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Integrated square error of nonparametric estimators of regression function

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Abstract. In the present paper, we consider the nonparametric regression and study the integrated square error for nonparametric estimate of the unknown function. For the case that the errors of the regression model are martingale differences, the asymptotic normality and consistency of the integrated square error are established. These results improve the works in Ioannides [5].

1. Introduction

Consider the nonparametric regression

$$Y_{n,i} = f(x_{n,i}) + \varepsilon_{n,i}, \quad i = 1, \cdots, n, \tag{1.1}$$

where $x_{n,1}, \cdots, x_{n,n}$ are some fixed points, $\varepsilon_{n,1}, \cdots, \varepsilon_{n,n}$ are observational errors with mean 0 and f is an unknown function to be estimated. We consider the general linear estimate of the form

$$\hat{f}_n(x) = \sum_{i=1}^{n} W_{n,i}(x) Y_{n,i}, \quad x \in [0, 1], \tag{1.2}$$

where the weight functions $W_{n,i}(x) = W_{n,i}(x, \tilde{x}_n)$, $i = 1, \cdots, n$, depend on the fixed design points $\tilde{x}_n = (x_{n,1}, \cdots, x_{n,n})$.

Hall [2, 3] studied a very general version of weighted integrated square error (ISE) of nonparametric estimators $\hat{f}_n$,

$$I_n := \mathrm{ISE} = \int_0^1 (\hat{f}_n(x) - f(x))^2 W(x) dx,$$

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where $W(x)$ is a weight function with $W(x) > 0$ for all $x \in [0,1]$. When $\hat{f}_n$ is a kernel estimator of the unknown function f, Hall [3] proved the laws of large numbers and central limit theorems of I_n. By using the martingale theory and the central limit theorem of the degenerate U-statistics with variable kernels, Hall [2] derived the central limit theorems for ISE of multivariate nonparametric density estimators. Nadaraya [7] proved the asymptotic normality of the ISE in the random design case, which is used to construct a test for testing the unknown regression function f. When the estimator $\hat{f}_n$ has the general linear form (1.2) and the errors $\varepsilon_{n,1}, \cdots, \varepsilon_{n,n}$ are independent identically distributed random variables, Ioannides [5] obtained the central limit theorem and law of large numbers for I_n. Wu et al. [9] studied the asymptotic properties for ISE of the general linear estimator in nonparametric regression assuming that the error process is a moving average process.

In the present paper, along with the works in Ioannides [5], we continue to study the asymptotic properties of I_n under the case that the errors are martingale differences. These results improve the works in Ioannides [5]. In the next section, the main results, the asymptotic normality and consistency for I_n, are stated. The proofs of the main results will be given in Section 3. Throughout this paper, the symbol C denotes a positive constant which is not necessarily the same one in each appearance. In order to simplify the notation, write x_i, ε_i, Y_i, W_i, h and $k_{i,j}$, instead of $x_{n,i}$, $\varepsilon_{n,i}$, $Y_{n,i}$, $W_{n,i}$, h_n and $k_{n,i,j}$, respectively. Without loss of generality, we assume that the weight function $W(x) \equiv 1$.

2. Main results

Firstly, we give the following assumptions to be used.

(A) Let $\{(\varepsilon_i, \mathcal{F}_i);\ 1 \le i \le n,\ n \ge 1\}$ be an array of martingale differences with

$$\sup_i \| E\left(|\varepsilon_i|^p | \mathcal{F}_{i-1}\right) \|_\infty < \infty \ \ a.s. \ \text{ for some } \ 2 < p \le 4$$

and

$$E(\varepsilon_i^2 | \mathcal{F}_{i-1}) = \sigma_i^2 \ \ a.s. \ \ \text{and} \ \ \sup_i \sigma_i^2 < \infty,$$

where $\{\sigma_i^2, i \ge 1\}$ is a sequence of finite positive constants.

(B) Assume that $\{h_n, n \ge 1\}$ is a sequence of positive constants, such that $0 < h := h_n \to 0$ and $nh^{3/2} \to \infty$.

(C) The function $f(x)$ satisfies Lipschitz condition with order 1 on $[0,1]$.

