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Generalized integral inequality and application on partial stability analysis

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Abstract. The method of Lyapunov is one of the most effective methods for the analysis of the partial stability of dynamical systems. Different authors develop the problem of partial practical stability based on Lyapunov techniques. In this paper, we investigate the partial practical stability of linear time–invariant perturbed systems based on the integral inequalities of the Gronwall type, in particular of Gamidov's type. We derive some sufficient conditions that guarantee global practical uniform exponential stability with respect to a part of the variables of linear time–invariant perturbed systems. Also, we have developed the local partial practical stability of nonlinear systems. Further, we provide two examples to support our findings.

1. Introduction

In 1892, Lyapunov, a Russian mathematician, mechanician and physicist, proposed the concept of the stability of motion. He provided general research methods in his doctoral dissertation [10] "The general problem of the stability of motion", in which he set the basis for the theory of stability.

In the study of nonlinear systems, in particular the study of dynamical systems, it is not possible to study the stability of all variables due to technological difficulties, the limitation of practical conditions, or it is not necessary to study all variables considering the actual need. In consequence, the study of the partial stability of differential equations becomes more important. As well, partial stability is used extensively in science and technology. For instance the absolute stability of famous Lurie adjusting systems can be changed into a problem of partial stability. In a word, it is of practical significance to study the partial stability of differential equations.

A. M. Lyapunov [11], the founder of the modern theory of stability, was the first who formulated the question of partial stability. Later work by V. V. Rumyantsev [12] attracted the attention of numerous mathematicians in the world to this issue.

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Lyapunov functions are used to prove the partial stability criteria of differential systems. A function is called a Lyapunov candidate function if it has the possibility to prove the stability of the differential system as an equilibrium, with the development of Lyapunov's first and second methods, more and more work is based on Lyapunov methods to study the stability with respect to a part of the variables of differential systems, see [13–15].

When origin is not a trivial solution, we can investigate the asymptotic stability of the solution to a small neighborhood of origin. The objective is to analyze the asymptotic stability of a system whose solution behavior is a small ball of state space or close to it. It is guaranteed that all state trajectories are bounded and close to a sufficiently small neighborhood of the origin. In this sense, the limit boundedness of solutions of systems, or the possibility of convergence of solutions often need to be analyzed on a ball centered on the origin, which is called "Practical Stability".

In particular, partial practical stability of differential equations is well developed in [2–4].

Recently, in [8], Ezzine et al. was the first who introduced and developed the concept of partial practical stability of differential equations via Lyapunov techniques.

Nevertheless, constructing an appropriate Lyapunov function is still a challenging task. The novelty of what we do is the development of the problem of partial practical stability of perturbed systems based on the explicit solution formed through integral inequalities of the Gronwall type, in particular Gamidov's inequality.

The present article provides a new class of integral inequalities of Gamidov's type stated by Ben Maklouf et al. in [1]. Then as an application, the linear time–invariant perturbed systems is considered, and the partial practical stability criteria is obtained.

The purpose of this paper is to state partial practical stability theorems for linear time–invariant perturbed systems. The authors use the method of integral inequality to establish a stability criterion. Also, the local partial practical stability of nonlinear systems.

The main structure of this paper is as follows: In Section 1, we introduce some needed preliminaries and definitions related to the partial stability. In Section 2, we present partial practical stability theorems for linear time–invariant perturbed systems by using the method of integral inequalities. In Section 3, the problem of local partial practical stability of nonlinear systems is discussed. As a concluding point, two examples are provided to illustrate the theoretical outcomes.

2. Preliminaries

We consider the following linear time–invariant system:

$$\dot{x}(t) = \mathcal{A}x(t), \tag{1}$$

where

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & 0 \\ 0 & \mathcal{A}_2 \end{pmatrix}, \quad x := (x_1, x_2) \in \mathrm{R}^{n_1} \times \mathrm{R}^{n_2},\ n_1 > 0,\ n_2 \geq 0,\ n_1 + n_2 = n.$$

- $\mathcal{A}_1$ is a constant $n_1 \times n_1$ matrix.
- $\mathcal{A}_2$ is a constant $n_2 \times n_2$ matrix.

The linear time–invariant system (1) might be expressed as the following:

$$\begin{cases} \dot{x}_1(t) = \mathcal{A}_1 x_1(t) \\ \\ \dot{x}_2(t) = \mathcal{A}_2 x_2(t), \end{cases} \tag{2}$$

with initial condition $x_0 := (x_{1_0}, x_{2_0}) \in \mathrm{R}^{n_1} \times \mathrm{R}^{n_2}$.

Definition 2.1. *[8] The equilibrium point $x = 0$ of linear time–invariant system* (2) *is said to be globally uniformly exponentially stable with respect to x_1, if there exists a pair of positive constants ζ_1 and ζ_2 such that for all $x_0 \in \mathrm{R}^n$, the following inequality is:*

$$\|x_1(t)\| \leq \zeta_1 \|x_0\| e^{-\zeta_2 t}, \quad \forall t \geq 0.$$

Assume that some parameters are excited or perturbed, and the perturbed system has the following form:

$$\begin{cases} \dot{x}_1(t) = \mathcal{A}_1(t)x_1(t) + \mathcal{G}(t, x_1(t), x_2(t)) \\ \\ \dot{x}_2(t) = \mathcal{A}_2 x_2(t), \end{cases} \tag{3}$$

with the same initial conditions and $\mathcal{G} : \mathrm{R}_+ \times \mathrm{R}^{n_1} \times \mathrm{R}^{n_2} \longrightarrow \mathrm{R}^{n_1}$ is a continuous function in (t, x_1, x_2), locally Lipschitz in x, uniformly in t.

We denote by $x(t, t_0, x_0) := (x_1(t, t_0, x_0), x_2(t, t_0, x_0))$ the solutions of the perturbed system (3) with initial condition $x(t_0) := x_0 := (x_{1_0}, x_{2_0})$.

Assume that there exists t such that $\mathcal{G}(t, 0, 0) \neq 0$, i.e., the perturbed system (3) does not have the trivial solution $x \equiv 0$.

The study of the stability with respect to a part of the variables of the solutions of the perturbed system (3) leads back to the study of the stability with respect to a part of variables of a ball centered at the origin:

$$\mathcal{B}_r := \{x \in \mathrm{R}^n : \|x\| \leq r\}, \ r > 0.$$

Definition 2.2. *[8]*

i) The ball $\mathcal{B}_r$ is said to be globally uniformly exponentially stable with respect to x_1, if there exists a pair of positive constants k_1 and k_2 such that for all $x_0 \in \mathrm{R}^n$, the following inequality is:

$$\|x_1(t)\| \leq k_1 \|x_0\| e^{-k_2 t} + r, \quad \forall t \geq 0. \tag{4}$$

ii) The perturbed system (3) is said to be globally practically uniformly exponentially stable with respect to x_1, if there exists $r > 0$ such that $\mathcal{B}_r$ is globally uniformly exponentially stable with respect to x_1.

