every $x \in X$, $f(N_\kappa(x)) \subset N_\lambda(f(x))$.

See also [5, 6], where similar notions are referred to as *immersions*, *gradually varied operators*, and *gradually varied mappings*.

A digital *isomorphism* (called *homeomorphism* in [1]) is a $(\kappa, \lambda)$-continuous surjection $f : X \to Y$ such that $f^{-1} : Y \to X$ is $(\lambda, \kappa)$-continuous.

The literature uses *path* polymorphically: a $(c_1, \kappa)$-continuous function $f : [0, m]_{\mathbb{Z}} \to X$ is a $\kappa$-path if $f([0, m]_{\mathbb{Z}})$ is a $\kappa$-path as described above from $f(0)$ to $f(m)$.

## 3. Han's variants on local isomorphisms

The definition [8] of a digital covering map was simplified to the following.

**Definition 3.1.** [4] *Let $p : (E, \kappa) \to (B, \lambda)$ be a continuous surjection of digital images. The map $p$ is a $(\kappa, \lambda)$* covering map *if and only if*

- *for every $b \in B$, there is an index set $M$ such that*
$$p^{-1}(N_\lambda(b)) = \bigcup_{i \in M} N_\kappa(e_i), \text{ where } e_i \in p^{-1}(b);$$
- *if $i, j \in M$, $i \neq j$, then $N_\kappa(e_i) \cap N_\kappa(e_j) = \emptyset$; and*
- *$p|_{N_\kappa(e_i)} : N_\kappa(e_i) \to N_\lambda(b)$ is a $(\kappa, \lambda)$-isomorphism.*

We find the following definition in Han's paper [10] (not in [7] despite the claims to the contrary in [10, 11]).

**Definition 3.2.** *A digitally continuous map $h : (X, \kappa) \to (Y, \lambda)$ is a* pseudo-local (PL) isomorphism *if for every $x \in X$, $h(N_\kappa(x)) \subset Y$ is $\lambda$-isomorphic to $N_\lambda(h(x)) \subset Y$.*

In his paper [7], Han gives the following.

**Definition 3.3.** *A digitally continuous map $h : (X, \kappa) \to (Y, \lambda)$ is a* local homeomorphism *[in more recent terminology, a* local isomorphism*] if for all $x \in X$, $h|_{N_\kappa(x)}$ is a $(\kappa, \lambda)$-homeomorphism [$(\kappa, \lambda)$-isomorphism] onto $N_\lambda(h(x))$.*

We have the following.

**Proposition 3.4.** *Let $h : (X, \kappa) \to (Y, \lambda)$ be a digitally continuous map. If $h$ is a local isomorphism then $h$ is a PL isomorphism.*

*Proof.* Elementary and left to the reader. □

**Theorem 3.5.** ([13], correcting an error of [7]) *Let $f : (X, \kappa) \to (Y, \lambda)$ be a continuous surjection. Then $f$ is a digital covering map if and only if $f$ is a local isomorphism.*

We will also discuss the following notion.

**Definition 3.6.** [9] *A function $h : (X, \kappa) \to (Y, \lambda)$ is a* weakly local (WL) isomorphism *if for all $x \in X$, $h|_{N(x,1)}$ is an isomorphism onto $h(N(x, 1))$.*

## 4. Theorem 3.15(3) of [11]

Part (3) of Theorem 3.15 of [11] states that

> Neither of a PL-$(k_0, k_1)$-isomorphism and a WL-$(k_0, k_1)$-isomorphism implies the other.

The assertion is correct, but Han's argument for the existence of $(X, k_0)$, $(Y, k_1)$, and a WL-$(k_0, k_1)$-isomorphism $h : X \to Y$ that is not a PL-$(k_0, k_1)$-isomorphism, is not as clear as it could be. In the following, we clarify Han's argument.

In his example, Han makes use of an unstated assumption, namely that $(Y, k_1)$ is connected. He also assumes $k_0 = k_1 = \kappa$, that

$$X \subset Y \text{ but } X \neq Y \tag{1}$$

and that $h$ is the inclusion map, trivially a WL-$(\kappa, \kappa)$-isomorphism.