(D) The weight functions $\{W_i(x),\ i = 1, \cdots, n\}$ satisfy the following conditions, for all $n \ge 1$:

(i) There exists a constant $C > 0$ such that

$$\sup_{x\in[0,1]} \max_{1\le i\le n} |W_i(x)| \le \frac{C}{nh}.$$

(ii) There exists a constant $C > 0$ such that

$$\sup_{x\in[0,1]} \sum_{i=1}^n |W_i(x)| \le C.$$

(iii) $\sup_{x\in[0,1]} \left| \sum_{i=1}^n W_i(x) - 1 \right| = O(n^{-1/2}h^{-1})$.

(iv) $\sup_{x\in[0,1]} \sum_{i=1}^n |W_i(x)| \cdot |f(x_i) - f(x)| I(|x_i - x| > n^{-1/2}h^{-1}) = O(n^{-1/2}h^{-1})$.

(v) There exists a constant $\sigma^2 > 0$ such that

$$\frac{\sum_{i=2}^{n} \sigma_i^2 \sum_{j=1}^{i-1} k_{i,j}^2 \sigma_j^2}{u_n} \to \sigma^2, \text{ as } n \to \infty,$$

where

$$k_{i,j} = \int_0^1 W_i(x) W_j(x) dx \text{ for } i, j = 1, \cdots, n$$

and

$$u_n = \sum_{i=2}^{n} \sum_{j=1}^{i-1} k_{i,j}^2.$$

(vi) Assume that the sequence (u_n) satisfies

$$\frac{1}{u_n n^2 h^3} = o(1), \quad \text{as } n \to \infty.$$

Remark 2.1. *From the conditions D(i) and (ii), it is easy to check that for every $n \geq 1$, $1 \leq i, j \leq n$,*

$$k_{i,j} \leq \frac{C}{(nh)^2} \quad \text{and} \quad \sum_{i=1}^{n} |k_{i,j}| = \sum_{j=1}^{n} |k_{i,j}| \leq \frac{C}{nh}.$$

Now we start to state our main results as follows.

Theorem 2.2. *Let the assumptions (A)-(D) hold, then*

$$\frac{I_n - EI_n}{\sqrt{u_n}} \xrightarrow{d} N(0, 4\sigma^2).$$

Theorem 2.3. *Under the assumptions in Theorem 2.2, let $\{a_n, n \geq 1\}$ be a sequence of positive constants such that $a_n = o(nh^{3/2})$, then we have*

$$a_n(I_n - EI_n) \xrightarrow{\mathbb{P}} 0.$$

Theorem 2.4. *Under the assumptions in Theorem 2.2, if*

$$\sum_{n \geq 1} \left(\frac{1}{(n^2 h^3)^{p/2}} \vee \frac{1}{(nh)^{p-1}} \right) < \infty,$$

then we have

$$I_n - EI_n \xrightarrow{a.s.} 0.$$

Remark 2.5. *In [5], Ioannides studied the asymptotic normality and consistency for I_n, when the errors $\varepsilon_{n,1}, \cdots, \varepsilon_{n,n}$ are independent identically distributed random variables with $\sup_i E|\varepsilon_i|^4 < \infty$. Hence our results extend the works in Ioannides [5] from independent case to martingale difference case and weaken the moment condition.*

Remark 2.6. *The condition D(vi) is weaker than the condition $u_n > cn^{-2}h^{-4}$ for some constant $c > 0$, which was assumed in Ioannides [5].*