Remark 2.3. *Eq.* (4) *implies that $x_1(t)$ will be bounded by a small bound $r > 0$, that is $\|x_1(t)\|$ will be small for sufficiently large t. It means that the solution given in* (4) *will be uniformly ultimately bounded for sufficiently large t. This means that solution given in* (4) *will be uniformly ultimately bounded for sufficiently large t. The factor k_2 in Eq.* (4) *is called the convergence speed, whereas the factor k_1 is called the transient estimate.*

It is also worth noticing that in the previous definition if we take $r = 0$, then we recover the standard concept of the global uniform exponential stability with respect to a part of the variables of the origin considered as an equilibrium point.

3. Global partial practical exponential stability

Our objective now is to state sufficient conditions to provide the global practical uniform exponential stability with respect to a part of the variables of the linear time–invariant perturbed system (3). In fact if we suppose that the perturbation term is bounded, then the origin is not necessarily an equilibrium point. For that reason, we will analyze the convergence of the solutions toward a neighborhood of origin.

Remark 3.1. *Different authors are well developed the problem of practical stability with respect to a part of the variables of differential equations via Lyapunov functions, see [6–8]. The construction of an appropriate Lyapunov functions is not always possible, which motivates us to look for another method. Our approach in this paper is to analyze the partial stability by using the explicit solution form and it is based on integral inequalities in particular of Gamidov's type.*

We suppose that the nominal system is globally uniformly exponentially stable with respect to x_1. A basic result in systems theory is that

$$\sigma(\mathcal{A}_1) \subset C^-,$$

where $\sigma(\mathcal{A}_1)$ denotes the set of eigenvalues of a the matrix $\mathcal{A}_1$. Under the condition, we obtain $\mathcal{R}e\lambda(\mathcal{A}_1) < 0$, where $\mathcal{R}e\lambda(\mathcal{A}_1)$ denotes the real parts of the eigenvalues of matrix $\mathcal{A}_1$.

First, we suppose the following assumption required for the stability purposes.

$(\mathcal{H}_1)$ Assume that $\mathcal{R}e\lambda(\mathcal{A}_1) < 0$. Note that the assumption $(\mathcal{H}_1)$ implies that

$$\|e^{t\mathcal{A}_1}\| \leq \alpha e^{-\beta t}, \quad \forall t \geq 0,$$

for a certain $\alpha > 0$ and

$$\beta \leq \min_{1\leq i\leq \eta} |\mathcal{R}e\lambda_i(\mathcal{A}_1)|.$$

$(\mathcal{H}_2)$ There exists a continuous non–negative known function $\varphi(t)$ such that

$$\|\mathcal{G}(t, x_1, x_2)\| \leq \varphi(t), \ \ \forall x = (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \quad \forall t \geq 0.$$

$(\mathcal{H}_3)$ The non–negative continuous function $\varphi(t)$ is such that

$$\lim_{t\to\infty} \varphi(t) = 0.$$

Theorem 3.2. *Under assumptions* $(\mathcal{H}_1), (\mathcal{H}_2)$ *and* $(\mathcal{H}_3)$ *the linear time–invariant perturbed system* (3) *is globally practically uniformly exponentially stable with respect to* x_1.

Proof. The solution of the sub–system x_1 with initial condition x_{1_0} of the linear time–invariant perturbed system (3) is expressed as follows:

$$x_1(t) = e^{t\mathcal{A}_1} x_{1_0} + \int_0^t e^{(t-s)\mathcal{A}_1} \mathcal{G}(s, x_1(s), x_2(s))ds, \quad \forall t \geq 0.$$

Thus, we obtain

$$\|x_1(t)\| = \|e^{t\mathcal{A}_1}\| \, \|x_{1_0}\| + \int_0^t \|e^{(t-s)\mathcal{A}_1}\| \, \|\mathcal{G}(s, x_1(s), x_2)\|ds, \quad \forall t \geq 0.$$

Based on assumption $(\mathcal{H}_1)$, one obtains

$$\begin{aligned}\|x_1(t)\| &\leq \alpha e^{-\beta t}\|x_{1_0}\| + \int_0^t \alpha e^{-\beta(t-s)}\|\mathcal{G}(s, x_1(s), x_2(s))\|ds \\ &\leq \alpha e^{-\beta t}\|x_{1_0}\| + \int_0^t \alpha e^{-\beta(t-s)}\|\mathcal{G}(s, x_1(s), x_2(s))\|ds.\end{aligned}$$

The assumption $(\mathcal{H}_2)$, yields

$$\begin{aligned}\|x_1(t)\| &\leq \alpha e^{-\beta t}\|x_{1_0}\| + \int_0^t \alpha e^{-\beta(t-s)}\varphi(s)ds\\ &\leq \alpha e^{-\beta t}\|x_{1_0}\| + \alpha e^{-\beta t}\int_0^t e^{\beta s}\varphi(s)ds.\end{aligned}$$

Assumption $(\mathcal{H}_3)$ yields that there exists $\widetilde{\varphi} > 0$ such that

$$\varphi(t) \leq \widetilde{\varphi}, \quad \forall t \geq 0.$$

Thus, it yields that

$$\begin{aligned}\|x_1(t)\| &\leq \alpha e^{-\beta t}\|x_{1_0}\| + \alpha\widetilde{\varphi}e^{-\beta t}\int_0^t e^{\beta s}ds\\ &\leq \alpha e^{-\beta t}\|x_{1_0}\| + \frac{\alpha\widetilde{\varphi}}{\beta}.\end{aligned}$$

That is, we see

$$\|x_1(t)\| \leq \alpha e^{-\beta t}\|x_{1_0}\| + \frac{\alpha\widetilde{\varphi}}{\beta}, \quad \forall t \geq 0.$$

As a consequence, the linear time–invariant perturbed system (3) is practically uniformly exponentially stable with respect to x_1. □

A simple extension can be made if we replace the hypothesis $(\mathcal{H}_3)$ with the following:

$(\mathcal{H}_3')$ The continuous non–negative function $\varphi(t)$ satisfies

$$\int_0^\infty \varphi(t) < \infty,$$

or

$$\varphi(t) \leq \overline{\varphi} < \infty, \quad \forall t \geq 0.$$

Theorem 3.3. *Under assumptions $(\mathcal{H}_1), (\mathcal{H}_2)$ and $(\mathcal{H}_3')$ the linear time–invariant perturbed system* (3) *is globally practically uniformly exponentially stable with respect to x_1.*

Proof. The solution of the sub–system x_1 with initial condition x_{1_0} of the perturbed system (3) is the following:

$$x_1(t) = e^{t\mathcal{A}_1}x_{1_0} + \int_0^t e^{(t-s)\mathcal{A}_1}\mathcal{G}(s, x_1(s), x_2(s))ds, \quad \forall t \geq 0.$$

