Filomat 37:27 (2023), 9357–9370
https://doi.org/10.2298/FIL2327357J

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Existence of periodic solution for double-phase parabolic problems with strongly nonlinear source

Hamza Jourhmane[a], Abderrazak Kassidi[a], Khalid Hilal[a], M'hamed Elomari[a]

[a]*Laboratory LMACS, Sultan Moulay Slimane University, Beni Mellal, Morocco*

Abstract. The aim of this paper is to study a degenerate double-phase parabolic problem with strongly nonlinear source under Dirichlet boundary conditions, proving the existence of a non-negative periodic weak solution. Our proof is based on the Leray-Schauder topological degree, which poses many problems for this type of equations, but has been overcome by using various techniques or well-known theorems. The system considered is a possible model for problems where the studied entity has different growth coefficients, p and q in our case, in different domains.

1. Introduction

Modelling a natural phenomena requires almost always the use of mathematical tools. It could be about economics [27, 33, 42, 45], epidemiology [25, 44] or many other domains that end up with differential equations as a mathematical approach to understand the problem. Sometimes, the equations have boundary or regularity conditions, but also behaviour or asymptotic conditions, periodicity is a popular example. Often, we are interested in a periodic solution of a periodic problem, which can appear in many domains, we mention as an example microbiology [18], relativistic physics [13] and radiative gas [32]. Many approaches can be used to solve this type of problems, for example [21, 22, 31] use the Leray-Schauder fixed point theorem, the sub- and super-solution method was used by [11, 12], and for more details see [1–3, 5] and the references therein.

In this paper, we prove the existence of a periodic solution for the degenerate evolution (p, q)-Laplacian equation of the form

$$(\mathcal{P})\quad \begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u\right) + h(x,t)u^m & \text{in } Q_\tau,\\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\tau),\\ u(x,t+\tau) = u(x,t), & (x,t) \in \Omega \times \mathbb{R}, \end{cases}$$

where $q \geq p \geq 2,\ \tau > 0,\ \Omega$ is a convex domain in $\mathbb{R}^N$ that is bounded and with smooth boundary $\partial\Omega,\ h(x,t)$ is continuous on $\overline{\Omega} \times \mathbb{R}$, periodic in t with period τ and positive in $Q_\tau = \Omega \times (0, \tau)$. We assume that $N > 1$,

2020 *Mathematics Subject Classification*. 35B10,35K55,35K65,47H11
Keywords. Topological degree; Periodic solution; Dirichlet conditions; Generalized Sobolev spaces
Received: 02 April 2023; Accepted: 21 May 2023
Communicated by Maria Alessandra Ragusa
Email addresses: hamzajourh@gmail.com (Hamza Jourhmane), abderrazakassidi@gmail.com (Abderrazak Kassidi), khalid.hilal.usms@gmail.com (Khalid Hilal), m.elomari@usms.ma (M'hamed Elomari)

$q-1 < p-1+\frac{p}{N}$ and we take m such that $q-1 < m < p-1+\frac{p}{N}$, and establish the existence of a non-negative non-trivial periodic solution.

This kind of problems, with the double phase Laplacian operator, were initially studied by Zhikov who introduced this class of operators, when he was describing a model for strongly anisotropic materials and was confronted with the functional

$$u \mapsto \int (|\nabla u|^p + |\nabla u|^q)dx$$

we refer the reader to [46, 47] and the references therein. In the last few decades, many authors studied functionals of this form concerning the regularity of local minimizers. We cite the works of Baroni-Colombo-Mingione [6–8], Baroni-Kuusi-Mingione [9], Colombo-Mingione [14, 15], Marcellini [34, 35], Ok [39, 40] and Ragusa-Tachikawa [41].

Several authors, for example [4, 11, 12] are interested in semi-linear equations of the type

$$\frac{\partial u}{\partial t} = \Delta u + f(x,t,u), \tag{1}$$

where f is periodic with respect to the time variable. In [12], Charkaoui et al. have studied the following special case

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(x,t) - G(x,t,\nabla u) & (x,t) \in Q_\tau, \\ u(x,t) = 0 & (x,t) \in \partial\Omega \times (0,\tau), \\ u(x,0) = u(x,\tau) & x \in \Omega, \end{cases}$$

where G is caratheodory and $f \in L^1(Q_\tau)$ nonnegative and periodic. By taking $u = (u_1, \ldots, u_M)$, $f = (f_1, \ldots, f_M)$ and $G = (G_1, \ldots, G_M)$, the same authors found a way to generalize their work in [11].

M. J. Esteban [20] showed that the problem associated to (1) has a nonnegative periodic solution in case $f(x,t,u) = h(x,t)u^m$, $h(x,t)$ being a positive time periodic function, as long as $1 < m < \frac{3N+8}{3N-4}$, and bettered the result in [19] by proving the same result but only if $1 < m < \frac{N}{N-2}$.

Inspired by a biological model where $u(x,t)$ represents the density of a species at the position x and the time t, R.Huang et al. in [29] have studied the case of the degenerate parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u^m = (a - \Phi[u])u & (x,t) \in Q_\tau, \\ u(x,t) = 0 & (x,t) \in \partial\Omega \times (0,\tau), \\ u(x,0) = u(x,\tau) & x \in \Omega, \end{cases}$$

where Δu^m models the manner of which the species studied tend to avoid clutter, m being a real number greater than 1, $\Phi[u] : L^2(\Omega)^+ \to \mathbb{R}^+$ is a bounded continuous functional and $a(x,t)$ is the maximum value reachable by the increasing ratio of the species at the position x and the time t.

Inspired by all the above cited references, we prove the existence of a nontrivial nonnegative periodic solution to problem $(\mathcal{P})$, by the means of the Leray-Schauder topological degree and the scaling or blow-up argument used in [19, 20, 28].

This paper is structered as follows: We start by defining the weak solution and announcing our main theorem in section 2. Sections 3, 4 and 5 will include the proof of the lemmas used to conclude the main result in section 6.

2. Weak solution and principal result

Taking into consideration the degeneracy of the equations studied, the problem $(\mathcal{P})$ does not have a classical solution in general, thus we consider its weak solutions in the following sense

Definition 2.1. *Let $C_\tau(\overline{Q_\tau})$ be the set of all functions in $C(\Omega \times R)$ which are periodic in t with period $\tau > 0$ and $q \in C_\tau(\overline{Q_\tau})$. A function $u \in L^q\left(0,\tau; W_0^{1,q}(\Omega)\right) \cap C_\tau(\overline{Q_\tau})$ is said to be a solution of the problem $(\mathcal{P})$ if u satisfies*

$$\int_{Q_\tau} \left\{ u\frac{\partial \psi}{\partial t} + |\nabla u|^{p-2}\nabla u \nabla \psi + |\nabla u|^{q-2}\nabla u \nabla \psi + h(x,t)u^m \psi \right\} dxdt = 0$$

for any $\psi \in C^1(\overline{Q_\tau})$ such that $\psi(.,0) = \psi(.,\tau)$ and $\psi(x,t) = 0$ if $(x,t) \in \partial\Omega \times (0,\tau)$.

Theorem 2.2. *The problem $(\mathcal{P})$ admits at least one nontrivial nonnegative solution*

$$u \in C\left(0,\tau; W_0^{1,q}(\Omega)\right) \cap C_\tau(\overline{Q_\tau}) \text{ with } \quad \frac{\partial u}{\partial t} \in L^2(Q_\tau),$$

provided that $N > 1$ and $q - 1 < m < p - 1 + \frac{p}{N}$.

In what follows only the proof of Theorem 1 will be given by means of the method of parabolic regularization. Precisely, we consider the regularized equations

$$(\mathcal{P}_\sigma) \qquad \frac{\partial u_\sigma}{\partial t} = \operatorname{div}\left(\left(\left|\nabla u_\sigma\right|^2 + \sigma\right)^{\frac{p-2}{2}} \nabla u_\sigma + \left(\left|\nabla u_\sigma\right|^2 + \sigma\right)^{\frac{q-2}{2}} \nabla u_\sigma\right) + h(x,t)u_\sigma^m$$

with small constant $\sigma > 0$. The desired solution of the problem $(\mathcal{P})$ is going to be nothing but the limit function of solutions of $(\mathcal{P}_\sigma)$.