Remark 2.7. *Theorem 2.3 gives the convergence rate of $I_n - EI_n$.*

3. Proofs of main results

In order to obtain the main results, we examine the following quantity,

$$
\begin{aligned}
I_n - EI_n &= \int_0^1 (\hat{f}_n(x) - f(x))^2 W(x)dx - \int_0^1 E(\hat{f}_n(x) - f(x))^2 dx \\
&= \int_0^1 (\hat{f}_n(x) - E\hat{f}_n(x) + E\hat{f}_n(x) - f(x))^2 dx \\
&\quad - \int_0^1 E(\hat{f}_n(x) - E\hat{f}_n(x))^2 dx - \int_0^1 (E\hat{f}_n(x) - f(x))^2 dx \\
&= \int_0^1 (\hat{f}_n(x) - E\hat{f}_n(x))^2 dx - \int_0^1 E(\hat{f}_n(x) - E\hat{f}_n(x))^2 dx \\
&\quad + 2\int_0^1 (\hat{f}_n(x) - E\hat{f}_n(x))(E\hat{f}_n(x) - f(x))dx \\
&= \int_0^1 \left(\sum_{i=1}^n W_i(x)\varepsilon_i\right)^2 dx - \int_0^1 E\left(\sum_{i=1}^n W_i(x)\varepsilon_i\right)^2 dx \\
&\quad + 2\int_0^1 \left(\sum_{i=1}^n W_i(x)\varepsilon_i\right)(E\hat{f}_n(x) - f(x))dx \\
&= \sum_{i=1}^n \left(\int_0^1 W_i^2(x)dx\right)\varepsilon_i^2 + 2\sum_{1\le j<i\le n}\left(\int_0^1 W_i(x)W_j(x)dx\right)\varepsilon_i\varepsilon_j \\
&\quad - \sum_{i=1}^n \left(\int_0^1 W_i^2(x)dx\right)(E\varepsilon_i^2) - 2\sum_{1\le j<i\le n}\left(\int_0^1 W_i(x)W_j(x)dx\right)(E\varepsilon_i\varepsilon_j) \\
&\quad + 2\sum_{i=1}^n \left(\int_0^1 W_i(x)(E\hat{f}_n(x) - f(x))dx\right)\varepsilon_i \\
&= \sum_{i=1}^n \left(\int_0^1 W_i^2(x)dx\right)(\varepsilon_i^2 - E\varepsilon_i^2) + 2\sum_{1\le j<i\le n}\left(\int_0^1 W_i(x)W_j(x)dx\right)\varepsilon_i\varepsilon_j \\
&\quad + 2\sum_{i=1}^n \left(\int_0^1 W_i(x)(E\hat{f}_n(x) - f(x))dx\right)\varepsilon_i \\
&= \sum_{i=1}^n k_{i,i}\left(\varepsilon_i^2 - E(\varepsilon_i^2|\mathcal{F}_{i-1}) + E(\varepsilon_i^2|\mathcal{F}_{i-1}) - E\varepsilon_i^2\right) + 2\sum_{i=2}^n\sum_{j=1}^{i-1} k_{i,j}\varepsilon_i\varepsilon_j + 2\sum_{i=1}^n d_i\varepsilon_i \\
&= \sum_{i=1}^n k_{i,i}\left(\varepsilon_i^2 - E(\varepsilon_i^2|\mathcal{F}_{i-1})\right) + 2\sum_{i=2}^n\sum_{j=1}^{i-1} k_{i,j}\varepsilon_i\varepsilon_j + 2\sum_{i=1}^n d_i\varepsilon_i \\
&\quad + \sum_{i=1}^n k_{i,i}\left(E(\varepsilon_i^2|\mathcal{F}_{i-1}) - E\varepsilon_i^2\right) \\
&=: I_{1n} + 2I_{2n} + 2I_{3n} + I_{4n},
\end{aligned}
$$

where

$$I_{1n} = \sum_{i=1}^{n} k_{i,i}\left(\varepsilon_i^2 - E(\varepsilon_i^2|\mathcal{F}_{i-1})\right), \quad I_{2n} = \sum_{i=2}^{n}\sum_{j=1}^{i-1} k_{i,j}\varepsilon_i\varepsilon_j,$$
$$I_{3n} = \sum_{i=1}^{n} d_i\varepsilon_i, \quad I_{4n} = \sum_{i=1}^{n} k_{i,i}\left(E(\varepsilon_i^2|\mathcal{F}_{i-1}) - E\varepsilon_i^2\right) \tag{3.1}$$

and

$$k_{i,j} = \int_0^1 W_i(x)W_j(x)dx, \text{ for } i,j = 1,\cdots,n,$$
$$d_i = \int_0^1 W_i(x)(E\hat{f}_n(x) - f(x))dx, \text{ for } i = 1,\cdots,n. \tag{3.2}$$

From the condition (A), we have

$$I_{4n} = 0 \ \ a.s. \tag{3.3}$$

Now, we need to recall the following central limit theorem of martingale to prove the asymptotic normality for $I_n - EI_n$.