Based on assumption $(\mathcal{H}_1)$, it yields that

$$\|x_1(t)\| \leq \alpha e^{-\beta t}\|x_{1_0}\| + \int_0^t \alpha e^{-\beta(t-s)}\|\mathcal{G}(s, x_1(s), x_2(s))\|ds.$$

Using $(\mathcal{H}_2)$, we obtain that

$$\begin{aligned}\|x_1(t)\| &\le \alpha e^{-\beta t}\|x_{1_0}\| + \int_0^t \alpha e^{-\beta(t-s)}\varphi(s)ds \\ &\le \alpha e^{-\beta t}\|x_{1_0}\| + \alpha e^{-\beta t}\int_0^t e^{\beta s}\varphi(s)ds.\end{aligned}$$

Indeed, since the non–negative continuous function $\varphi(t)$ satisfies assumption $(\mathcal{H}_3')$, there exists $\kappa > 0$ such that

$$e^{-\beta t}\int_0^t e^{\beta s}\varphi(s)ds \le \kappa, \quad \forall t \ge 0,$$

where $\kappa = \min\left(\frac{\overline{\varphi}}{\beta}, \|\varphi\|_1\right)$, thus one gets

$$\|x_1(t)\| \le \alpha e^{-\beta t}\|x_{1_0}\| + \alpha\kappa, \quad \forall t \ge 0.$$

As a result, we arrive at

$$\|x_1(t)\| \le \alpha e^{-\beta t}\|x_0\| + \alpha\kappa, \quad \forall t \ge 0.$$

Hence, the linear time–invariant perturbed system is globally practically uniformly exponentially stable with respect to x_1. □

In the sequel our objective is to prove the practical uniform exponential stability with respect to a part of the variables of the linear time–invariant perturbed system (3) by using the generalized integral inequality of the Gamidov type.

In [9], Gamidov has demonstrated the following generalized integral inequality:

Lemma 3.4. *[9] Assume that*

$$\mathcal{U}(t) \le \phi(t) + \nu\int_0^t \xi(s)\mathcal{U}^d(s)ds,$$

where all functions are continuous and non–negative on $[0, b)$, $0 < d < 1$, *and* $b,\ \nu > 0$. *Then there exists a constant* $\rho > 0$ *such that*

$$\mathcal{U}(t) \le \phi(t) + \nu\rho^d\left(\int_0^t \xi^{\frac{1}{1-d}}(s)ds\right)^{1-d}.$$

Later on Abdellatif et al. [1] improved the previous lemma for $b = \infty$.

Lemma 3.5. *[1] Assume that*

$$\mathcal{U}(t) \le \phi(t) + \nu\int_0^t \xi(s)\mathcal{U}^d(s)ds,$$

where all functions are continuous and non–negative on $[0, \infty)$, $0 < d < 1$, *and* $\nu > 0$.
Then there exists a continuous non–negative function $\rho > 0$ *such that*

$$\mathcal{U}(t) \le \phi(t) + \nu\rho^d(t)\left(\int_0^t \xi^{\frac{1}{1-d}}(s)ds\right)^{1-d}.$$

In particular for $\xi(t) = e^{(1-d)\beta t}$ and $\phi(t) \leq \overline{m}$, they have established the next lemma.

Lemma 3.6. *[1] Assume that*

$$\mathcal{U}(t) \leq \overline{m} + \nu \int_0^t e^{(1-d)\beta s}\mathcal{U}^d(s)ds,$$

where $\mathcal{U}$ *is continuous and non–negative on* $[0,\infty)$, $0 < d < 1$, *and* $\overline{m},\ \nu > 0$.
Then

$$\mathcal{U}(t) \leq 2^{\frac{d}{1-d}}\overline{m} + \left(\frac{2^d\nu}{\beta}\right)^{\frac{1}{1-d}} e^{\beta t}.$$

A simple extension may be made if we replace $(\mathcal{H}_2)$ with the following hypothesis about the perturbed term.

$(\mathcal{H}'_2)$ There exists two continuous positive functions $\Gamma_1(t)$ and $\Gamma_2(t)$ such that

$$\|\mathcal{G}(t,x_1,x_2)\| \leq \Gamma_1(t)\|x_1\|^d + \Gamma_2(t), \quad 0 < d < 1,\ \forall x = (x_1,x_2) \in \mathrm{R}^{n_1} \times \mathrm{R}^{n_2},\ \forall t \geq 0,$$

where the function $\Gamma_1(\cdot)$ satisfies

$$\Gamma_1(t) \leq \widetilde{\Gamma}_1, \quad \forall t \geq 0,$$

and the function $\Gamma_2(\cdot)$ satisfies

$$\int_0^\infty e^{\beta s}\Gamma_2(s)ds = \widetilde{\Gamma}_2 < \infty.$$

Theorem 3.7. *Suppose that assumptions* $(\mathcal{H}_1)$ *and* $(\mathcal{H}'_2)$ *are satisfied, then the linear time–invariant perturbed system* (3) *is globally practically uniformly exponentially stable with respect to* x_1.

Proof. The solution of the sub–system x_1 with initial condition x_{1_0} of the perturbed system (3) is the following:

$$x_1(t) = e^{t\mathcal{A}_1}x_{1_0} + \int_0^t e^{(t-s)\mathcal{A}_1}\mathcal{G}(s,x_1(s),x_2(s))ds, \quad \forall t \geq 0.$$

Based on assumption $(\mathcal{H}_1)$, it yields that

$$\|x_1(t)\| \leq \alpha e^{-\beta t}\|x_{1_0}\| + \int_0^t \alpha e^{-\beta(t-s)}\|\mathcal{G}(s,x_1(s),x_2(s))\|ds.$$

By assumption $(\mathcal{H}'_2)$ we obtain that

$$\begin{aligned}\|x_1(t)\| &\leq \alpha e^{-\beta t}\|x_{1_0}\| + \alpha e^{-\beta t}\int_0^t e^{\beta s}\left(\Gamma_1(s)\|x_1(s)\|^d + \Gamma_2(s)\right)ds \\ &\leq \alpha e^{-\beta t}\|x_{1_0}\| + \alpha e^{-\beta t}\int_0^t \left(\widetilde{\Gamma}_1 e^{\beta s}\|x_1(s)\|^d + e^{\beta s}\Gamma_2(s)\right)ds.\end{aligned}$$

Multiply each side with $e^{\beta t}$, one obtains

$$e^{\beta t}\|x_1(t)\| \leq \alpha\|x_{1_0}\| + \alpha\widetilde{\Gamma}_1\int_0^t e^{\beta s}\|x_1(s)\|^d ds + \alpha\int_0^t e^{\beta s}\Gamma_2(s)ds.$$