Proposition 2.3. *Assuming the conditions of Theorem 2.2 hold, the following problem has a nonnegative solution u_σ*

$$\begin{cases} \dfrac{\partial u_\sigma}{\partial t} = \operatorname{div}\left(\left(\left|\nabla u_\sigma\right|^2 + \sigma\right)^{\frac{p-2}{2}} \nabla u_\sigma + \left(\left|\nabla u_\sigma\right|^2 + \sigma\right)^{\frac{q-2}{2}} \nabla u_\sigma\right) + h(x,t)u_\sigma^m, \text{ in } \Omega \times \mathbb{R}, \\ u_\sigma(x,t) = 0, & (t,x) \in \partial\Omega \times (0,T), \\ u_\sigma(x,0) = u_\sigma(x,T), & x \in \Omega, \end{cases}$$

and there exist positive numbers r and R independent of σ such that,

$$r \leq \max_{\overline{Q_\tau}} u_\sigma(x,t) \leq R.$$

To prove this proposition, we apply the topological degree theory. To do that, we will study an equation with one-parameter, which attach the semi-linear operator used to an easier operator, the Laplacian:

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\left(\nu|\nabla u|^2 + \sigma\right)^{\frac{p-2}{2}} \nabla u + \left(\nu|\nabla u|^2 + \sigma\right)^{\frac{q-2}{2}} \nabla u\right) + g(x,t), \tag{2}$$

where $\nu \in [0,1]$ and $g \in C_\tau(\overline{Q_\tau})$. Section 3 will hold the proof that for any $\nu \in [0,1]$ and $g \in C_\tau(\overline{Q_\tau})$, the periodic problem associated to (2) has one and only one solution $u \in C_\tau(\overline{Q_\tau})$ and the map $\mathcal{V} : [0,1] \times C_\tau(\overline{Q_\tau}) \to C_\tau(\overline{Q_\tau})$ defined by $u = \mathcal{V}(\nu, g)$ is compact and so is the map $\mathcal{V}(\nu, \Phi(u))$ with $\Phi(u) = h(x,t)u_+^m$. The small parameter $\nu \in [0,1]$ in the leading term of the equations makes proving the compactness more difficult. We will prove the proposition using the inequality $\deg\left(\mathcal{I} - \mathcal{V}(1,\Phi(\cdot)), B_R(0)\backslash B_r(0), 0\right) \neq 0$, such as $B_\rho(0)$ is the ball of $C_\tau(\overline{Q_\tau})$ with radius ρ and zero as its origin. First, we shall demonstrate in section 4 that there exists a radius $r > 0$ unrelated to σ, such that $\deg\left(\mathcal{I} - \mathcal{V}(1,\Phi(\cdot)), B_r(0), 0\right) = 1$. After that, we will substantiate, in section 5, that $\deg\left(\mathcal{I} - \mathcal{V}(1,\Phi(\cdot)), B_R(0), 0\right) = 0$ for some large real number $R > r$ unrelated to σ. After this, proving the proposition will only come to establishing an upper bound for the solutions. As we mentioned previously, our technique to get an upper bound is the blow-up argument (scaling argument) which was extensively used in [19, 20, 28], and others. To summarize, in section 6, Theorem 2.2 is proved thanks to the proposition.

3. Proprieties of the map $\mathcal{V}$

To simplify, we assume that $h(x,t)$ is Hölder continuous in the subsequent sections. In fact, by a process of approximation, this assumption can be removed.

Lemma 3.1. *For any $\nu \in [0,1]$ and $g \in C_\tau(\overline{Q_\tau})$, the periodic problem related to* (2) *has a unique solution $u \in C\left(0,\tau; W_0^{1,q}(\Omega)\right) \cap C_\tau(\overline{Q_\tau})$, $\partial u/\partial t \in L^2(Q_\tau)$, and u satisfies*

$$\|u\|_\infty = \|\mathcal{V}(\nu,g)\|_\infty \le C\left(\frac{\|g\|_\infty}{\delta}\right)^{\delta+1} \quad \text{for any} \quad 0<\delta<1 \tag{3}$$

$$\left\|\frac{\partial u}{\partial t}\right\|_2 \le C\left\|g\right\|_2 \tag{4}$$

where the constant C depends only upon N, σ, p and q. Here and below, we use $\|g\|_a$ to denote the L^a-norm of a function g.

Proof. In the particular case $\nu = 0$ the reader can check [19, 20]. From now on $\nu \neq 0$ is assumed. Applying the result of [43], the periodic problem associated to (2) has one and only one solution $u \in L^q\left(0,\tau; W_0^{1,q}(\Omega)\right)$ for any $g \in C_\tau(\overline{Q_\tau})$ and $\nu \in (0,1]$. The results of [16], allow us to claim that $u \in C^m(\overline{Q_\tau})$, and $\nabla u \in C^m(\overline{Q_\tau})$. Next, we seek to estimate the uniform norm of the solutions. If we replace the test function in the integral equality satisfied by u, with $|u|^{r_k}u$, we get

$$\frac{d}{dt}\|u(t)\|_{r_k+2}^{r_k+2} + C(\sigma,p,q)\left\|\nabla\left(|u|^{r_k/2}u\right)\right\|_2^2 \le \|g\|_\infty \|u\|_{r_k+2}^{r_k+1},$$

where $r_1 = 1$ and for any $k \ge 1$, $r_k = 2r_{k-1} + 2 = 2^k - 2$, $u(t) = u(\cdot,t)$, and the positive constant $C(\sigma,p,q)$ related only to σ, p and q. Setting $w_k = |u|^{\frac{r_k}{2}}u$, we have

$$\frac{d}{dt}\|w_k\|_2^2 + C(\sigma,p,q)\|\nabla w_k\|_2^2 \le \|g\|_x \|w_k\|_2^{\frac{2(r_k+1)}{(r_k+2)}}.$$

Now we establish the estimate (3), by adopting the Moser iteration technique, see [37].
Lastly, if we just take $\frac{\partial u}{\partial t}$ as a test function, (4) can be derived easily. □

Lemma 3.2. *The functional $\mathcal{V} : [0,1] \times C_\tau(\overline{Q_\tau}) \to C_\tau(\overline{Q_\tau})$ is well defined and compact.*

Proof. Let's start by showing that $u = \mathcal{V}(\nu,g) \in C_\tau(\overline{Q_\tau})$ for all $\nu \in [0,1]$ and $g \in C_\tau(\overline{Q_\tau})$. If $\nu = 0$, by Theorem 10.1 of [30] and since u is time periodic, we obtain

$$|u(x_1,t_1) - u(x_2,t_2)| \le \gamma\left(|x_1-x_2| + |t_1-t_2|^{\frac{1}{2}}\right)^\beta, \tag{5}$$

where γ and β are positive constants that depend upon N, σ, p, q, as well as the upper bound of $\|u\|_\infty$ and, by Lemma 3.1, that of $\|g\|_\infty$ too. Now, if $\nu \neq 0$, then, according to (2), the function $v = \nu^{\frac{1}{2}}u$ satisfies

$$\frac{\partial v}{\partial t} = \operatorname{div}\left(\left(|\nabla v|^2+\sigma\right)^{\frac{p-2}{2}}\nabla v + \left(|\nabla v|^2+\sigma\right)^{\frac{q-2}{2}}\nabla v\right) + \nu^{\frac{1}{2}}g(x,t) \tag{6}$$

Noticing the time periodicity of v, and applying the result of [17] we conclude that v is Hölder continuous in $\overline{Q_\tau}$. Furthermore, if we apply Theorem 10.1 in [30] to v, then go back to u to get a similar inequality than (5), and by Arzelà-Ascoli theorem the image of any bounded set of $[0,1] \times C_\tau(\overline{Q_\tau})$ by the map $\mathcal{V}$ is a compact set of $C_\tau(\overline{Q_\tau})$.

To prove the continuity of $\mathcal{V}$, we take $\nu_k \to \nu, g_k \to g$ as $k \to \infty$ and $u_k = V(\nu_k, g_k)$. By the means of the inequalities (3) and (5) we get the existence of $u \in C_\tau(\overline{Q_\tau})$ such that

$$u_k(x,t) \to u(x,t) \quad \text{uniformly in } Q_\tau \tag{7}$$

u_k could mean its own subsequence if needed. To prove that $u = \mathcal{V}(\nu, g)$, we proceed just like in [48].
It suffices to multiply (2) by u_k, and integrate over Q_τ, to obtain

$$\int_{Q_\tau} \left(\left(\nu_k |\nabla u_k|^2 + \sigma\right)^{\frac{p-2}{2}} + \left(\nu_k |\nabla u_k|^2 + \sigma\right)^{\frac{q-2}{2}} \right) |\nabla u_k|^2 \, dxdt \leq C$$

and hence

$$\int_{Q_\tau} \nu_k^{\frac{p-2}{2}} |\nabla u_k|^p \, dxdt \leq C \text{ and } \int_{Q_\tau} \nu_k^{\frac{q-2}{2}} |\nabla u_k|^q \, dxdt \leq C, \tag{8}$$

$$\int_{Q_\tau} \sigma^{\frac{p-2}{2}} |\nabla u_k|^2 \, dxdt \leq C \text{ and } \int_{Q_\tau} \sigma^{\frac{q-2}{2}} |\nabla u_k|^2 \, dxdt \leq C. \tag{9}$$