Proposition 3.1. *[4, Corollary 3.1] Let $\{S_{i,n}, \mathcal{F}_{i,n}; 1 \le i \le k_n, n \ge 1\}$ be a zero-mean, square integrable martingale array with difference $\{X_{i,n}; 1 \le i \le k_n, n \ge 1\}$. Supposed that the following conditions hold:*

(1) *The σ-fields are nested; that is, $\mathcal{F}_{i,n} \subseteq \mathcal{F}_{i+1,n}$ for $1 \le i \le k_n,\ n \ge 1$.*

(2) *For all $\varepsilon > 0$,*

$$\sum_{i=1}^{k_n} E\left(X_{i,n}^2 I(|X_{i,n}| > \varepsilon)|\mathcal{F}_{i-1,n}\right) \xrightarrow{\mathbb{P}} 0$$

(3)

$$\sum_{i=1}^{k_n} E\left(X_{i,n}^2|\mathcal{F}_{i-1,n}\right) \xrightarrow{\mathbb{P}} 1.$$

Then we have

$$S_n = \sum_{i=1}^{k_n} X_{i,n} \xrightarrow{d} N(0,1).$$

Lemma 3.2. *[6] Let $\{(X_i, \mathcal{F}_i); 1 \le i \le n\}$ be a finite sequence of a martingale difference and assume that there exists a positive constant M such that for all $i \ge 1$, $E|X_i|^p < M$ for some $1 < p \le 2$. Let $x > 0$, then we have*

$$\mathbb{P}\left(\max_{1\le i\le n}|S_i| > nx\right) \le \frac{1}{n^p x^p}E|S_n|^p \le \frac{b_p^p}{x^p n^p}\sum_{i=1}^{n} E|X_i|^p \le \frac{M}{x^p} b_p^p n^{1-p},$$

where $S_n = \sum_{i=1}^n X_i$, $b_p = 18pq^{\frac{1}{2}}$ and q is such that $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 3.3. *Let $\{(X_i, \mathcal{F}_i); 1 \le i \le n\}$ be a finite sequence of a martingale difference and assume that for all $1 \le i \le n$, $E|X_i|^p < \infty$ for some $p \ge 2$. Let $x > 0$, then we have*

$$E\left|\sum_{i=1}^{n} X_i\right|^p \le b_p^p\left(\sum_{i=1}^{n}(E|X_i|^p)^{2/p}\right)^{p/2},$$

where $b_p = 18pq^{\frac{1}{2}}$ and q is such that $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By the Burkholder's inequality (cf. Hall and Heyde [4]), we have

$$E\left|\sum_{i=1}^{n} X_i\right|^p \leq (18pq^{1/2})^p E\left|\sum_{i=1}^{n} X_i^2\right|^{p/2}.$$

In addition, by the Minkowski's inequality, we get

$$E\left|\sum_{i=1}^{n} X_i^2\right|^{p/2} \leq \left(\sum_{i=1}^{n} (E|X_i|^p)^{2/p}\right)^{p/2}.$$

Hence the desired results can be obtained. □

Lemma 3.4. *Let the assumptions (A), (B), (D)(i), (D)(ii), (D)(v) and (D)(vi) hold, then we have*

$$\frac{1}{\sqrt{u_n}} I_{2n} \xrightarrow{d} N(0, \sigma^2).$$

Proof. For each $2 \leq i \leq n$, define

$$Z_i = \varepsilon_i \sum_{j=1}^{i-1} k_{i,j}\varepsilon_j, \tag{3.4}$$

then it is easy to see that Z_i is $\mathcal{F}_i$-measurable for each i and $E(Z_i|\mathcal{F}_{i-1}) = 0$. Hence $\{(Z_i, \mathcal{F}_i); 2 \leq i \leq n,\ n \geq 1\}$ is an array of martingale difference and

$$\frac{1}{\sqrt{u_n}} I_{2n} = \frac{1}{\sqrt{u_n}} \sum_{i=2}^{n}\sum_{j=1}^{i-1} k_{i,j}\varepsilon_i\varepsilon_j = \frac{1}{\sqrt{u_n}} \sum_{i=2}^{n} Z_i.$$