Let set the function $\mathcal{U}(t)$ by $e^{\beta t}\|x_1(t)\|$, we get

$$\mathcal{U}(t) \leq \alpha\|x_{1_0}\| + \alpha\widetilde{\Gamma}_1 \int_0^t e^{(1-d)\beta s}\mathcal{U}^d(s)ds + \alpha \int_0^t e^{\beta s}\Gamma_2(s)ds.$$

Since $\int_0^{\infty} e^{\beta s}\Gamma_2(s)ds = \widetilde{\Gamma}_2 < \infty$, we have

$$\mathcal{U}(t) \leq \alpha\|x_{1_0}\| + \alpha\widetilde{\Gamma}_1 \int_0^t e^{(1-d)\beta s}\mathcal{U}^d(s)ds + \alpha\widetilde{\Gamma}_2.$$

Accordingly,

$$\mathcal{U}(t) \leq \bar{m} + \nu \int_0^t e^{(1-d)\beta s}\mathcal{U}^d(s)ds,$$

where $\overline{m} = \alpha\|x_{1_0}\| + \alpha\widetilde{\Gamma}_2$, and $\nu = \alpha\widetilde{\Gamma}_1$.

From the Gamidov inequality (Lemma 3.6), it yields that

$$\mathcal{U}(t) \leq 2^{\frac{d}{1-d}}\overline{m} + \left(\frac{2^d\nu}{\beta}\right)^{\frac{1}{1-d}} e^{\beta t}.$$

We can see that

$$e^{\beta t}\|x_1(t)\| \leq 2^{\frac{d}{1-d}}\overline{m} + \left(\frac{2^d\nu}{\beta}\right)^{\frac{1}{1-d}} e^{\beta t}.$$

Then we obtain

$$\|x_1(t)\| \leq 2^{\frac{d}{1-d}}\alpha\|x_0\|e^{-\beta t} + 2^{\frac{d}{1-d}}\alpha\widetilde{\Gamma}_2 + \left(\frac{2^d\alpha\widetilde{\Gamma}_1}{\beta}\right)^{\frac{1}{1-d}}, \quad \forall x = (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \quad \forall t \geq 0,$$

thus the linear time–invariant perturbed system (3) is globally practically uniformly exponentially stable with respect to x_1. □

Lastly, for the standard case, we assume that the perturbed term $\mathcal{G}(t, x)$ meets the following hypothesis.

($\mathcal{A}_2$) There exists a continuous non–negative known function $\psi(t)$ such that

$$\|\mathcal{G}(t, x_1, x_2)\| \leq \psi(t)\|x_1\|, \quad \forall x = (x_1, x_2) \in \mathrm{R}^{n_1} \times \mathrm{R}^{n_2}, \;\; \forall t \geq 0,$$

where $\psi(t)$ is bounded and satisfies

$$\sup_{t\geq 0} \psi(t) \leq \Theta\beta, \quad 0 < \Theta < \frac{1}{\alpha}.$$

Theorem 3.8. *Suppose that assumptions* ($\mathcal{H}_1$) *and* ($\mathcal{A}_2$) *are satisfied, then the linear time–invariant perturbed system* (3) *is globally uniformly exponentially stable with respect to* x_1.

Proof. Similar to the previous theorem, we have

$$x_1(t) = e^{t\mathcal{A}_1}x_{1_0} + \int_0^t e^{(t-s)\mathcal{A}_1}\mathcal{G}(s, x_1(s), x_2(s))ds, \quad \forall t \geq 0.$$

Combining $(\mathcal{H}_1)$ and $(\mathcal{A}_1)$, one obtains

$$\begin{aligned}\|x_1(t)\| &\leq \alpha e^{-\beta t}\|x_{1_0}\| + \int_0^t \alpha e^{-\beta(t-s)}\|\mathcal{G}(s, x_1(s), x_2(s))\|ds \\ &\leq \alpha e^{-\beta t}\|x_{1_0}\| + \int_0^t \alpha e^{-\beta(t-s)}\psi(s)\|x_1(s)\|ds.\end{aligned}$$

Multiplying both sides by $e^{\beta t}$, one gets

$$\|x_1(t)\|e^{\beta t} \leq \alpha\|x_{1_0}\| + \int_0^t \alpha e^{\beta s}\psi(s)\|x_1(s)\|ds.$$

We suppose that $\eta(t) = \|x_1(t)\|e^{\beta t}$, one gets

$$\eta(t) \leq \alpha\|x_{1_0}\| + \alpha\int_0^t \psi(s)\eta(s)ds.$$

Applying Gronwall's lemma [5], we get

$$\eta(t) \leq \alpha\|x_{1_0}\|e^{\alpha\int_0^t \psi(s)ds} \leq \alpha\|x_{1_0}\|\exp(\alpha\Theta\beta t).$$

Thus, we deduce that

$$\|x_1(t)\| \leq \alpha\|x_0\|\exp\left(-\beta(1-\alpha\Theta)t\right), \quad \forall x = (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \ \forall t \geq 0.$$

Hence the linear time–invariant perturbed system (3) is globally uniformly exponentially stable with respect to x_1. □

In the sequel we will consider the following linear time–invariant perturbed system associated to the system (3):

$$\begin{cases}\dot{x}_1(t) = \mathcal{A}_1(t)x_1(t) + \mathcal{G}(t, x_1(t), x_2(t)) + \lambda(t, x_1(t), x_2(t)) \\ \\ \dot{x}_2(t) = \mathcal{A}_2x_2(t),\end{cases} \tag{5}$$

where $\lambda : \mathbb{R}_+ \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ is a continuous function in (t, x_1, x_2), Lipschitz in x, uniformly in t.

$(\mathcal{A}_3)$ There exists a non–negative constant Λ such that

$$\|\lambda(t, x_1, x_2)\| \leq \Lambda\|x_1(t)\|, \quad \forall x = (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}.$$

Theorem 3.9. *Suppose that assumptions $(\mathcal{H}_1)$, $(\mathcal{H}_2')$ and $(\mathcal{A}_3)$ are satisfied, then the linear time–invariant perturbed system* (5) *is globally practically uniformly exponentially stable with respect to x_1.*

To prove the previous theorem, we need to recall a generalized integral inequality of the Gamidov type, proved in [1].

Lemma 3.10. *[1] Assume that*

$$\mathcal{U}(t) \le \overline{m} + \int_0^t \left(\nu e^{(1-d)\beta s} \mathcal{U}^d(s) + \iota\, \mathcal{U}(s) \right) ds,$$

where $\mathcal{U}$ *is continuous and non–negative on* $[0, \infty)$, $0 < d < 1$, *and* $\overline{m},\ \nu > 0$, *and* $\beta,\ \iota$ *such that* $0 \le \iota < \beta$.