Note that, C will represent a constant that can have different values. To simplify we write $\nabla_i u$ for $\partial u/\partial x_i$.
Since

$$\left| \left(\nu_k |\nabla u_k|^2 + \sigma\right)^{\frac{p-2}{2}} \nabla_i u_k \right|^{\frac{p}{p-1}} \leq \left(\nu_k |\nabla u_k|^2 + \sigma\right)^{\frac{p(p-2)}{2(p-1)}} |\nabla u_k|^{p(p-1)} \leq C \left(\nu_k^{\frac{p(p-2)}{2(p-1)}} |\nabla u_k|^p + \sigma^{\frac{p(p-2)}{2(p-1)}} |\nabla u_k|^{\frac{p}{p-1}} \right),$$

and

$$\left| \left(\nu_k \left|\nabla u_k\right|^2 + \sigma\right)^{\frac{q-2}{2}} \nabla_i u_k \right|^{\frac{q}{q-1}} \leq \left(\nu_k \left|\nabla u_k\right|^2 + \sigma\right)^{\frac{q(q-2)}{2(q-1)}} \left|\nabla u_k\right|^{q(q-1)} \leq C \left(\nu_k^{\frac{q(q-2)}{2(q-1)}} \left|\nabla u_k\right|^q + \sigma^{\frac{q(q-2)}{2(q-1)}} \left|\nabla u_k\right|^{\frac{q}{q-1}} \right),$$

by means of (8) and (9) we get

$$\begin{aligned} \int_{Q_\tau} \left| \left(\nu_k \left|\nabla u_k\right|^2 + \sigma\right)^{\frac{p-2}{2}} \nabla_i u_k \right|^{\frac{p}{p-1}} dxdt &\leq C \nu_k^{\frac{p(p-2)}{2(p-1)}} \int_{Q_\tau} |\nabla u_k|^p \, dxdt + C \sigma^{\frac{p(p-2)}{2(p-1)}} \int_{Q_\tau} |\nabla u_k|^{\frac{p}{p-1}} \, dxdt \\ &\leq C \nu_k^{\frac{p-2}{2(p-1)}} \int_{Q_\tau} \nu_k^{\frac{p-2}{2}} |\nabla u_k|^2 \, dxdt + C \sigma^{\frac{p(p-2)}{2(p-1)}} \left(\frac{C}{\sigma^{\frac{p-1}{2}}} \right)^{\frac{p}{2(p-1)}} \leq C, \end{aligned}$$

and

$$\begin{aligned} \int_{Q_\tau} \left| \left(\nu_k |\nabla u_k|^2 + \sigma\right)^{\frac{q-2}{2}} \nabla_i u_k \right|^{\frac{q}{q-1}} dxdt &\leq C \nu_k^{\frac{q(q-2)}{2(q-1)}} \int_{Q_\tau} |\nabla u_k|^q \, dxdt + C \sigma^{\frac{q(q-2)}{2(q-1)}} \int_{Q_\tau} |\nabla u_k|^{\frac{q}{q-1}} \, dxdt \\ &\leq C \nu_k^{\frac{q-2}{2(q-1)}} \int_{Q_\tau} \nu_k^{\frac{q-2}{2}} |\nabla u_k|^2 \, dxdt + C \sigma^{\frac{q(q-2)}{2(q-1)}} \left(\frac{C}{\sigma^{(q-1)/2}} \right)^{\frac{q}{2(q-1)}} \leq C, \end{aligned}$$

guaranteeing the existence of $\xi_i \in L^{\frac{p}{p-1}}(Q_\tau) \cap L^{\frac{q}{q-1}}(Q_\tau)$ where

$$\left(\left(\nu_k |\nabla u_k|^2 + \sigma\right)^{\frac{p-2}{2}} + \left(\nu_k |\nabla u_k|^2 + \sigma\right)^{\frac{q-2}{2}} \right) \nabla_i u_k \rightharpoonup \xi_i \quad \text{weakly in } L^{\frac{p}{p-1)}}(Q_\tau) \cap L^{\frac{q}{q-1}}(Q_\tau)$$

subsequences are noted the same as their original sequence. Thus, it is not hard to see that

$$\int_{Q_\tau} u \frac{\partial \psi}{\partial t} dxdt = \int_{Q_\tau} \xi \nabla \psi dxdt - \int_{Q_\tau} g \psi dxdt \tag{10}$$

for any $\psi \in C_0^\infty(Q_\tau)$, where $\xi = (\xi_1, \ldots, \xi_N)$. To conclude, we need to show that

$$\int_{Q_\tau} \xi \nabla \psi dxdt = \int_{Q_\tau} \Big(\big(\nu|\nabla u|^2 + \sigma\big)^{\frac{p-2}{2}} + \big(\nu|\nabla u|^2 + \sigma\big)^{\frac{q-2}{2}}\Big)\nabla u \nabla \psi dxdt \tag{11}$$

For starters, the following quantity is not negative

$$\int_{Q_\tau} \Big[\Big(\big(\nu_k\big|\nabla u_k\big|^2 + \sigma\big)^{\frac{p-2}{2}} + \big(\nu_k\big|\nabla u_k\big|^2 + \sigma\big)^{\frac{q-2}{2}}\Big)\nabla\big(\nu_k^{\frac{1}{2}} u_k\big) - \Big(\big(|\nabla v|^2 + \sigma\big)^{\frac{p-2}{2}} + \big(|\nabla v|^2 + \sigma\big)^{\frac{q-2}{2}}\Big)\nabla v\Big]\Big[\nabla\big(\nu_k^{\frac{1}{2}} u_k\big) - \nabla v\Big] dxdt \tag{12}$$

for all $v \in L^q\left(0, \tau; W_0^{1,q}(\Omega)\right)$. In fact, let $R(\mathcal{X}) = \Big(\big(|\mathcal{X}|^2 + \sigma\big)^{\frac{p-2}{2}} + \big(|\mathcal{X}|^2 + \sigma\big)^{\frac{q-2}{2}}\Big)\mathcal{X}$; it suffices to note that

$$R'(\mathcal{X}) = \left(\big(|\mathcal{X}|^2 + \sigma\big)^{\frac{p-2}{2}} + \big(|\mathcal{X}|^2 + \sigma\big)^{\frac{q-2}{2}}\right)\mathcal{I} + \left((p-2)\big(|\mathcal{X}|^2 + \sigma\big)^{\frac{p-4}{2}} + (q-2)\big(|\mathcal{X}|^2 + \sigma\big)^{\frac{q-4}{2}}\right)\mathcal{X}\mathcal{X}^T$$

is a positive definite matrix, so that we have

$$\Big(R(\nabla(\nu_k^{\frac{1}{2}} u_k)) - R(\nabla v)\Big)\Big(\nabla\big(\nu_k^{\frac{1}{2}} u_k\big) - \nabla v\Big) \geq 0,$$

and (12) follows. From the periodicity of u_k and all the equations it satisfies we have

$$\int_{Q_\tau} \Big(\big(\nu_k\big|\nabla u_k\big|^2 + \sigma\big)^{\frac{p-2}{2}} + \big(\nu_k\big|\nabla u_k\big|^2 + \sigma\big)^{\frac{q-2}{2}}\Big)\big|\nabla u_k\big|^2 dxdt = \int_{Q_\tau} g_k u_k dxdt$$

combined with (12) derive

$$\begin{aligned}\int_{Q_\tau} g_k u_k dxdt \geq &\int_{Q_\tau} \Big(\big(\nu_k\big|\nabla u_k\big|^2 + \sigma\big)^{\frac{p-2}{2}} + \big(\nu_k\big|\nabla u_k\big|^2 + \sigma\big)^{\frac{q-2}{2}}\Big)\nabla u_k \nabla v dxdt \\ &+ \int_{Q_\tau} \Big(\big(\nu_k|\nabla v|^2 + \sigma\big)^{\frac{p-2}{2}} + \big(\nu_k|\nabla v|^2 + \sigma\big)^{\frac{q-2}{2}}\Big)\nabla v \nabla\big(u_k - v\big) dxdt\end{aligned}$$

If we let $k \to \infty$, we get

$$\int_{Q_\tau} gudxdt \geq \int_{Q_\tau} \xi \nabla v dxdt + \int_{Q_\tau} \Big(\big(\nu_k|\nabla v|^2 + \sigma\big)^{\frac{p-2}{2}} + \big(\nu_k|\nabla v|^2 + \sigma\big)^{\frac{q-2}{2}}\Big)\nabla v \nabla\big(u_k - v\big) dxdt. \tag{13}$$