From Proposition 3.1, it is enough to check that for any $\epsilon > 0$,

$$\frac{1}{u_n}\sum_{i=2}^{n} E\left(Z_i^2 I(|Z_i| > \epsilon\sqrt{u_n})|\mathcal{F}_{i-1}\right) \xrightarrow{\mathbb{P}} 0 \tag{3.5}$$

and

$$\frac{1}{u_n}\sum_{i=2}^{n} E\left(Z_i^2|\mathcal{F}_{i-1}\right) \xrightarrow{\mathbb{P}} \sigma^2. \tag{3.6}$$

For $p > 2$, from Lemma 3.3, we have

$$\begin{aligned} E|Z_i|^p =& E\left(\left|\sum_{j=1}^{i-1} k_{i,j}\varepsilon_j\right|^p E\left(|\varepsilon_i|^p|\mathcal{F}_{i-1}\right)\right) \\ \leq& CE\left|\sum_{j=1}^{i-1} k_{i,j}\varepsilon_j\right|^p \leq \left(\sum_{j=1}^{i-1} k_{i,j}^2 (E|\varepsilon_j|^p)^{2/p}\right)^{p/2} \leq C\left(\sum_{j=1}^{i-1} k_{i,j}^2\right)^{p/2}. \end{aligned} \tag{3.7}$$

So, by using Remark 2.1, we get

$$\begin{aligned} \frac{1}{u_n^{p/2}}\sum_{i=2}^{n} E|Z_i|^p \leq& C\frac{1}{u_n^{p/2}}\sum_{i=2}^{n}\left(\sum_{j=1}^{i-1} k_{i,j}^2\right)^{p/2} \\ \leq& C\frac{n/(nh)^{3p/2}}{1/(n^2h^3)^{p/2}} \leq \frac{C}{n^{\frac{p}{2}-1}} \to 0, \end{aligned}$$

which implies (3.5). In order to obtain (3.6), since

$$\begin{aligned}
&\sum_{i=2}^{n} E\left(Z_i^2|\mathcal{F}_{i-1}\right)\\
&= \sum_{i=2}^{n} E\left(\left(\varepsilon_i \sum_{j=1}^{i-1} k_{i,j}\varepsilon_j\right)^2 \middle|\mathcal{F}_{i-1}\right)\\
&= \sum_{i=2}^{n} E(\varepsilon_i^2|\mathcal{F}_{i-1}) \sum_{j=1}^{i-1} k_{i,j}^2\varepsilon_j^2 + 2\sum_{i=2}^{n} E(\varepsilon_i^2|\mathcal{F}_{i-1}) \sum_{j=1}^{i-2}\sum_{l=j+1}^{i-1} k_{i,j}k_{i,l}\varepsilon_j\varepsilon_l\\
&= \sum_{i=2}^{n} \sigma_i^2 \sum_{j=1}^{i-1} k_{i,j}^2\varepsilon_j^2 + 2\sum_{i=2}^{n} \sigma_i^2 \sum_{j=1}^{i-2}\sum_{l=j+1}^{i-1} k_{i,j}k_{i,l}\varepsilon_j\varepsilon_l\\
&:= U_{n1} + U_{n2} \qquad a.s.,
\end{aligned}$$

it is enough to show

$$\frac{1}{u_n}U_{n1} \xrightarrow{\mathbb{P}} \sigma^2 \text{ and } \frac{1}{u_n}U_{n2} \xrightarrow{\mathbb{P}} 0.$$