Then

$$\mathcal{U}(t) \le 2^{\frac{d}{1-d}} \overline{m} e^{\iota t} + \left(\frac{2^d \nu}{\beta - \nu} \right)^{\frac{1}{1-d}} e^{\beta t}.$$

Proof of Theorem 3.9. The solution of the sub–system x_1 with initial condition x_{1_0} of the perturbed system (5) has the following form:

$$x_1(t) = e^{t\mathcal{A}_1} x_{1_0} + \int_0^t e^{(t-s)\mathcal{A}_1} \left(\mathcal{G}(s, x_1(s), x_2(s)) + \lambda(s, x_1(s), x_2(s)) \right) ds, \quad \forall t \ge 0.$$

Since the unperturbed nominal system is globally uniformly exponentially stable, one obtains

$$\|x_1(t)\| \le \alpha e^{-\beta t} \|x_{1_0}\| + \int_0^t \alpha e^{-\beta(t-s)} \|\mathcal{G}(s, x_1(s), x_2(s)) + \lambda(s, x_1(s), x_2(s))\| ds.$$

Based on assumption $(\mathcal{A}_3)$, for $\Lambda = \dfrac{\Theta\beta}{\alpha},\ 0 < \Theta < 1$, it follows that

$$\|\lambda(t, x_1, x_2)\| \le \frac{\Theta\beta}{\alpha} \|x_1(t)\|, \quad \forall x = (x_1, x_2) \in \mathrm{R}^{n_1} \times \mathrm{R}^{n_2},\ \forall t \ge 0. \tag{6}$$

Applying assumption $(\mathcal{H}_2')$ and Eq. (12), we see that

$$\|x_1(t)\| \le \alpha e^{-\beta t} \|x_{1_0}\| + \alpha e^{-\beta t} \int_0^t e^{\beta s} \left(\Gamma_1(s) \|x_1(s)\|^d + \frac{\Theta\beta}{\alpha} \|x_1(s)\| + \Gamma_2(s) \right) ds.$$

Multiply each side with $e^{\beta t}$, yields

$$\|x_1(t)\| e^{\beta t} \le \alpha \|x_{1_0}\| + \alpha \int_0^t e^{\beta s} \left(\Gamma_1(s) \|x_1(s)\|^d + \frac{\Theta\beta}{\alpha} \|x_1(s)\| \right) ds + \alpha \int_0^t e^{\beta s} \Gamma_2(s) ds.$$

Let $\mathcal{U}(t) = e^{\beta t} \|x_1(t)\|$, and with the assumption $(\mathcal{H}_2')$, we arrive at

$$\begin{aligned} \mathcal{U}(t) &\le \alpha \|x_{1_0}\| + \int_0^t \left(e^{(1-d)\beta s} \alpha \widetilde{\Gamma}_1 \mathcal{U}^d(s) + \Theta\beta \mathcal{U}(s) \right) ds + \alpha \int_0^\infty e^{\beta s} \Gamma_2(s) ds \\ &\le \alpha \|x_{1_0}\| + \int_0^t \left(e^{(1-d)\beta s} \alpha \widetilde{\Gamma}_1 \mathcal{U}^d(s) + \Theta\beta \mathcal{U}(s) \right) ds + \alpha \widetilde{\Gamma}_2. \end{aligned}$$

Thus,

$$\mathcal{U}(t) \le \overline{m} + \int_0^t \left(\nu e^{(1-d)\beta s} \mathcal{U}^d(s) + \iota\, \mathcal{U}(s) \right) ds,$$

where $\overline{m} = \alpha\|x_{1_0}\| + \alpha\widetilde{\Gamma}_2,\ \nu = \alpha\widetilde{\Gamma}_1,$ and $\iota = \Theta\beta$.

By applying the Gamidov lemma 3.10, one gets

$$\mathcal{U}(t) \leq 2^{\frac{d}{1-d}}\overline{m}e^{\iota t} + \left(\frac{2^d\nu}{\beta-\iota}\right)^{\frac{1}{1-d}} e^{\beta t}.$$

As a result, we have

$$e^{\beta t}\|x_1(t)\| \leq 2^{\frac{d}{1-d}}\overline{m}e^{\iota t} + \left(\frac{2^d\nu}{\beta-\iota}\right)^{\frac{1}{1-d}} e^{\beta t}.$$

That is,

$$\|x_1(t)\| \leq 2^{\frac{d}{1-d}}\alpha\|x_0\|e^{-(1-\Theta)\beta t} + \left(\frac{2^d\alpha\widetilde{\Gamma}_1}{(1-\Theta)\beta}\right)^{\frac{1}{1-d}} + 2^{\frac{d}{1-d}}\alpha\widetilde{\Gamma}_2, \quad \forall x = (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2},\ \forall t \geq 0.$$

Consequently the perturbed system (5) is globally practically uniformly exponentially stable with respect to x_1. □

Example 3.11. *Let us consider the following system:*

$$\begin{cases} \dot{x}_1(t) = \mathcal{A}_1 x_1(t) + \mathcal{G}(t, x_1(t), x_2(t) \\ \\ \dot{x}_2(t) = \mathcal{A}_2 x_2(t), \end{cases} \tag{7}$$

where $x = (x_1, x_2) \in \mathbb{R}^4$, $x_1 = (z_1, z_2) \in \mathbb{R}^2$, $x_2 = (z_3, z_4) \in \mathbb{R}^2$.

$$\mathcal{A}_1 = \begin{pmatrix} -2 & 0 \\ -2 & -2 \end{pmatrix}, \quad \mathcal{A}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{G}(t, x_1, x_2) = \begin{pmatrix} \frac{\sqrt{|x_1|}}{1+t^2}\cos^2(z_3) + e^{-\vartheta t} \\ 0 \end{pmatrix}, \ \vartheta > 2,$$

with initial condition $x_0 = (x_{1_0}, x_{2_0})$, $x_{1_0} = (z_{1_0}, z_{2_0})$, *and* $x_{2_0} = (z_{3_0}, z_{4_0})$.

Notice that, with the perturbed term $\mathcal{G}$, *the fact that the function* $\sqrt{|x_1|}$ *is not Lipschitzian around zero is not problematic for the uniqueness of the solutions because the concept of practical stability is based on the convergence of solutions towards a neighborhood of the origin.*

The solution of the sub–system with respect to the variable x_2 *of the perturbed system* (7), *is provided by the next:*

$$x_2(t) = \begin{pmatrix} z_{30}(t, w)\cos(t) - z_{40}(t, w)\sin(t) \\ z_{30}(t, w)\sin(t) + z_{40}(t, w)\cos(t) \end{pmatrix}.$$

That is,

$$\|x_2(t)\| = \|x_{2_0}\|$$

. *Hence, for all* $t \geq 0$, *and all* $x_{2_0} \in \mathbb{R}^2$ *with* $\|x_{2_0}\| \leq m'$, *one obtains*$\|x_2(t)\| \leq m'$. *Thus, the state* $x_2(t)$ *is globally uniformly bounded as can seen from the Figure.* 1.