On the other hand, taking $\psi = u$ in (10) gives

$$\int_{Q_\tau} \xi \nabla u dxdt = \int_{Q_\tau} gudxdt \tag{14}$$

Together with (14) and (13) yield

$$\int_{Q_\tau} \Big(\xi_i - \Big(\big(\nu|\nabla v|^2 + \sigma\big)^{\frac{p-2}{2}} + \big(\nu|\nabla v|^2 + \sigma\big)^{\frac{q-2}{2}}\Big)\nabla_i v\Big)\Big(\nabla_i u - \nabla_i v\Big) dxdt \geq 0.$$

Letting $v = u - \lambda\psi$ with $\lambda > 0$, $\psi \in C_0^\infty(Q_\tau)$, we get

$$\int_{Q_\tau} \Big(\xi_i - \Big(\big(\nu|\nabla(u - \lambda\psi)|^2 + \sigma\big)^{\frac{p-2}{2}} + \big(\nu|\nabla(u - \lambda\psi)|^2 + \sigma\big)^{\frac{q-2}{2}}\Big)\nabla_i(u - \lambda\psi)\Big)\nabla_i \psi dxdt \geq 0.$$

Taking $\lambda \to 0$ yields

$$\int_{Q_\tau} \Big(\xi_i - \Big(\big(\nu|\nabla u|^2 + \sigma\big)^{\frac{p-2}{2}} + \big(\nu|\nabla u|^2 + \sigma\big)^{\frac{q-2}{2}}\Big)\Big)\nabla_i u \nabla_i \psi dxdt \geq 0. \tag{15}$$

With a very similar manner we can prove that the converse inequality also holds, which makes (11) true. □

4. Topological degree on $B_r(0)$

Lemma 4.1. *Assuming the conditions of Theorem 2.2 hold,* $\deg\left(\mathcal{I} - \mathcal{V}(1,\Phi(\cdot)), B_r(0), 0\right) = 1$ *for some* $r > 0$ *independent of* σ.

Proof. Notice that the map $\mathcal{V}(1, \nu\Phi(u))$ is compact because $\mathcal{V}$ is compact and Φ is continuous. By the homotopy invariance of degree

$$\deg\left(\mathcal{I} - \mathcal{V}(1,\Phi(\cdot)), B_r(0), 0\right) = \deg\left(\mathcal{I}, B_r(0), 0\right) = 1, \tag{16}$$

assuming that

$$\mathcal{V}(1, \nu\Phi(u)) \neq u \quad \text{for } \nu \in [0,1], \quad u \in \partial B_r(0). \tag{17}$$

Which we will prove by taking

$$r = \Big(\frac{1}{MC_0^q|\Omega|^{1-\frac{q}{q'}}}\Big)^{\frac{1}{m-q+1}},$$

where $q' = \frac{Nq}{N-q}$ if $q < N$, $q' = q+1$ if $q \geq N$, $M = \max\limits_{\overline{Q_\tau}} h(x,t)$, Denote u_ν the periodic solution of

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\left(\left|\nabla u\right|^2 + \sigma\right)^{\frac{p-2}{2}} \nabla u + \left(\left|\nabla u\right|^2 + \sigma\right)^{\frac{q-2}{2}} \nabla u\right) + \nu h(x,t) u_+^m \tag{18}$$

with the Dirichlet boundary condition. Using the maximum principle and the continuity of u_ν, we have $u_\nu(x,t) \geq 0$. If we multiply (18) by u_ν and integrate over Q_τ we get

$$K = \nu \int_{Q_\tau} h(x,t) u_\nu^{m+1} dxdt - \int_{Q_\tau} \left(\left(\left|\nabla u_\nu\right|^2 + \sigma\right)^{\frac{p-2}{2}} + \left(\left|\nabla u_\nu\right|^2 + \sigma\right)^{\frac{q-2}{2}}\right)\left|\nabla u_\nu\right|^2 dxdt = 0 \tag{19}$$

for $\nu \in [0,1]$. In what follows, we will be using the embedding theorems

$$\|u_\nu\|_{q'} \leq C_1 \|\nabla u_\nu\|_p \quad \text{and} \quad \|u_\nu\|_{q'} \leq C_2 \|\nabla u_\nu\|_q .$$

In the following, we denote $C_0 = \max(C_1, C_2)$. In the case where $q < N$, from (19) we have

$$K \leq M \int_{Q_\tau} u_\nu^{m+1} dxdt - \frac{1}{C_0^q} \int_0^\tau \left(\|u_\nu\|_{q'}^p + \|u_\nu\|_{q'}^q\right) dt \tag{20}$$

If $m+1 \geq q'$, then

$$K \leq M \max_{\overline{Q_\tau}} u_\nu^{m+1-q'} \int_0^\tau \|u\|_{q'}^{q'} dt - \frac{2}{C_0^q} \int_0^\tau \|u_\nu\|_{q'}^q dt \leq \int_0^\tau \|u_\nu\|_{q'}^q \left(M|\Omega|^{\frac{q'-q}{q'}} \max_{\overline{Q_\tau}} u_\nu^{m+1-\gamma} - \frac{2}{C_0^q}\right) dt. \tag{21}$$

If (17) were not true, then we would have $u_\nu \in \partial B_r(0)$. Therefore

$$\max_{\overline{Q_\tau}} u_\nu(x,t) = r = \left(\frac{1}{M|\Omega|^{1-\frac{q}{q'}}}\right)^{\frac{1}{m+1-q}}$$

and the last integral in (21) equals $-\int_0^\tau \|u_\nu\|_{q'}^p\, dt < 0$, contradicting with the equality (19).
In case $q < m+1 < q'$, we use the Hölder inequality for the first integral on the right part of (20)

$$\int_\Omega u_\nu^{m+1} dx \leq \|u_\nu\|_{q'}^{m+1} |\Omega|^{\frac{q'-1-m}{q'}}$$

and obtain

$$K \leq M|\Omega|^{\frac{q'-1-m}{q'}} \int_0^\tau \|u_\nu\|_{q'}^{m+1}\, dt - \frac{2}{C_0^q}\int_0^\tau \|u_\nu\|_{q'}^q\, dt = \int_0^\tau \left(M|\Omega|^{\frac{q'-q}{q'}} \|u_\nu\|_{q'}^{m+1-q} - \frac{2}{C_0^q}\right)\|u_\nu\|_{q'}^q\, dt \tag{22}$$

If we assume (17) to be false in this case, we would then have $u_\nu \in \partial B_r(0)$, implying

$$\max_{\overline{Q_\tau}} u_\nu(x,t) = r = \left(\frac{1}{M|\Omega|^{1-\frac{q}{q'}}}\right)^{\frac{1}{m+1-q}}$$

which makes the last integral in (22) equal to $-\int_0^\tau \|u_\nu\|_{q'}^q\, dt < 0$. This inconsistency implies $u_\nu \notin \partial B_r(0)$.

If $q \geq N$, It is easy to check that $r = \left(1/M|\Omega|^{1-\frac{q}{q'}}\right)^{\frac{1}{m-q+1}}$ with $q' = q+1$ satisfies (17) similarly as the previous case. Hence lemma 4.1 is proved. □

5. Topological degree on $B_R(0)$

From here and below, λ will denote an eigenvalue of $-\Delta$ in Ω with the homogenous Dirichlet boundary condition and ψ_λ a positive eigenfunction related to λ.

Lemma 5.1. *Set u_ν as a nonnegative periodic solution of*

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\left(\nu|\nabla u|^2+\sigma\right)^{\frac{p-2}{2}}\nabla u + (\nu\,|\nabla u_\nu|^2+\sigma)^{\frac{q-2}{2}}\nabla u\right) + h(x,t)u^m + (1-\nu)\left(\lambda(\sigma^{\frac{p-2}{2}}+\sigma^{\frac{q-2}{2}})u+1\right) \tag{23}$$

with the Dirichlet boundary value condition of problem ($\mathcal{P}$), where $0 \leq \nu \leq 1$. Assuming that the conditions of Theorem 2.2 hold, there is constant $L > 0$ unrelated to ν, and

$$\|u\|_\infty \leq L.$$

In case $\nu = 1$, L would be unrelated to σ as well.