For any $\epsilon > 0$, from Lemma 3.2, we have

$$\begin{aligned}
\mathbb{P}\left(\frac{1}{u_n}|U_{n1} - EU_{n1}| > \epsilon\right) &\le \frac{C}{u_n^{p/2}} E\left|\sum_{i=2}^{n} \sigma_i^2 \sum_{j=1}^{i-1} k_{i,j}^2\varepsilon_j^2 - \sum_{i=2}^{n} \sigma_i^2 \sum_{j=1}^{i-1} k_{i,j}^2 E\varepsilon_j^2\right|^{\frac{p}{2}}\\
&= \frac{C}{u_n^{p/2}} E\left|\sum_{i=2}^{n} \sigma_i^2 \sum_{j=1}^{i-1} k_{i,j}^2(\varepsilon_j^2 - \sigma_j^2)\right|^{\frac{p}{2}}\\
&= \frac{C}{u_n^{p/2}} E\left|\sum_{i=2}^{n} \sigma_i^2 \sum_{j=1}^{i-1} k_{i,j}^2\left(\varepsilon_j^2 - E(\varepsilon_j^2|\mathcal{F}_{j-1})\right)\right|^{\frac{p}{2}}\\
&= \frac{C}{u_n^{p/2}} E\left|\sum_{j=1}^{n-1}\left(\sum_{i=j+1}^{n} \sigma_i^2 k_{i,j}^2\right)\left(\varepsilon_j^2 - E(\varepsilon_j^2|\mathcal{F}_{j-1})\right)\right|^{\frac{p}{2}}\\
&\le \frac{C}{u_n^{p/2}} \sum_{j=1}^{n-1}\left(\sum_{i=j+1}^{n} \sigma_i^2 k_{i,j}^2\right)^{\frac{p}{2}} E\left|\varepsilon_j^2 - E(\varepsilon_j^2|\mathcal{F}_{j-1})\right|^{p/2}\\
&\le \frac{C}{u_n^{p/2}} \sum_{j=1}^{n-1}\left(\sum_{i=j+1}^{n} k_{i,j}^2\right)^{\frac{p}{2}}\\
&\le C\frac{n/(nh)^{3p/2}}{1/(n^2h^3)^{p/2}} \le \frac{C}{n^{\frac{p}{2}-1}} \to 0,
\end{aligned} \tag{3.8}$$

which, together with the condition D(v), implies that

$$\frac{1}{u_n}U_{n1} \xrightarrow{\mathbb{P}} \sigma^2.$$

Now for $1 < l < i < n$, define

$$T_{i,l} = \varepsilon_l \sum_{j=1}^{l-1} k_{i,j}\varepsilon_j,$$

then, for every $1 < i < n$, $\{(T_{i,l}, \mathcal{F}_i); 2 \le l \le n-1,\ n \ge 1\}$ is an array of martingale difference, and from Lemma 3.3, we have

$$E|T_{i,l}|^p = E\left|\varepsilon_l \sum_{j=1}^{l-1} k_{i,j}\varepsilon_j\right|^p = E\left(\left|\sum_{j=1}^{l-1} k_{i,j}\varepsilon_j\right|^p E\left(|\varepsilon_l^p| \middle| \mathcal{F}_{l-1}\right)\right)$$
$$\le CE\left|\sum_{j=1}^{l-1} k_{i,j}\varepsilon_j\right|^p \le C\left(\sum_{j=1}^{l-1} k_{i,j}^2 \left(E\left|\varepsilon_j\right|^p\right)^{2/p}\right)^{p/2} \le C\left(\sum_{j=1}^{l-1} k_{i,j}^2\right)^{p/2},$$

which, by the Minkowski inequality and Lemma 3.3, implies

$$E\left|\sum_{i=2}^{n} \sigma_i^2 \sum_{j=1}^{i-2}\sum_{l=j+1}^{i-1} k_{i,j}k_{i,l}\varepsilon_j\varepsilon_l\right|^p = E\left|\sum_{i=2}^{n} \sigma_i^2 \sum_{l=2}^{i-1} \varepsilon_l k_{i,l} \sum_{j=1}^{l-1} k_{i,j}\varepsilon_j\right|^p$$
$$= E\left|\sum_{i=2}^{n} \sigma_i^2 \sum_{l=2}^{i-1} k_{i,l}T_{i,l}\right|^p \le \left(\sum_{i=2}^{n} \sigma_i^2 \left(E\left|\sum_{l=2}^{i-1} k_{i,l}T_{i,l}\right|^p\right)^{1/p}\right)^p$$
$$\le C\left(\sum_{i=2}^{n} \left(\sum_{l=2}^{i-1} \left(E\left|k_{i,l}T_{i,l}\right|^p\right)^{2/p}\right)^{1/2}\right)^p$$
$$\le C\left(\sum_{i=2}^{n} \left(\sum_{l=2}^{i-1} k_{i,l}^2 \sum_{j=1}^{l-1} k_{i,j}^2\right)^{1/2}\right)^p \le C\frac{n^p}{(nh)^{3p}}.$$