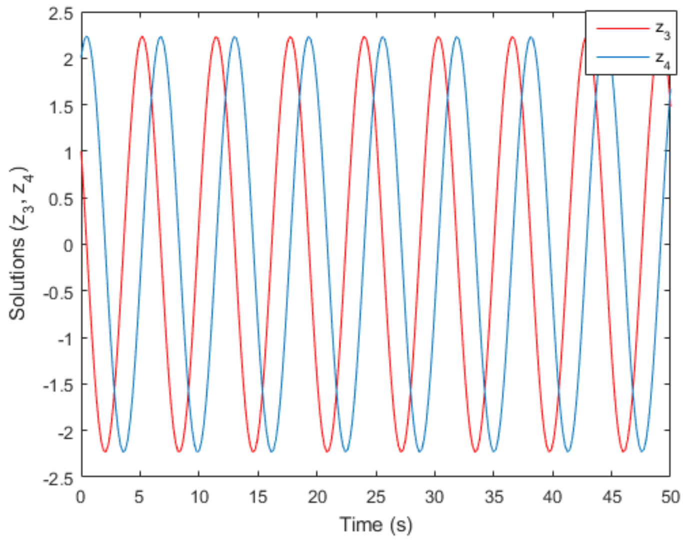

Figure 1: Time evolution of the states $z_3(t)$ and $z_4(t)$ of the system (7)

The system (7) *can be considered as a perturbed system of the following linear time–invariant system:*

$$\begin{cases} \dot{x}_1(t) = \mathcal{A}_1 x_1(t) \\ \\ \dot{x}_2(t) = \mathcal{A}_2 x_2(t), \end{cases} \tag{8}$$

The nominal system is globally uniformly exponentially with respect to x_1, *since the sub–system* $\dot{x}_1(t) = \mathcal{A}_1 x_1(t)$ *satisfies* $\mathcal{R}e\lambda(\mathcal{A}_1) < 0$. *Figure* 2 *shows the trajectories of the sub–system* x_1 *of the linear time–invariant system* (8).

Otherwise, we have

$$\|\mathcal{G}(t, x_1, x_2)\| \leq \frac{\sqrt{|x_1|}}{1+t^2} + e^{-\vartheta t}.$$

Consequently assumption $(\mathcal{H}_2')$ *is satisfied with* $\Gamma_1(t) = \dfrac{1}{1+t^2}$, $\Gamma_2(t) = e^{-\vartheta t}$.

Applying Theorem 3.8 we deduce that the linear time–invariant perturbed system (7) *is globally practically uniformly exponentially stable with respect to* x_1. *For visual simulation, we select* $\vartheta = 3$. *Figure* 3 *shows the trajectories of the sub–system* x_1 *of the perturbed system* (7).

Remark 3.12. *We remark that we cannot prove the global practical uniform exponential stability in the standard sense, i.e., with respect to all the variables, since the sub–system with respect to the variable* x_2 *is globally uniformly bounded.*

Remark 3.13. *Note that we cannot prove the practical uniform exponential stability for the previous example with respect to all the variables, since the solutions of the sub–system with respect to the variables* y_3 *and* y_4 *are periodic, which implies that are globally uniformly bounded.*

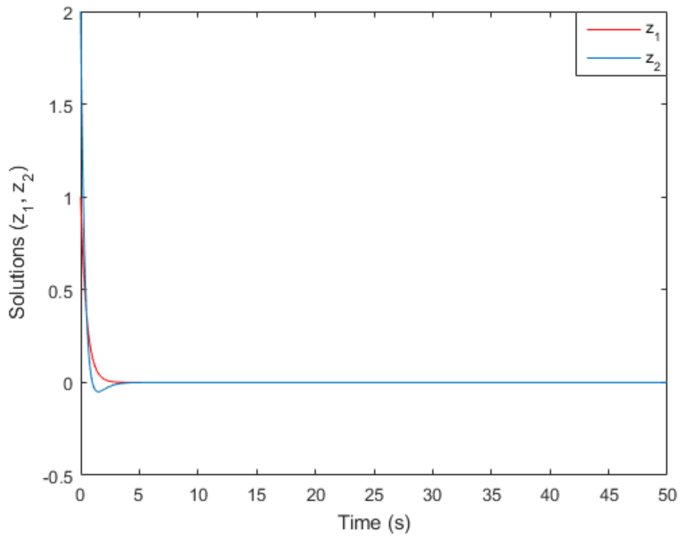

Figure 2: Time evolution of the states $z_1(t)$ and $z_2(t)$ of the nominal system (8)

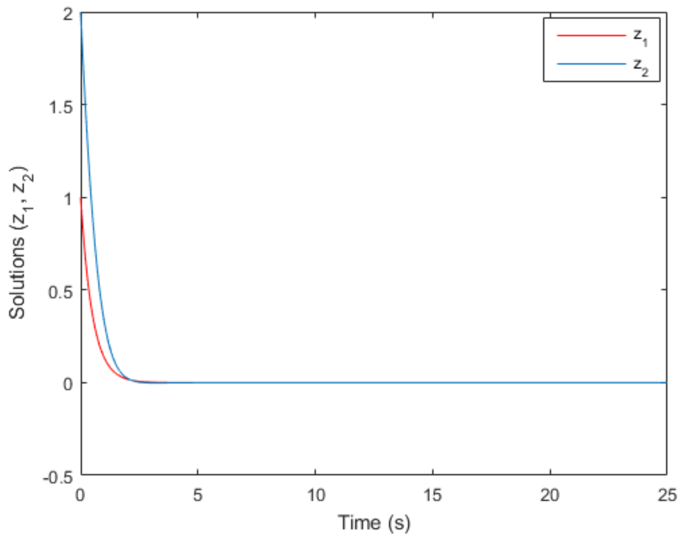

Figure 3: Time evolution of the states $z_1(t)$ and $z_2(t)$ of the perturbed system (7)

4. Local partial practical stability of nonlinear system

We consider the following nonlinear system:

$$\begin{cases} \dot{x}_1(t) = \mathcal{F}_1(x_1(t)) \\ \dot{x}_2(t) = \mathcal{F}_2(x_2(t)), \end{cases} \tag{9}$$

where $x = (x_1, x_2) \in \mathrm{R}^{n_1} \times \mathrm{R}^{n_2}$ is the state vector, $\mathcal{F}_1$, $\mathcal{F}_2$ are two smooth functions defined respectively on R^{n_1}, R^{n_2}, with $\mathcal{F}_1(0) = 0$, $\mathcal{F}_2(0) = 0$.