Proof. Set $0 < \sigma \leq 1$. Assume that u_ν has no bound. Meaning there are sequences $(\nu_n)_{n\geq 0} \subset [0,1]$ and $(u_n)_{n\geq 0}$, where

$$M_n = \max_{\overline{Q_\tau}} u_n(x,t) = u_n(x_n,t_n) \to \infty \quad \text{ as } n \to \infty.$$

We assume that $\nu_n \to \nu_\infty$ and $(x_n,t_n) \to (x_\infty,t_\infty)$ as $n \to \infty$.
We start by proving that $\nu_n \neq 0$ for all n. In fact, if $\nu_n = 0$ for some n, then (23) becomes

$$\frac{\partial u}{\partial t} = (\sigma^{\frac{p-2}{2}}+\sigma^{\frac{q-2}{2}})\Delta u + h(x,t)u^m + \lambda(\sigma^{\frac{p-2}{2}}+\sigma^{\frac{q-2}{2}})u+1. \tag{24}$$

Multiplying (24) by ψ_λ, integrating over Q_τ, and noticing the periodicity of u, we obtain

$$0 = \int_{Q_\tau} \frac{\partial \psi_\lambda u}{\partial t} dxdt = \int_{Q_\tau} (\sigma^{\frac{p-2}{2}}+\sigma^{\frac{q-2}{2}})\psi_\lambda \Delta u dxdt + \int_{Q_\tau} h(x,t)u^m\psi_\lambda dxdt + \lambda\int_{Q_\tau}(\sigma^{\frac{p-2}{2}}+\sigma^{\frac{q-2}{2}})u\psi_\lambda dxdt$$
$$+\tau\int_\Omega \psi_\lambda dx = \int_{Q_\tau} h(x,t)u^m\psi_\lambda dxdt + \tau\int_\Omega\psi_\lambda dx > 0$$

which is a contradiction.
For all n define μ_n, z, s, and v_n as

$$\mu_n^{\frac{q}{m-q+1}} M_n = 1, \quad z = \frac{x-x_n}{\mu_n}, \quad s = \frac{t-t_n}{\mu_n^{\frac{(m-1)q}{m-q+1}}} \text{ and } v_n(z,s) = \mu_n^{\frac{q}{m-q+1}} u_n(x,t).$$

Noticing that Ω is convex, we get $\delta_0 > 0$ so that we have $\operatorname{dist}(x_n, \partial\Omega) \geq \delta_0$ from [36] and [26]. Thus, the function $v_n(z,s)$ has proper meaning in the set

$$D_{n,\delta_0} = D\Big(\frac{\delta_0}{2\mu_n}\Big) \times \Big(\frac{-\tau}{\mu_n^{\frac{(m-1)q}{m-q+1}} \nu_n^{\frac{q-2}{2}}}, \frac{\tau}{\mu_n^{\frac{(m-1)q}{m-q+1}} \nu_n^{\frac{q-2}{2}}}\Big),$$

such that $D(\ell)$ is the ball of $\mathbb{R}^N$ that has ℓ as radius and 0 as its center. In D_{n,δ_0}, the function $w_n(z,s) = \nu_n^{\frac{1}{2}} v_n(z,s)$ verifies

$$\frac{\partial w_n}{\partial s} = \mu_n^{\frac{q^2(p-2)(1-m)}{(m-p+1)(m-q+1)}} \operatorname{div}\Big(\Big(\big|\nabla w_n\big|^2 + \mu_n^{\frac{2p}{m-q+1}}\sigma\Big)^{\frac{p-2}{2}} \nabla w_n\Big) + \mu_n^{\frac{q(m-1)(q-2)}{2(m-q+1)}} \operatorname{div}\Big(\Big(\big|\nabla w_n\big|^2 + \mu_n^{\frac{2q}{m-q+1}}\sigma\Big)^{\frac{q-2}{2}} \nabla w_n\Big)$$
$$+ h\Big(x_n + \mu_n z, t_n + s\mu_n^{\frac{(m-1)q}{m-q+1}}\Big) v_n^{m-1} w_n + \big(1-\nu_n\big)\Big(\lambda\sigma^{\frac{q-2}{2}} \mu_n^{\frac{q(m-1)}{m-q+1}} w_n + \nu_n^{\frac{1}{2}} \mu_n^{\frac{mq}{m-q+1}}\Big)$$

Since $\|v_n\|_\infty = v_n(0,0) = 1$, we have $\|w_n\|_\infty = w_n(0,0) = \nu_n^{\frac{1}{2}}$. For any given $\delta > 0$, let

$$S_1 = D(2\delta) \times \Big(\frac{-2d}{\nu_n^{\frac{q-2}{2}}}, \frac{2d}{\nu_n^{\frac{q-2}{2}}}\Big) \quad \text{and} \quad S_2 = D(\delta) \times \Big(\frac{-d}{\nu_n^{\frac{q-2}{2}}}, \frac{d}{\nu_n^{\frac{q-2}{2}}}\Big)$$

Since $\mu_n \to 0$ as $n \to \infty$, we see that $S_2 \subset S_1 \subset D_{n,\delta_0}$.
Applying Theorem 1.1 in [16] and noticing that $N > 1$, we get

$$|w_n(z_1, s_1) - w_n(z_2, s_2)| \leq \gamma\left(|z_1 - z_2| + |s_1 - s_2|^{\frac{1}{2}}\right)^\beta,$$

which implies that there exists a function $v \in C\left(\mathbb{R}^N \times \mathbb{R}\right)$ such that

$$v_n(z,s) \to v(z,s) \quad \text{in } C_{\text{loc}}\left(\mathbb{R}^N \times \mathbb{R}\right) \tag{25}$$

and a domain Q_τ containing $(0,0)$ with $Q \subset S_2$, such that for any $(z,s) \in Q_\tau$

$$v_n(z,s) \geq \frac{1}{2}. \tag{26}$$

Let $\chi \in C_0^\infty\left(\mathbb{R}^N \times \mathbb{R}\right)$ be a smooth cut-off function defined in $D(2r) \times (2T_1 - T_2, 2T_2 - T_1)$ such that

$$\chi(x,t) = 1 \text{ in } D(r) \times (T_1, T_2), \quad |D\chi| \leq \frac{C}{r} \quad \text{and} \quad \left|\frac{\partial \chi}{\partial s}\right| \leq \frac{C}{T_2 - T_1}.$$

If we multiply, by $w_n\chi^\theta$ $(\theta > q)$, the equation verified by w_n, and integrate over D_n, σ_0, we obtain

$$\frac{1}{2}\int_{D_{n,},\delta_0} \frac{\partial w_n^2}{\partial s}\chi^\theta dzds + \int_{D_{n,\delta_0}} \Big[\mu_n^{\frac{q^2(p-2)(1-m)}{(m-p+1)(m-q+1)}}\Big(\big|\nabla w_n\big|^2 + \mu_n^{\frac{2p}{m-q+1}}\sigma\Big)^{\frac{p-2}{2}} + a\mu_n^{\frac{q(m-1)(q-2)}{2(m-q+1)}}\Big(\big|\nabla w_n\big|^2 + \mu_n^{\frac{2q}{m-q+1}}\sigma\Big)^{\frac{q-2}{2}}\Big]$$
$$\times \frac{\partial w_n}{\partial z_i}\frac{\partial\big(w_n\chi^\theta\big)}{\partial z_i} dzds = \int_{A_{n,\delta_0}} \Big[hv_n^{m-1}w_n + \big(1-\nu_n\big)\Big(\lambda\sigma^{\frac{q-2}{2}}\mu_n^{\frac{q(m-1)}{m-q+1}} w_n + \nu_n^{\frac{1}{2}}\mu_n^{\frac{mq}{m-q+1}}\Big)\Big] w_n\chi^\theta dzds \tag{27}$$

provided n is big enough to ensure that D_{n,δ_0} contains $D(2r) \times (2T_1 - T_2, 2T_2 - T_1)$.
Notice that

$$\Big|\int_{D_{n,\delta_0}} \frac{\partial w_n^2}{\partial s}\chi^\theta dzds\Big| = \Big|\int_{D_{n,\delta_0}} \Big(\frac{\partial\big(w_n^2\chi^\theta\big)}{\partial s} - rw_n^2\chi^{\theta-1}\frac{\partial\chi}{\partial s}\Big) dzds\Big| = \Big|\int_{D_{n,\delta_0}} \theta w_n^2\chi^{\theta-1}\frac{\partial\chi}{\partial s} dzds\Big|$$
$$\leq \nu_n \frac{C}{T_2 - T_1} \operatorname{meas}\Big(D(2r) \times \big(2T_1 - T_2, 2T_2 - T_1\big)\Big) = C\nu_n r^N. \tag{28}$$