Hence we have

$$\frac{1}{u_n^p} E|U_{n2}|^p = o\left(\frac{n^p/(nh)^{3p}}{1/n^{2p}h^{3p}}\right) = o(1),$$

which implies

$$\frac{1}{u_n} U_{n2} \xrightarrow{\mathbb{P}} 0.$$

From the above discussions, Lemma 3.4 can be obtained. □

Lemma 3.5. *Let the assumptions (A), (B) and (D) hold, then*

$$\frac{1}{\sqrt{u_n}} I_{1n} \xrightarrow{\mathbb{P}} 0.$$

If, in addition, the assumption (C) holds, then

$$\frac{1}{\sqrt{u_n}} I_{3n} \xrightarrow{\mathbb{P}} 0.$$

Proof. For any $\epsilon > 0$, by using Lemma 3.2, we have

$$\mathbb{P}\left(\frac{1}{\sqrt{u_n}}|I_{1n}| > \epsilon\right) = \mathbb{P}\left(\left|\sum_{i=1}^{n} k_{i,i}(\varepsilon_i^2 - E(\varepsilon_i^2|\mathcal{F}_{i-1}))\right| > \epsilon\sqrt{u_n}\right)$$
$$\le \frac{C}{u_n^{p/4}} E\left|\sum_{i=1}^{n} k_{i,i}(\varepsilon_i^2 - E(\varepsilon_i^2|\mathcal{F}_{i-1}))\right|^{\frac{p}{2}}$$
$$\le \frac{C}{u_n^{p/4}} \sum_{i=1}^{n} k_{i,i}^{\frac{p}{2}} E\left|\varepsilon_i^2 - E(\varepsilon_i^2|\mathcal{F}_{i-1})\right|^{\frac{p}{2}}$$
$$\le \frac{C}{u_n^{p/4}} \sum_{i=1}^{n} k_{i,i}^{\frac{p}{2}} \le C\frac{1/(nh)^{p-1}}{1/(n^2h^3)^{p/4}} \le C\frac{h^{1-\frac{p}{4}}}{n^{\frac{p}{2}-1}} \to 0.$$

Next we consider the term I_{3n}. From the conditions (C), D(ii), D(iii) and D(iv), we have

$$\begin{aligned}
&\left|\sum_{i=1}^{n} W_i(x)f(x_i) - f(x)\right| \\
\leq &\left|\sum_{i=1}^{n} |W_i(x)||f(x_i) - f(x)|I(|x_i - x| > n^{-1/2}h^{-1})\right| \\
&+ \left|\sum_{i=1}^{n} |W_i(x)||f(x_i) - f(x)|I(|x_i - x| \leq n^{-1/2}h^{-1})\right| \\
&+ f(x)\left|\sum_{i=1}^{n} W_i(x) - 1\right| \\
= &O(n^{-1/2}h^{-1}).
\end{aligned}$$