We may use a linear approximation to explore the partial stability problem for such systems, since $\mathcal{F}_1$ and $\mathcal{F}_2$ are smooth functions, locally on a certain neighborhood of the origin $\mathcal{V}(0)$, thus we may write $\mathcal{F}_1$ and $\mathcal{F}_2$ as follows:

$$\mathcal{F}_1(x_1(t)) = \widetilde{\mathcal{A}}_1 x_1(t) + \varphi_1(x_1(t)),$$

and

$$\mathcal{F}_2(x_2(t)) = \widetilde{\mathcal{A}}_2 x_2(t) + \varphi_2(x_2(t)),$$

where $\widetilde{\mathcal{A}}_1 = \dfrac{\partial \mathcal{F}_1(0)}{\partial x_1}$, $\widetilde{\mathcal{A}}_2 = \dfrac{\partial \mathcal{F}_2(0)}{\partial x_2}$, with $\lim\limits_{x_1 \to 0} \dfrac{\|\varphi_1(x_1)\|}{\|x_1\|} = 0$, and $\lim\limits_{x_2 \to 0} \dfrac{\|\varphi_2(x_2)\|}{\|x_2\|} = 0$.

Next we prove the local partial uniform exponential stability with respect to a small ball for nonlinear system. We will investigate the asymptotic behavior of the solutions with respect to a part of the variables in the sense that the trajectories converge to a small ball centered at the origin $\mathcal{B}(0, r)$, $r > 0$ small enough in such away $\mathcal{B}(0, r) \subset \mathcal{B}(0, \tau)$ and for all solutions starting from $\mathcal{B}(0, \tau) \setminus \mathcal{B}(0, r)$, will approach partially exponentially to $\mathcal{B}(0, r)$ for t large enough. We consider the nonlinear system in the presence of the perturbation term $\mathcal{G}(t, x)$.

Theorem 4.1. *Suppose that* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2')$ *are satisfied, then there exists* $r > 0$ *such that* $\mathcal{B}_r$ *is partially uniformly exponentially stable with respect to system:*

$$\begin{cases} \dot{x}_1(t) = \mathcal{F}_1(x_1(t)) + \mathcal{G}(t, x_1(t), x_2(t)) \\ \\ \dot{x}_2(t) = \mathcal{F}_2(x_2(t)). \end{cases} \tag{10}$$

Proof. The perturbed system (10) may be expressed as the following for all $x \in \mathcal{V}(0)$:

$$\begin{cases} \dot{x}_1(t) = \widetilde{\mathcal{A}}_1 x_1(t) + \varphi_1(x_1(t)) + \mathcal{G}(t, x_1(t), x_2(t)) \\ \\ \dot{x}_2(t) = \widetilde{\mathcal{A}}_2 x_2(t) + \varphi_2(x_2(t)). \end{cases}$$

The solution of the sub–system x_1 with initial condition x_{1_0} of the above system is given by the following:

$$x_1(t) = e^{t\widetilde{\mathcal{A}}_1} x_{1_0} + \int_0^t e^{(t-s)\widetilde{\mathcal{A}}_1} \left(\mathcal{G}(s, x_1(s), x_2(s)) + \varphi_1(x_1(s))\right) ds, \quad \forall t \geq 0.$$

Similar to the the proof of Theorem (3.7) and based on assumption $(\mathcal{H}_1)$, one obtains

$$\|x_1(t)\| \leq \alpha e^{-\beta t} \|x_{1_0}\| + \int_0^t \alpha e^{-\beta(t-s)} \|\mathcal{G}(s, x_1(s), x_2(s)) + \varphi_1(x_1(s))\| ds.$$

Since,

$$\lim_{x_1 \to 0} \frac{\|\varphi_1(x_1)\|}{\|x_1\|} = 0,$$

then for a given constant $l > 0$, there exists $l_0 > 0$, such that $\forall x \in \mathcal{B}(0, l_0) \subset \mathcal{B}(0, l)$, for all $t \geq 0$, we obtain

$$\|\varphi_1(x_1)\| \leq l\|x_1\|.$$

For $l = \dfrac{\Theta\beta}{\alpha}$, it follows that

$$\|\varphi_1(x_1)\| \leq \frac{\Theta\beta}{\alpha}, \quad 0 < \Theta < 1.$$

By Assumption $\mathcal{H}_2'$, it follows immediately that

$$\|x_1(t)\| \leq \alpha e^{-\beta t}\|x_{1_0}\| + \int_0^t \alpha e^{-\beta(t-s)}\left(\Gamma_1\|x_1(s)\|^d + \Gamma_2(s) + \frac{\Theta\beta}{\alpha}\|x_1(s)\|\right)ds.$$

Multiplying both sides by $e^{\beta t}$, one see

$$e^{\beta t}\|x_1(t)\| \leq \alpha\|x_{1_0}\| + \int_0^t \alpha e^{\beta s}\left(\Gamma_1(s)\|x_1(s)\|^d + \frac{\Theta\beta}{\alpha}\|x_1(s)\|\right)ds + \alpha\int_0^t e^{\beta s}\Gamma_2(s)ds.$$

Let $\mathcal{U}(t) = e^{\beta t}\|x_1(t)\|$, then we obtain

$$\mathcal{U}(t) \leq \alpha\|x_{1_0}\| + \int_0^t \left(\alpha e^{(1-d)\beta s}\Gamma_1(s)\mathcal{U}^d(s) + \Theta\beta\mathcal{U}(s)\right)ds + \alpha\int_0^\infty e^{\beta s}\Gamma_2(s)ds.$$

By assumption ($\mathcal{H}_2'$), we find that

$$\mathcal{U}(t) \leq \alpha\|x_{1_0}\| + \int_0^t \left(e^{(1-d)\beta s}\alpha\widetilde{\Gamma}_1\mathcal{U}^d(s) + \Theta\beta\mathcal{U}(s)\right)ds + \alpha\widetilde{\Gamma}_2. \tag{11}$$

Thus, one obtains

$$\mathcal{U}(t) \leq \overline{m} + \int_0^t \left(\nu e^{(1-d)\beta s}\mathcal{U}^d(s) + \iota\,\mathcal{U}(s)\right)ds,$$

where $\overline{m} = \alpha\|x_{1_0}\| + \alpha\widetilde{\Gamma}_2,\ \nu = \alpha\widetilde{\Gamma}_1,$ and $\iota = \Theta\beta$.