On the other hand, we obtain

$$\int_{D_{n,\delta_0}} \mu_n^{\frac{q^2(p-2)(1-m)}{(m-p+1)(m-q+1)}} \Big(\big|\nabla w_n\big|^2 + \mu_n^{\frac{2p}{m-q+1}}\sigma\Big)^{\frac{p-2}{2}} \frac{\partial w_n}{\partial z_i}\frac{\partial (w_n\chi^\theta)}{\partial z_i} dzds$$
$$\leq \int_{D_{n,\delta_0}} \Big(\big|\nabla w_n\big|^2 + \mu_n^{\frac{2p}{m-q+1}}\sigma\Big)^{\frac{p-2}{2}} \frac{\partial w_n}{\partial z_i}\Big(\frac{\partial w_n}{\partial z_i}\chi^\theta + r\chi^{\theta-1} w_n \frac{\partial \chi}{\partial z_i}\Big) dzds$$
$$\leq \frac{1}{2}\int_{D_{n,\delta_0}} \chi^\theta |\nabla w_n|^p\, dzds + \theta \int_{D_n\cdot\delta_0} \Big(\big|\nabla w_n\big|^2 + \mu_n^{\frac{2p}{m-q+1}}\sigma\Big)^{\frac{p-2}{2}} \frac{\partial w_n}{\partial z_i}\chi^{\theta-1} w_n \frac{\partial \chi}{\partial y_i} dzds \quad (29)$$

Similarly, we have

$$\int_{D_{n,\delta_0}} a\mu_n^{\frac{q(m-1)(q-2)}{2(m-q+1)}} \Big(\big|\nabla w_n\big|^2 + \mu_n^{\frac{2q}{(m-q+1)}}\sigma\Big)^{\frac{q-2}{2}} \frac{\partial w_n}{\partial z_i}\frac{\partial (w_n\chi^\theta)}{\partial z_i} dzds$$
$$\leq \frac{1}{2}\int_{D_{n,\delta_0}} \chi^\theta |\nabla w_n|^q\, dzds + \theta \int_{D_n,\delta_0} \Big(|\nabla w_n|^2 + \mu_n^{\frac{2q}{m-q+1}}\sigma\Big)^{\frac{q-2}{2}} \frac{\partial w_n}{\partial z_i}\chi^{\theta-1} w_n \frac{\partial \chi}{\partial y_i} dzds \quad (30)$$

and

$$\int_{D_{n,\delta_0}} \Big[hv_n^{m-1} w_n + \big(1-\nu_n\big)\Big(\lambda\sigma^{\frac{(q-2)}{2}}\mu_n^{\frac{q(m-1)}{m-q+1}} w_n + \nu_n^{\frac{1}{2}}\mu_n^{\frac{mq}{m-q+1}}\Big)\Big] w_n\chi^\theta dzds \leq C\nu_n \int_{D_{n,\delta_0}} \chi^\theta dzds \leq C\nu_n r^N\big(T_2 - T_1\big). \quad (31)$$

Furthermore, and using Young's inequality

$$\Big|\theta \int_{D_{n},\delta_0} \Big(\big|\nabla w_n\big|^2 + \mu_n^{\frac{2p}{m-q+1}}\sigma\Big)^{\frac{p-2}{2}} \frac{\partial w_n}{\partial z_i}\chi^{\theta-1} w_n \frac{\partial \chi}{\partial z_i} dyds\Big| \leq C\int_{D_{n},\delta_0} \Big(\big|\nabla w_n\big|^{p-2} + \mu_n^{\frac{p(p-2)}{m-q+1}}\sigma^{\frac{p-2}{2}}\Big)\chi^{\theta-1} w_n |\nabla w_n| |\nabla\chi| dzds$$
$$\leq \frac{1}{4}\int_{D_{n,\delta_0}} \chi^\theta |\nabla w_n|^p\, dzds + C\int_{D_{n,\delta_0}} w_n^p \chi^{\theta-p}|\nabla\chi|^p dyds + \frac{1}{4}\int_{D_{n,\delta_0}} \mu_n^{\frac{p(p-2)}{m-q+1}}\sigma^{\frac{p-2}{2}}(\chi^\theta|\nabla w_n|^2 + C w_n^2\chi^{\theta-2}|\nabla\chi|^2) dzds, \quad (32)$$

and with a similar manner

$$\Big|\theta \int_{D_n,\delta_0} \Big(|\nabla w_n|^2 + \mu_n^{\frac{2q}{m-q+1}}\sigma\Big)^{\frac{q-2}{2}} \frac{\partial w_n}{\partial z_i}\chi^{\theta-1} w_n \frac{\partial \chi}{\partial z_i} dyds\Big| \leq \frac{1}{4}\int_{D_{n,\delta_0}} \chi^\theta |\nabla w_n|^q\, dzds + C\int_{D_{n,\delta_0}} w_n^p\chi^{\theta-q}|\nabla\chi|^q dyds$$
$$+ \frac{1}{4}\int_{D_{n,\delta_0}} \mu_n^{\frac{q(q-2)}{m-q+1}}\sigma^{\frac{q-2}{2}}(\chi^\theta|\nabla w_n|^2 + C w_n^2\chi^{\theta-2}|\nabla\chi|^2) dzds. \quad (33)$$

Combining the inequalities (27)-(33) yields

$$\int_{D_n,\delta_0} \chi^\theta |\nabla w_n|^p\, dzds + \int_{D_{n,\delta_0}} \mu_n^{\frac{p(p-2)}{m-q+1}}\sigma^{\frac{p-2}{2}}\chi^\theta |\nabla w_n|^2\, dzds \leq C\nu_n r^N + C\nu_n s^N (T_2 - T_1) + C\nu_n^{\frac{p}{2}} r^N\big(T_2 - T_1\big)\Big(\frac{C}{r}\Big)^p$$
$$+ C\nu_n\sigma^{\frac{p-2}{2}} r^N (T_2 - T_1)\Big(\frac{C}{r}\Big)^2 \mu_n^{\frac{p(p-2)}{m-q+1}} = C_1\nu_n,$$

similarly

$$\int_{D_n,\delta_0} \chi^\theta |\nabla w_n|^q\, dzds + \int_{D_{n,\delta_0}} \mu_n^{\frac{q(q-2)}{m-q+1}}\sigma^{\frac{q-2}{2}}\chi^\theta |\nabla w_n|^2\, dzds \leq C_2\nu_n,$$

such that the constants C_1 and C_2 relate only to r and $T_1 - T_2$, so we get for all $r > 0$, and $T_2 > T_1$

$$\nu_n^{\frac{p-2}{2}} \int_{T_1}^{T_2} \int_{B_r} |\nabla v_n|^p \, dzds \leq C, \tag{34}$$

$$\mu_n^{\frac{p(p-2)}{m-q+1}} \sigma^{\frac{p-2}{2}} \int_{T_1}^{T_2} \int_{B_r} |\nabla v_n|^2 \, dzds \leq C, \tag{35}$$

$$\nu_n^{\frac{q-2}{2}} \int_{T_1}^{T_2} \int_{B_r} |\nabla v_n|^q \, dzds \leq C \tag{36}$$

and

$$\mu_n^{\frac{q(q-2)}{m-q+1}} \sigma^{\frac{q-2}{2}} \int_{T_1}^{T_2} \int_{B_r} |\nabla v_n|^2 \, dzds \leq C. \tag{37}$$

If $\nu_\infty = 0$, then for any $\psi \in C_0^\infty\left(\mathbb{R}^N \times \mathbb{R}\right)$

$$\begin{aligned}
&\Big| \int_{D_{n,\delta_0}} \left(\nu_n \left|\nabla v_n\right|^2 + \mu_n^{\frac{2p}{m-q+1}} \sigma\right)^{\frac{p-2}{2}} \frac{\partial v_n}{\partial z_n} \frac{\partial \psi}{\partial y z_i} dzds \Big| \leq C \int_{D_{n,\delta_0}} \left(\nu_n^{\frac{p-2}{2}} \left|\nabla v_n\right|^{p-2} + \mu_n^{\frac{p(p-2)}{m-q+1}} \sigma^{\frac{p-2}{2}}\right) \left|\nabla v_n\right| |\nabla \psi| dzds \\
&\leq C \Big(\int_{\operatorname{supp} \psi} \nu_n^{\frac{p-2}{2}} \left|\nabla v_n\right|^p dyds \Big)^{\frac{p-1}{p}} \Big(\int_{\operatorname{supp} \psi} \nu_n^{\frac{p-2}{2}} |\nabla \psi|^p dzds \Big)^{\frac{1}{p}} \\
&\quad + C \Big(\int_{\operatorname{supp} \psi} \mu_n^{\frac{p(p-2)}{m-q+1}} \sigma^{\frac{p-2}{2}} \left|\nabla v_n\right|^2 dzds \Big)^{\frac{1}{2}} \Big(\int_{\operatorname{supp} \psi} \mu_n^{\frac{p(p-2)}{m-q+1}} \sigma^{\frac{p-2}{2}} |\nabla \psi|^2 dzds \Big)^{\frac{1}{2}} \\
&\leq C_\psi \nu_n^{\frac{p-2}{2}} + C_\psi \sigma^{\frac{p-2}{4}} \mu_n^{\frac{p(p-2)}{m-q+1}},
\end{aligned}$$