Note that since $(Y, \kappa)$ is connected, (1) implies there is a $\kappa$-path $\{y, y'\} \subset Y$ such that $y \in X$, $y' \in Y \setminus X$. Therefore,

$$h(N(X, y, \kappa)) = N(X, y, \kappa) \text{ is a proper subset of } N(X, y, \kappa) \cup \{y'\} \subset N(Y, h(y), \kappa).$$

Hence $h$ is not a PL-$(\kappa, \kappa)$-isomorphism.

## 5. Theorem 3.20 of [11]

Let $(X, \kappa)$ and $(Y, \lambda)$ be digital simple closed curves of $\ell_1$ and $\ell_2$ points, respectively. Theorem 3.20 of [11] states that $(X, \kappa)$ embeds into $(Y, \lambda)$ if and only if $\ell_1 = \ell_2$. Since a connected nonempty subset of $(Y, \lambda)$ is either $(Y, \lambda)$ itself or is isomorphic to a digital interval - which is not even of the same digital homotopy type as $(X, \kappa)$ - Han's assertion is an easy consequence of the much older Theorem 5.1 of [3], which states that $(X, \kappa)$ and $(Y, \lambda)$ have the same digital homotopy type if and only if $\ell_1 = \ell_2$.

## 6. Han's pseudo-covering maps in [11]

Han defines a digital pseudo-covering as follows.

**Definition 6.1.** [9] *Let $p : (E, \kappa) \to (B, \lambda)$ be a surjection such that for every $b \in B$,*

1. *there is an index set $M$ such that $p^{-1}(N_\kappa(b, 1)) = \bigcup_{i \in M} N_\kappa(e_i, 1)$, where $e_i \in p^{-1}(b)$;*
2. *if $i, j \in M$ and $i \neq j$, then $N_\kappa(e_i, 1) \cap N_\kappa(e_j, 1) = \emptyset$; and*
3. *$p|_{N_\kappa(e_i,1)} : N_\kappa(e_i, 1) \to N_\lambda(b, 1)$ is a WL-isomorphism for all $i \in M$.*

*Then $p$ is a* pseudo-covering *map.*

However, A. Pakdaman shows in [12] that Han's definition does not effectively give us a new object of study. In particular, Pakdaman shows the following.

**Theorem 6.2.** *A digital pseudo-covering map as defined in Definition 6.1 is in fact a digital covering map.*

**Definition 6.3.** [8] *Let $p : (E, \kappa) \to (B, \lambda)$ be $(\kappa, \lambda)$-continuous. Let $f : [0, m]_\mathbb{Z} \to B$ be $(c_1, \lambda)$-continuous. A $(c_1, \kappa)$-continuous function $\tilde{f} : [0, m]_\mathbb{Z} \to E$ such that $p \circ \tilde{f} = f$ is a* (digital) path lifting of $f$. *If for every $b_0 \in B$, every $e_0 \in p^{-1}(b_0)$, and every path $f$ such that $f(0) = b_0$,*

*there is a unique lifting $\tilde{f}$ such that $\tilde{f}(0) = e_0$,*

*then $p$ has the* unique path lifting property.

**Theorem 6.4.** [8] *A digital covering map has the unique path lifting property.*

Next, we show that several of the variants of covering maps that we have discussed are equivalent.

**Theorem 6.5.** *Let $p : (X, \kappa) \to (Y, \lambda)$ be a continuous surjection. Then the following are equivalent.*

1. *$p$ is a digital covering map.*
2. *$p$ is a local isomorphism.*
3. *$p$ is a pseudo-covering in the sense of Definition 6.1.*
4. *$p$ is a WL-isomorphism with the unique path lifting property.*

*Proof.* That 1) and 2) are equivalent is stated in Theorem 3.5.

That 1) implies 3) follows from Definitions 3.1 and 6.1.

That 3) implies 1) is stated in Theorem 6.2.

It follows from Theorem 6.4 that 1) implies 4).