Using the Gamidov lemma 3.10, it follows from Eq. (11) that

$$\mathcal{U}(t) \leq 2^{\frac{d}{1-d}}\overline{m}e^{\iota t} + \left(\frac{2^d\nu}{\beta-\iota}\right)^{\frac{1}{1-d}} e^{\beta t}.$$

As a result, we have

$$e^{\beta t}\|x_1(t)\| \leq 2^{\frac{d}{1-d}}\overline{m}e^{\iota t} + \left(\frac{2^d\nu}{\beta-\iota}\right)^{\frac{1}{1-d}} e^{\beta t}.$$

That is,

$$\|x_1(t)\| \leq 2^{\frac{d}{1-d}}\alpha\|x_0\|e^{-(1-\Theta)\beta t} + \left(\frac{2^d\alpha\widetilde{\Gamma}_1}{(1-\Theta)\beta}\right)^{\frac{1}{1-d}} + 2^{\frac{d}{1-d}}\alpha\widetilde{\Gamma}_2, \quad \forall x = (x_1, x_2) \in \mathrm{R}^{n_1} \times \mathrm{R}^{n_2},\ \forall t \geq 0.$$

As a result the ball $\mathcal{B}_r$ with $r = \left(\dfrac{2^d\alpha\widetilde{\Gamma}_1}{(1-\Theta)\beta}\right)^{\frac{1}{1-d}} + 2^{\frac{d}{1-d}}\alpha\widetilde{\Gamma}_2$ is uniformly exponentially stable with respect to the nonlinear system (10) with respect to x_1. □

Example 4.2. *We consider the following nonlinear system:*

$$\begin{cases} \dot{x}_1(t) = \mathcal{F}_1(x_1(t)) + \mathcal{G}(t, x_1(t), x_2(t)) \\ \\ \dot{x}_2(t) = \mathcal{F}_2(x_2(t)), \end{cases} \tag{12}$$

where $x = (x_1, x_2) \in \mathrm{R}^4,\ x_1 = (z_1, z_2) \in \mathrm{R}^2,\ x_2 = (z_3, z_4) \in \mathrm{R}^2,$

$$\mathcal{F}_1(x_1) = \begin{cases} z_1 \cos(z_1) - 2\sin(z_1) \\ \\ -z_1 - \sin(z_2)\cos(z_2) \end{cases},\ \mathcal{F}_2(x_2) = \begin{cases} \cos(z_4) \\ \\ -\cos(z_3) \end{cases},\ \mathcal{G}(t, x) = \begin{cases} \eta\sqrt{|y_1|} + e^{-2t},\ \eta > 0 \\ \\ 0. \end{cases}$$

with initial condition $x_0 = (x_{1_0}, x_{2_0})$, $x_{1_0} = (z_{1_0}, z_{2_0})$, *and* $x_{2_0} = (z_{3_0}, z_{4_0})$.

The state x_2 *of the system* (12) *is globally uniformly bounded as we have already explained in example 3.11, the nonlinear system* (12) *can be regarded as a perturbed system of the following form:*

$$\begin{cases} \dot{x}_1(t) = \mathcal{F}_1(x_1(t)) \\ \\ \dot{x}_2(t) = \mathcal{F}_2(x_2(t)). \end{cases} \tag{13}$$

The linear approximation for the associated unperturbed system is the following:

$$\begin{cases} \dot{x}_1(t) = \widetilde{\mathcal{A}}_1 x_1(t) \\ \\ \dot{x}_2(t) = \widetilde{\mathcal{A}}_2 x_2(t), \end{cases} \tag{14}$$

where

$$\widetilde{\mathcal{A}}_1 = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix},\quad \widetilde{\mathcal{A}}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Notice that, the previous approximation can be done in a small neighborhood of the origin, i.e., there exists $\tau > 0$, *such that for all* $x \in \mathcal{B}(0, \tau)$ *the passage from nonlinear system to linear system is possible. The point is to find a constant* $r > 0$ *small enough in such away* $\mathcal{B}(0, r) \subset \mathcal{B}(0, \tau)$ *and all solutions starting from* $\mathcal{B}(0, \tau) \setminus \mathcal{B}(0, r)$, *will approach partially exponentially to* $\mathcal{B}(0, r)$ *for t large enough.*

Indeed, since $\mathcal{R}e\lambda(\widetilde{\mathcal{A}}_1) < 0$, *then the unperturbed nominal linear time–invariant system* (14) *is globally uniformly exponentially with respect to* x_1, *as appears from Figure* 5.

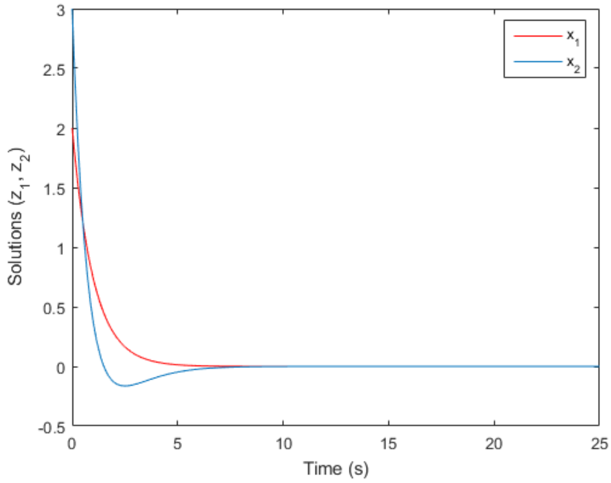

Figure 4: Time evolution of the states $z_1(t)$ and $z_2(t)$ of the unperturbed nominal linear system (14)

In addition, we have

$$\|\mathcal{G}(t,x)\|^2 = \eta \sqrt{|x_1|} + e^{-2t}.$$

where $\Gamma_1(t) = \eta$ *and* $\Gamma_2(t) = e^{-2t}$*, which satisfy both conditions of assumption* $(\mathcal{H}_2')$*.*

Hence by applying Theorem 4.1, the nonlinear system is practically uniformly exponentially stable with respect to x_1*, as can be seen in Figure* 5*, for* $\eta = 3$*.*

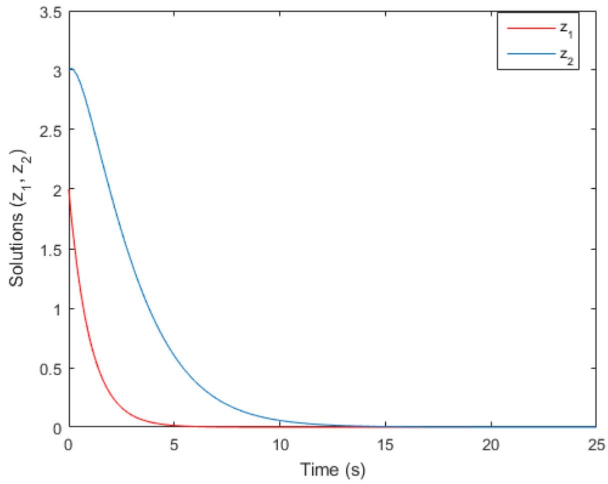

Figure 5: Time evolution of the states $z_1(t)$ and $z_2(t)$ of the nonlinear system (12)

5. Conclusion

In this article, we use the method of integral inequalities to establish partial practical stability criteria for certain classes of linear time–invariant perturbed systems. The proof procedure is explained in details by using integral inequalities of the Gronwall type, in particular of Gamidov's type. At the same time, we also develop the problem of local partial stability of nonlinear systems. We also give two examples to illustrate our theoretical results.

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