and C_ψ is a positive number related only to ψ. Consequently

$$\int_{D_n,\delta_0} \left(\nu_n \left|\nabla v_n\right|^2 + \mu_n^{\frac{4p}{m-q+1}} \sigma\right)^{\frac{p-2}{2}} \nabla v_n \nabla \psi dzds \to 0$$

as $n \to \infty$. Following the same steps we can prove that

$$\int_{D_{n,\delta_0}} \left(\nu_n \left|\nabla v_n\right|^2 + \mu_n^{\frac{4p}{m-q+1}} \sigma\right)^{\frac{q-2}{2}} \nabla v_n \nabla \psi dzds \to 0$$

as $n \to \infty$.
Using Lebesgue's theorem and (25), we obtain

$$\int v \frac{\partial \psi}{\partial s} dzds + h\left(x_\infty, t_\infty\right) \int v^m \psi dzds = 0 \quad \text{ for any } \psi \in C_0^\infty\left(\mathbb{R}^N \times R\right),$$

which implies

$$\frac{\partial v\left(z_0, s\right)}{\partial s} = h\left(x_\infty, t_\infty\right) v^m\left(z_0, s\right) \tag{38}$$

almost everywhere. Using the fact that v is continuous, we show that the equality (38) is in fact true for $s \in (-\infty, \infty)$. But, we take from (25) and (26) the existence of z_0 where $v\left(z_0, 0\right) > 0$ and therefore there is no global solution for (38) such that $m > p - 1 \geq 1$. The present contradiction implies that $\nu_\infty = 0$ is impossible. Now, since $\nu_\infty \neq 0$ it follows from (34)-(37), that

$$\Big[\left(\nu_n \left|\nabla v_n\right|^2 + \mu_n^{\frac{4p}{m-q+1}} \sigma\right)^{\frac{p-2}{2}} + \left(\nu_n \left|\nabla v_n\right|^2 + \mu_n^{\frac{4q}{m-q+1}} \sigma\right)^{\frac{q-2}{2}}\Big] \nabla_i v_n \rightharpoonup \xi_i$$

in $L_{loc}^{\frac{p}{p-1}}\left(\mathbb{R}^N\times\mathbb{R}\right)\cap L_{loc}^{\frac{q}{q-1}}\left(\mathbb{R}^N\times\mathbb{R}\right)$. Using a reasoning like the one of section 3 we can prove that

$$\xi_i = (\nu_\infty^{\frac{p-2}{2}}|\nabla v|^{p-2} + \nu_\infty^{\frac{q-2}{2}}|\nabla v|^{q-2})\nabla_i v.$$

Since $\mu_n \to 0$ as $n \to \infty$ and the inequality

$$\int \Big(\Big[\big(\nu_n\big|\nabla v_n\big|^2 + \mu_n^{\frac{4p}{m-q+1}}\sigma\big)^{\frac{p-2}{2}} + \big(\nu_n\big|\nabla v_n\big|^2 + \mu_n^{\frac{4q}{m-q+1}}\sigma\big)^{\frac{q-2}{2}}\Big]\nabla v_n - \Big[\big(\nu_n\big|\nabla w_n\big|^2 + \mu_n^{\frac{4p}{m-q+1}}\sigma\big)^{\frac{p-2}{2}} + \big(\nu_n\big|\nabla w_n\big|^2 + \mu_n^{\frac{4q}{m-q+1}}\sigma\big)^{\frac{q-2}{2}}\Big]\nabla w\Big)\big(\nabla v_n - \nabla w\big)dzds \geq 0$$

is fulfilled, we derive that $v \in C\left(\mathbb{R}^N\times\mathbb{R}\right)\cap L^q_{\mathrm{loc}}\left(\mathbb{R}; W^{1,q}_{\mathrm{loc}}\left(\mathbb{R}^N\right)\right)$ and verify the following equation

$$\frac{\partial v}{\partial s} = s_\infty^{\frac{p-2}{2}}\operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right) + s_\infty^{\frac{q-2}{2}}\operatorname{div}\left(|\nabla v|^{q-2}\nabla v\right) + h\big(x_\infty, t_\infty\big)v^m, \quad (z,s)\in\mathbb{R}^N\times\mathbb{R}. \tag{39}$$

Still, from [23, 24], we conclude that the Cauchy problem for (39) can't have a global nontrivial nonnegative solution where $q-1 < m < p-1+\frac{p}{N}$. The contradiction means that the first claim of our lemma holds. The second claim of the lemma can be proved by analogy. □

Lemma 5.2. *Assuming the conditions of Theorem 2.2 hold, there is $R > r$ where* $\deg(\mathcal{I} - \mathcal{V}(1,\Psi(\cdot)), B_R(0), 0) = 0$.

Proof. Let $\hat{\Psi}(u)(x,t) = h(x,t)u_+^m + \lambda u_+ + 1$ for $u \in C_\tau(\overline{Q_\tau})$ and $G(\nu,u) = \mathcal{V}(\nu, \nu\Phi(u) + (1-\nu)\hat{\Psi}(u))$. Based on Lemma 3.1 and Lemma 3.2, $G(\nu,\cdot)$ is well defined from $C_\tau(\overline{Q_\tau})$ to $C_\tau(\overline{Q_\tau})$ and compact. Thus, applying the homotopy invariance of degree, we arrive at

$$\deg(\mathcal{I} - \mathcal{V}(1,\Psi(\cdot)), B_R(0), 0) = \deg\left(\mathcal{I} - \mathcal{V}(0,\hat{\Phi}(\cdot)), B_R(0), 0\right),$$

assuming that

$$G(\nu,u) \neq u \quad \text{for all } u \in \partial B_R(0),\ \nu \in [0,1].$$

In fact, Lemma 5.1 proves the validity of the inequality for $R > \max\{L, r\}$.
Furthermore, similarly to the proof of Lemma 5.1, (24) cannot have a nonnegative periodic solution with the Dirichlet boundary value condition. Hence $\deg\left(\mathcal{I} - \mathcal{V}(0,\hat{\Psi}(\cdot)), B_R(0), 0\right) = 0$ implying $\deg\left(\mathcal{I} - \mathcal{V}(1,\Psi(\cdot)), B_R(0), 0\right) = 0$. □

6. Proof of the main theorem

To conclude, we use the previous lemmas and the proposition to prove Theorem 2.2. According to Lemma 4.1 and Lemma 5.2, we have

$$\deg\left(\mathcal{I} - \mathcal{V}(1,\Phi(\cdot)), B_R(0)\backslash B_r(0), 0\right) = -1,$$

hence there is at least one periodic solution u_σ of (2) with Dirichlet boundary conditions, with

$$r \leq \max_{\overline{Q_\tau}} u_\sigma(x,t) \leq R \quad \text{for any } 0 < \sigma < 1.$$

This gives an end to the proof of the first proposition. By the uniform boundedness of u_σ and Theorem 3 in [38] to get the uniform Hölder continuity of u_σ and then apply the Arzelà-Ascoli theorem to deduct the existence of a function $u \in C_\tau(\overline{Q_\tau})$ and a subsequence of $\{u_\sigma\}$, without loss of generality also noted $\{u_\sigma\}$, so that we have

$$u_\sigma \to u \quad \text{uniformly in } \overline{Q_\tau}.$$

Note that $r \leq \max_{\overline{Q_\tau}} u(x,t) \leq R$. By Lemma 3.1 and the proof of Lemma 3.2, the following inequalities hold

$$\left\| \frac{\partial u_\sigma}{\partial t} \right\|_2 \leq C_1, \quad \int_{Q_\tau} \left(\nu \, |\nabla u_\sigma|^2 + \sigma \right)^{\frac{p-2}{2}} |\nabla u_\sigma|^2 \, dxdt \leq C_2 \quad \text{and} \quad \int_{Q_\tau} \left(\nu \, |\nabla u_\sigma|^2 + \sigma \right)^{\frac{q-2}{2}} |\nabla u_\sigma|^2 \, dxdt \leq C_3,$$

where C_1, C_2 and C_3 are independent of σ. An argument similar to section 3 can be adopted to prove that u is a periodic solution of the problem $(\mathcal{P})$ in the same sense announced in definition 2.1. The proof of Theorem 1 is now complete.

References

[1] A. Abbassi, C. Allalou, A. Kassidi, *Existence of weak solutions for nonlinear p-elliptic problem by topological degree*, Nonlinear Dyn. Syst. Theory **20(3)** (2020), 229–241.