To show 4) implies 2): suppose $p$ is a WL-isomorphism with the unique path lifting property. Let $x \in X$, $p(x) = y \in Y$, $y' \in N_\lambda(y) \setminus \{y\}$. Then $\{y, y'\}$ is a path in $(Y, \lambda)$, hence lifts to a unique path $\{x, x'\}$ in $(X, \kappa)$ with $p(x) = y$, $p(x') = y'$. Thus $N_\lambda(y) \subset p(N_\kappa(x))$. Since continuity implies $p(N_\kappa(x)) \subset N_\lambda(y)$, we have $p(N_\kappa(x)) = N_\lambda(y)$. Since $p$ is a WL-isomorphism, we have that $p$ is a local isomorphism. □

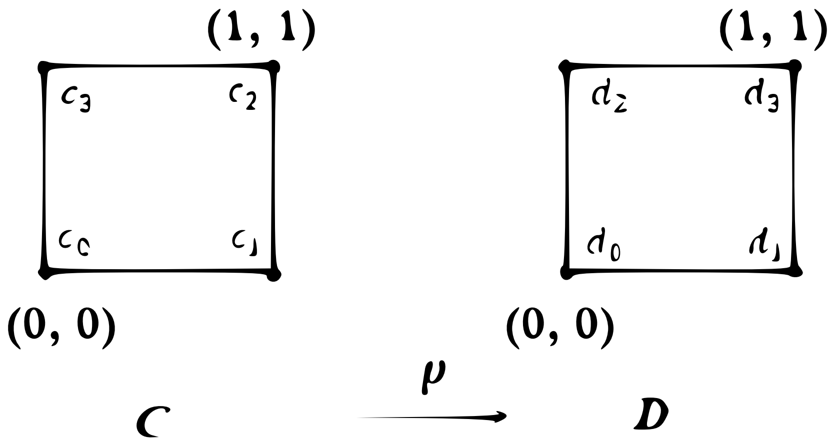


Figure 1: The function $p(c_i) = d_i$ of Han's Example 4.3(4) of [9]; discussed here in Remark 6.6

**Remark 6.6.** *Han's Example 4.3(4) of [9] considers (please note here "$c_1$" is the label of a point, so we will avoid using this notation for 4-adjacency)* $C = \{c_i\}_{i=0}^3$, $D = \{d_i\}_{i=0}^3$, *where*

$$(0,0) = c_0 = d_0, \quad (1,0) = c_1 = d_1, \quad (1,1) = c_2 = d_3, \quad (0,1) = c_3 = d_2,$$

*See Figure 1. Han's claim, that p is not a pseudo-4-covering (a pseudo-covering when A and B both use 4-adjacency), is correct, but this example should not have been considered since p is not 4-continuous:*

$$c_0 \leftrightarrow_4 c_3 \quad \textit{but} \quad p(c_0) = d_0 \not\leftrightarrow_4 d_3 = p(c_3).$$

*Since p is (4,8)-continuous, perhaps Han intended to show that p is not a (4,8)-pseudocovering as defined at Definition 6.1. This can be done by observing that*

$$\#N_4(c_0) = 3 \neq 4 = \#N_8(d_0) = \#N_8(p(c_0)).$$

*Therefore, p is not a (4,8)-local isomorphism, so by Theorem 6.5 is not a (4,8)-pseudocovering as defined at Definition 6.1.*

In the first paragraph of page 5104 of [11], Han attributes the definition of a digital pointed continuous function to his paper [8]. The definition should be attributed to the earlier paper [2].

Pakdaman modifies Han's Definition 6.1 as follows.

**Definition 6.7.** [12] *Let* $p : (E,\kappa) \to (B,\lambda)$ *be a surjection of digital images. Suppose for all* $b \in B$ *we have the following.*

1. *for some index set M,* $\bigcup_{i \in M} N_\kappa(e_i, 1) \subset p^{-1}(N_\lambda(b,1))$ *where* $e_i \in p^{-1}(b)$;
2. *if* $i, j \in M$ *and* $i \neq j$ *then* $N_\kappa(e_i, 1) \cap N_\kappa(e_i, 1) = \emptyset$*; and*
3. *for all* $i \in M$, $p|N_\kappa(e_i,1) : N_\kappa(e_i,1) \to p(N_\kappa(e_i,1))$ *is a* $(\kappa,\lambda)$*-isomorphism.*

*Then p is a* $(\kappa,\lambda)$-pseudocovering map.

Pakdaman proceeds to compare unique path lifting results for pseudocovering maps based on Definition 6.7 with those asserted by Han in [11] based on Definition 6.1. He showed that Definition 6.7 gives something not equivalent to a covering map, since such a pseudocovering need not have the unique path lifting property.

## 7. Further remarks

We have discussed various flaws in Han's paper [11]. We have shown that several variants of digital covering maps that were presented in [11] are in fact equivalent.

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