Hence, for any $\epsilon > 0$, from Lemma 3.3, we have

$$\begin{aligned}
&\mathbb{P}\left(\frac{1}{\sqrt{u_n}}|I_{3n}| > \epsilon\right) = \mathbb{P}\left(\left|\sum_{i=1}^{n} d_i\varepsilon_i\right| > \epsilon\sqrt{u_n}\right) \\
\leq &\frac{C}{u_n^{p/2}} E\left|\sum_{i=1}^{n} \varepsilon_i \int_0^1 W_i(x)(E\hat{f}_n(x) - f(x))dx\right|^p \\
\leq &\frac{C}{u_n^{p/2}}\left(\sum_{i=1}^{n} (E|\varepsilon_i|^p)^{2/p}\left(\int_0^1 W_i(x)(E\hat{f}_n(x) - f(x))dx\right)^2\right)^{p/2} \\
\leq &\frac{C}{u_n^{p/2}}\left(\sum_{i=1}^{n}\left(\int_0^1 W_i(x)(E\hat{f}_n(x) - f(x))dx\right)^2\right)^{p/2} \\
\leq &\frac{C}{u_n^{p/2}}\left(\sum_{i=1}^{n}\left(\int_0^1 W_i^2(x)dx\right)\left(\int_0^1 (E\hat{f}_n(x) - f(x))^2dx\right)\right)^{p/2} \\
= &\frac{C}{u_n^{p/2}}\left(\sum_{i=1}^{n}\left(\int_0^1 W_i^2(x)dx\right)\left(\int_0^1\left(\sum_{i=1}^{n} W_i(x)f(x_i) - f(x)\right)^2 dx\right)\right)^{p/2} \\
\leq &\frac{C}{u_n^{p/2}}\left(\sum_{i=1}^{n}\left(\int_0^1 W_i^2(x)dx\right)\right)^{p/2}\left(\frac{1}{nh^2}\right)^{p/2} \\
\leq &C\frac{1/(n^2h^3)^{p/2}}{u_n^{p/2}} = o(1).
\end{aligned}$$

From the above discussions, the desired results can be obtained. □

Proof. **[Proof of Theorem 2.2]** From the decomposition $I_n - EI_n = I_{1n} + 2I_{2n} + 2I_{3n} + I_{4n}$, Lemma 3.4, Lemma 3.5 and (3.3), Theorem 2.2 can be proved. □

Proof. **[Proof of Theorem 2.3]** Let $\{Z_i, 2 \leq i \leq n\}$ be defined in (3.4). For any $\epsilon > 0$, we have

$$\mathbb{P}(a_n|I_{2n}| > \epsilon) \leq Ca_n^2 E\left(\sum_{i=2}^{n} Z_i\right)^2$$

$$=Ca_n^2\sum_{i=2}^{n}E\left[\left(\sum_{j=1}^{i-1}k_{i,j}\varepsilon_j\right)^2E(\varepsilon_i^2|\mathcal{F}_{i-1})\right]$$
$$\leq Ca_n^2\sum_{i=2}^{n}\sum_{j=1}^{i-1}k_{i,j}^2=O\left(\frac{a_n^2}{(nh)^2}\right)=o(h).$$

From Lemma 3.5, for any $\epsilon>0$, we have

$$\mathbb{P}\left(a_n|I_{1n}|>\epsilon\right)\leq Ca_n^{\frac{p}{2}}\sum_{i=1}^{n}k_{i,i}^{\frac{p}{2}}=o\left(\frac{(nh^{3/2})^{p/2}}{(nh)^{p-1}}\right)=o(1)$$

and

$$\mathbb{P}\left(a_n|I_{3n}|>\epsilon\right)\leq Ca_n^p\left(\sum_{i=1}^{n}\left(\int_0^1W_i^2(x)dx\right)\right)^{p/2}\left(\frac{1}{nh^2}\right)^{p/2}$$
$$=O\left(\frac{a_n^p}{(n^2h^3)^{p/2}}\right)=o(1).$$

From the above discussions, the desired results can be obtained. □

Proof. **[Proof of Theorem 2.4]** Let $\{Z_i,2\leq i\leq n\}$ be defined in (3.4). For any $\epsilon>0$, from Lemma 3.3 and (3.7), we have

$$\mathbb{P}\left(|I_{2n}|>\epsilon\right)\leq CE\left|\sum_{i=2}^{n}Z_i\right|^p\leq C\left(\sum_{i=2}^{n}\left(E\left|Z_i\right|^p\right)^{2/p}\right)^{p/2}$$
$$\leq\left(\sum_{i=2}^{n}\sum_{j=1}^{i-1}k_{i,j}^2\right)^{p/2}\leq\frac{C}{(nh)^p}.$$

From the proof of Theorem 2.3, we have

$$\mathbb{P}\left(|I_{1n}|>\epsilon\right)\leq\frac{C}{(nh)^{p-1}}$$

and

$$\mathbb{P}\left(|I_{3n}|>\epsilon\right)\leq\frac{C}{(n^2h^3)^{p/2}}.$$

□

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