[2] A. Abbassi, C. Allalou, A. Kassidi, *Existence results for some nonlinear elliptic equations via topological degree methods*, Journal of Elliptic and Parabolic Equations **7(1)** (2021,) 121–136.

[3] A. Abbassi, C. Allalou, A. Kassidi, *Topological degree methods for a Neumann problem governed by nonlinear elliptic equation*, Moroccan Journal of Pure and Applied Analysis **6(2)** (2020), 231–242.

[4] A. Aberqi, J. Bennouna, O. Benslimane, MA Ragusa, *Existence Results for Double Phase Problem in Sobolev–Orlicz Spaces with Variable Exponents in Complete Manifold*. Mediterr. J. Math. **19 (4)** (2022), 158.

[5] C. Allalou, A. Abbassi, A. Kassidi, *The discontinuous nonlinear Dirichlet boundary value problem with p-Laplacian*, Azerb. J. Math **11(2)** (2021), 60–77.

[6] P. Baroni, M. Colombo, G. Mingione, *Harnack inequalities for double phase functionals*, Nonlinear Anal. **121**, (2015), 206–222.

[7] P. Baroni, M. Colombo, G. Mingione, *Non-autonomous functionals, borderline cases and related function classes*, St. Petersburg Math. J. **27** (2016) 347–379.

[8] P. Baroni, M. Colombo, G. Mingione, *Regularity for general functionals with double phase*, Calc. Var. Partial Differential Equations 57(2), Art. **62** (2018), p 48.

[9] P. Baroni, T. Kuusi, G. Mingione, *Borderline gradient continuity of minima*, J. Fixed Point Theory Appl. **15(2)** (2014), 537–575.

[10] J. Berkovits, *Extension of the Leray-Schauder degree for abstract Hammerstein type mappings*, Journal of Differential Equations vol. **234(1)** (2007), 289–310.

[11] A. Charkaoui, G. Kouadri, NE. Alaa, *Some Results on The Existence of Weak Periodic Solutions For Quasilinear Parabolic Systems With L^1 Data*, Bol. Soc. Paran. Mat., **40** (2022), 1–15.

[12] A. Charkaoui, G. Kouadri, O. Selt, NE. Alaa, *Existence results of weak periodic solution for some quasilinear parabolic problem with L^1 data*, Annals of the University of Craiova, Mathematics and Computer Science Series Volume **46(1)** (2019), 66–77.

[13] J. Chen, Z. Zhang, G. Chang, J. Zhao, *Periodic Solutions to Klein–Gordon Systems with Linear Couplings*, Adv. Nonlinear Stud. **21(3)** (2021), 633–660.

[14] M. Colombo, G. Mingione, *Bounded minimisers of double phase variational integrals*, Arch. Ration. Mech. Anal. **218(1)** (2015), 219-273.

[15] M. Colombo, G. Mingione, *Regularity for double phase variational problems*, Arch. Ration. Mech. Anal. **215(2)** (2015), 443–496.

[16] E. Di Benedetto, *Degenerate Parabolic Equations*, Springer-Verlag, New York (1993).

[17] E. Di Benedetto, *On the local behavior of solutions of degenerate parabolic equations with measurable coefficients*, Ann. Scuola Norm. Sup. Pisa. Cl. Sci. **(4) 13** (1986), 487–535.

[18] M, Di Francesco, A. Lorz, P. Markowich, *Chemotaxis-fluid coupled model for swimming bacteria with nonlinear diffusion: global existence and asymptotic behavior*, Discrete Contin. Dyn. Syst. **28** (2010), 1437–1453.

[19] M. Esteban, *A remark on the existence of positive periodic solutions of superlinear parabolic problems*, Proc. Amer. Math. Soc. **102** (1988), 131–136.

[20] M. Esteban, *On periodic solutions of superlinear parabolic problems*, Trans. Amer. Math. Soc. **293** (1986), 171–189.

[21] MA. Farid, EM. Marhrani, M. Aamri, *Leray–Schauder fixed point theorems for block operator matrix with an application*, Journal of Mathematics (2021).

[22] M. Fečkan, J. Wang, Y. Zhou, *Periodic solutions for nonlinear evolution equations with non-instantaneous impulses*, Nonautonomous Dynamical Systems **1(1)** (2014).

[23] H. Fujita, *On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **16** (1966), 105–113.

[24] Z. Junning, *On the Cauchy problem and initial traces for the evolution p-Laplacian equations with strongly nonlinear sources*, J. Differential Equations **121** (1995), 329–383.

[25] JS. Gans, *The economic consequences of R = 1: towards a workable behavioural epidemiological model of pandemics*, Review of Economic Analysis **14(1)** (2022), 3–25.

[26] B. Gidas, WM. Ni, L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 203–243.

[27] FJ, Hasanov, MH. Al Rasasi, SS. Alsayaary, Z. Alfawzan, *Money demand under a fixed exchange rate regime: the case of Saudi Arabia*, Journal of Applied Economics, **25(1)** (2022), 385–411.

[28] N. Hirano, N. Mizoguch, *Positive unstable periodic solutions for super-linear parabolic equations*, Proc. Amer. Math. Soc. **123** (1995), 1487–1495.

[29] R. Huang, Y. Wang, Y. Ke, *Existence of non-trivial non-negative periodic solutions for a class of degenerate parabolic equations with non-local terms*, Dis. Cont. Dyn. Systems-series B. **5(4)** (2005), 1005–1014.

[30] OA. Ladyzhenskaja, VA. Solonnikov, NN. Ural'tzeva, *Linear and quasilinear equations of parabolic type*, in Trans. Math. Merro., Vol. **23**, Amer. Math. Soc., Providence, RI, (1968).

[31] AC. Lazer, PJ.McKenna, *Large-amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis*, Siam Review, **32(4)** (1990), 537–578.

[32] K. Li, L. Ruan, A. Yang, *Periodic entropy solution to a conservation law with nonlocal source arising in radiative gas*, Journal of Mathematical Analysis and Applications 512(1) (2022), 126–117.

[33] I. Marouani, T. Guesmi, H. Hadj Abdallah, BM. Alshammari, K. Alqunun, AS. Alshammari, S. Rahmani, *Combined economic emission dispatch with and without consideration of PV and wind energy by using various optimization techniques*, a review, Energies, **15(12)** (2022), 4472.

[34] P. Marcellini, *Regularity and existence of solutions of elliptic equations with p,q-growth conditions*, J. Differential Equations **90(1)** (1991), 1–30.

[35] P. Marcellini, *The stored-energy for some discontinuous deformations in nonlinear elasticity, Partial differential equations and the calculus of variations*, Vol. II, vol. **2** (1989), 767–786.

[36] N. Mizoguchi, *Periodic solutions for degenerate diffusion equations*, Indiana Univ. Math. J. **44** (1995), 413–432.

[37] M. Nakao, *Periodic solutions of some nonlinear degenerate parabolic equations*, J. Math. Anal. Appl. **104** (1984), 554–567.

[38] Y. Ohara, *L^∞-estimates of solutions of some nonlinear degenerate parabolic equations*, Nonlinear Anal. **18** (1992), 413–426.

[39] J. Ok, *Partial regularity for general systems of double phase type with continuous coefficients*, Nonlinear Anal., **177** (2018), 673–698.

[40] J. Ok, *Regularity for double phase problems under additional integrability assumptions*, Nonlinear Anal.**194**, 111408 (2020).

[41] MA. Ragusa, A. Tachikawa, *Regularity for minimizers for functionals of double phase with variable exponents*, Adv. Nonlinear Anal. **9(1)** (2020), 710–728.

[42] E. Scharfenaker, *Statistical equilibrium methods in analytical political economy*, Journal of Economic Surveys, **36(2)** (2022), 276–309.

[43] TI. Seidman, *Periodic solutions of a nonlinear parabolic equation*, J. Differential Equations **19**, (1975), 242–257.

[44] A. Soares, CM. Caloi, RC. Bassanezi, *Numerical simulations of the SEIR epidemiological model with population heterogeneity to assess the Efficiency of Social Isolation in Controlling COVID-19 in Brazil*, Trends in Computational and Applied Mathematics. **23(2)** (2022), 257–272.

[45] SA. Temghart, A. Kassidi, C. Allalou, A. Abbassi, *On an elliptic equation of Kirchhoff type problem via topological degree*, Nonlinear Studies, **28(4)** (2021).

[46] VV. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Izv. Akad. Nauk SSSR, Ser. Mat. **50(4)** (1986), 675–710.

[47] VV. Zhikov, *On some variational problems*, Russ. J. Math. Phys.**5(1)** (1997), 105–116.

[48] W. Zhuoqun, Z. Junning, Y. Jingxue, L. Huilai, *Nonlinear Diffusion Equations*, Jilin Univ. Press, Changcun, (1996).